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THE NUCLEOLUS AS A NONCOOPERATIVE GAME SOLUTION

M. Shubik and H. P. Young

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THE NUCLEOLUS AS A NONCOOPERATIVE GAME SOLUTION[†]

M. Shubik and H.P. Young*

INTRODUCTION

In this paper we present a new approach to measuring the value of players in a voting game (i.e., a simple game) in terms of the prices competing lobbyists would pay to buy control of the players. One natural model of this situation, which was conceived of independently by the present authors, is to assume that two lobbyists compete for the players at each vote, and that the player does the bidding of whoever pays him the most. This leads to a two-person zero-sum game with discontinuities in the payoff function that is related to so-called Colonel Blotto games [3,4,7] -- but which differs from these in the complementarities that exist between different sets of players. The expected amounts offered to the *voters* in an equilibrium solution to this two-person game constitute a noncooperative value of the underlying voting game. The computation of this value involves technical difficulties however, since in most cases equilibrium solutions in *pure* strategies do not exist. In fact, in certain cases no equilibrium exists at all. A further analysis of this model and the corresponding value may be found in [14,15].

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In this article we propose an alternative model of the same problem which is more tractable in that an equilibrium solution in *pure strategies* typically exists. In this second model the lobbyists do not bid for the players on each bill, but rather bid for "shares" of the players at the start of a legislative session; the shares then determine how many times the players may be used by each lobbyist to form coalitions during the session. The imputed value to the players then turns out to be essentially the nucleolus when the game is "decisive."

1. BASIC DEFINITIONS

A *voting game* (or *simple game*) Γ is a set N of players together with a list W of the sets of players whose "yea" votes are sufficient to pass a measure. In general we assume that

$$(1) \quad \begin{aligned} &\phi \notin W, \quad N \in W, \\ &S \in W \text{ and } S \subseteq T \text{ implies } T \in W \end{aligned}$$

A class of voting games of particular interest in the sequel are the *decisive* games: those games in which, for every subset S of players, exactly one of S and $N-S$ wins [2]. The prototype of this class of games is simple majority rule with an odd number of votes (i.e., no "ties" are allowed). A second example is a weighted voting game with a simple majority quota. In general, we say that Γ is a *weighted voting game* with *representation* $(q; w_1, w_2, \dots, w_n)$ if player i has weight w_i and a set S wins if and only if $\sum_{i \in S} w_i \geq q$ where q is the *quota*. A weighted voting game Γ is said to be *homogeneous* [12] if it has a representation $(q; w_1, \dots, w_n)$ such that every minimal winning set has weight precisely equal to q .

Example 1

Consider the weighted voting game on four players $(3; 2, 1, 1, 1)$. There are four minimal winning coalitions described by

$$\{I, II\} \quad , \quad \{I, III\} \quad , \quad \{I, IV\} \quad \text{and} \quad \{II, III, IV\}$$

where I, II, III, IV are the names of the players. This game is easily seen to be decisive. Note also that a weighted voting game may have several different representations: for instance the above game can also be represented by $(2.5; 2, 1, 1, 1)$ to give just one possibility.

2. DESCRIPTION OF THE MODEL

Consider two lobbyists A and B who compete for control of the players (voters) $N = \{1, 2, \dots, n\}$ in a voting game Γ . A *strategy* of lobbyist A is to offer x_i "dollars" to voter i subject

to a budget limitation $\sum x_i \leq a$. Thus A's strategy space is the set \underline{X} of all vectors satisfying

$$\underline{X} = \{ \underline{x} = (x_1, x_2, \dots, x_n) \geq 0 \text{ and } \sum x_i \leq a \} .$$

Similarly, B's strategies are the vectors

$$\underline{Y} = \{ \underline{y} = (y_1, y_2, \dots, y_n) \geq 0 \text{ and } \sum y_i \leq b \} .$$

The Blotto-type formulation of the game is then to postulate that lobbyist A gains control of player i on a particular vote if $x_i > y_i$, B gains control if $y_i > x_i$, and there is a standoff or "tie" if $x_i = y_i$. We assume that A's objective is to *pass* a bill, that is, to put together a winning coalition of players. The objective of B is to *block* the bill, so B tries to put together a coalition of players whose complement does *not* win (a *blocking* coalition).

The *payoff* to A is defined to be +1 if the measure passes and -1 if it fails; the payoff to B is +1 if it fails and -1 if it passes. Hence a two-person, zero-sum game is defined. However, in general, an equilibrium solution in pure strategies for this game will not exist, except for the case when one of the lobbyists (the "fat" lobbyist) has substantially more resources than the other. In this case a solution in pure strategies obtains that is essentially the nucleolus [14].

Here we consider an alternative model of the problem that also leads essentially to the nucleolus as a pure strategy solution when the lobbyists have *equal* resources. In this second model, instead of the lobbyists competing for the voters on each bill (or vote), we suppose that the lobbyists compete at the beginning of the legislative session for "shares" of the voters that will apply to that whole session.

Specifically, we imagine that every voter is willing to sell himself a certain number of times, t , during the session. At the beginning of the session the lobbyists make bids for each of the voters that will determine how many times each lobbyist is able to use that player to form a coalition. For the present we

restrict the analysis to two lobbyists, although the model can easily be extended to the case of more than two lobbyists.

Given strategies \underline{x} and \underline{y} for A and B as above, lobbyist A gets an $x_i/(x_i + y_i)$ share of voter i , and B gets $y_i/(x_i + y_i)$. This means that A gets to use player i $x_i t/(x_i + y_i)$ times to form a coalition, and B gets to use player i $y_i t/(x_i + y_i)$ times. (If $x_i = y_i = 0$ we consider both shares to be $\frac{1}{2}$.) The objective of lobbyist A is to form as many winning coalitions as possible during the session using his shares of the players. Similarly, the objective of B is to form as many blocking coalitions as possible using his shares. For a decisive voting game this is the same as A's objective, since the winning coalitions are the same as the blocking coalitions. We also implicitly assume that the lobbyists never compete directly by trying to form coalitions for the same bill.

Given strategies \underline{x} and \underline{y} , let the shares be denoted by

$$\alpha_i = x_i/(x_i + y_i) \quad \text{for A}$$

$$\beta_i = y_i/(x_i + y_i) = 1 - \alpha_i \quad \text{for B .}$$

To find out how many coalitions he can form with his shares, A solves a linear program defined as follows. Let \hat{W} be the set of all *minimal* winning coalitions, and let M be the $(0,1)$ -incidence matrix of minimal winning coalitions versus players. That is, the rows of M represent minimal winning coalitions S and the columns represent the players i , and there is a 1 in position (S,i) if player i is in coalition S and a 0 otherwise. Let A use coalition S λ_S times; then his objective is to find numbers λ_S for each $S \in \hat{W}$ such that

$$\sum_{S \in \hat{W}} \lambda_S = \max$$

(2) subject to $\lambda M \leq (\alpha_1 t, \alpha_2 t, \dots, \alpha_n t)$

$$\lambda \geq 0 .$$

Similarly, B's objective is to solve

$$\begin{aligned}
 \sum_{S \in \hat{W}} \lambda_S &= \max \\
 (3) \quad \text{subject to } \lambda M &\leq (\beta_1 t, \beta_2 t, \dots, \beta_n t) \\
 \lambda &\geq 0 .
 \end{aligned}$$

In terms of actual applications, of course, the numbers λ_S and shares $\alpha_i t$ should be *integer*, but this can always be guaranteed by suitable choice of t (assuming the shares α_i are rational). For computing the noncooperative value, however, all that really matters is the *relative* value of the variables, so we may as well assume that $t = 1$.

Given any shares $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, let $v(\underline{\alpha})$ be the optimal value of (2). For given \underline{x} and \underline{y} and resulting shares $\underline{\alpha}$, the *payoff* to A is defined to be $v(\underline{\alpha})$ and the payoff to B is defined to be $v(1 - \underline{\alpha})$. This determines a two-person non-zero sum game, which we call the *session lobbying game* associated with Γ .

Let $(\underline{x}, \underline{y})$ be an equilibrium point for the session lobbying game. Then values may be imputed in a natural way to the voters by considering the total expected amount each is offered: thus, the *value* of voter i is $x_i + y_i$.

3. THE EQUILIBRIUM SOLUTION

The noncooperative value defined in this way is essentially the nucleolus of the game. More precisely, the *least core* of the voting game Γ is defined to be the set of all imputations $x = (x_1, x_2, \dots, x_n)$ such that the largest "excess value" of any set is minimized [11]. Adopting the usual convention that the value of any winning set is +1, we have:

$$\begin{aligned}
 \min e \\
 (4) \quad \text{subject to } 1 - \sum x_i &\leq e \quad \text{for all } S \in \hat{W} . \\
 x_i &\geq 0 \quad , \quad \sum_{i=1}^n x_i = 1
 \end{aligned}$$

If this program is nondegenerate, which will typically be the case, then the optimal solution is of course precisely the nucleolus [6,8]. In the case of degeneracy the least core forms a closed convex set that contains the nucleolus. We say that Γ is *nondegenerate* if the least core consists of one point (the nucleolus); otherwise Γ is *degenerate*.

An equivalent and somewhat more convenient formulation of the linear program (4) is the following:

$$(5) \quad \begin{aligned} & \min \sum w_i \\ & \text{subject to } \sum_{i \in S} w_i \geq a \quad \text{for each } S \in \hat{W}, \\ & \quad \quad \quad \underline{w} \geq 0, \end{aligned}$$

where $a > 0$. If \underline{w} is optimal for (5), then clearly for any $\lambda > 0$, $\lambda \underline{w}$ is optimal when a is replaced by λa . Therefore let us choose $a = a^* > 0$ so that the optimum value of (5) is 1, and let \underline{w}^* be a corresponding optimal solution. The maximum excess of any set is then $1 - a^*$. If \underline{w}' is any imputation in the least core with excess $1 - a' < 1 - a^*$, then for all $S \in \hat{W}$

$$\sum_{i \in S} w'_i \geq a' > a^*$$

whence \underline{w}^* could *not* have been optimal to (5) with $a = a^*$. Therefore $a' = a^*$ and the least core is given by the optimal solutions to (5) (with $a = a^*$).

Theorem. If Γ is a nondegenerate decisive voting game with nucleolus \underline{w}^* then $(\underline{w}^*, \underline{w}^*)$ is the unique equilibrium for the session lobbying game, and the values of the players are proportional to $w_1^*, w_2^*, \dots, w_n^*$.

Proof. Let Γ be a nondegenerate, decisive voting game with nucleolus \underline{w}^* . With the incidence matrix M defined as before, and for suitably chosen a^* , \underline{w}^* is the unique optimal solution to the linear program.

$$\begin{aligned}
 & \min \sum w_i \\
 (6) \quad & \text{subject to } M\tilde{w} \geq a^*(1,1,\dots,1) \\
 & \tilde{w} \geq 0 \quad .
 \end{aligned}$$

Now let A and B, with budgets $a = 1$, play the session lobbying game associated with Γ . Consider any pair of pure strategies \tilde{x} and \tilde{y} . Letting $\alpha_i = x_i/(x_i + y_i)$, the payoff to A is $v(\alpha) = \sum_{S \in \hat{W}} \lambda'_S$, where $\tilde{\lambda}'$ is an optimal solution to

$$\begin{aligned}
 & \max \sum_{S \in \hat{W}} \lambda_S \\
 (7) \quad & \text{subject to } \tilde{\lambda}M \leq \alpha \\
 & \tilde{\lambda} \geq 0 \quad .
 \end{aligned}$$

The payoff to B is $v(1 - \alpha) = \sum_{S \in \hat{W}} \lambda''_S$ for some $\tilde{\lambda}''$ optimal to

$$\begin{aligned}
 & \max \sum_{S \in \hat{W}} \lambda_S \\
 (8) \quad & \text{subject to } \tilde{\lambda}M \leq 1 - \alpha \\
 & \tilde{\lambda} \geq 0 \quad .
 \end{aligned}$$

Therefore $\lambda' + \lambda''$ is feasible for the linear program

$$(9) \quad \begin{aligned} & \max \sum_{S \in \hat{W}} \lambda_S \\ & \text{subject to } \lambda_M \leq (1, 1, \dots, 1) = \underline{1} \\ & \lambda \geq \underline{0} \end{aligned}$$

whence

$$(10) \quad v(\underline{1}) \geq \sum_{S \in \hat{W}} \lambda'_S + \sum_{S \in \hat{W}} \lambda''_S = v(\underline{\alpha}) + v(\underline{1} - \underline{\alpha}) .$$

Now suppose $(\underline{x}, \underline{y})$ is an equilibrium pair. Then the payoff to A under $(\underline{y}, \underline{y})$ must be no larger than that under $(\underline{x}, \underline{y})$. Since under $(\underline{y}, \underline{y})$ A's share of each player is $1/2$, we have

$$v(1/2, 1/2, \dots, 1/2) = 1/2 v(\underline{1}) \leq v(\underline{\alpha}) .$$

Similarly, the payoff to B under $(\underline{x}, \underline{x})$ is no larger than under $(\underline{x}, \underline{y})$, so

$$v(1/2, 1/2, \dots, 1/2) = 1/2 v(\underline{1}) \leq v(\underline{1} - \underline{\alpha}) .$$

Adding, and using (10), we see that

$$(11) \quad v(\underline{1}) \leq v(\underline{\alpha}) + v(\underline{1} - \underline{\alpha}) \leq v(\underline{1}) ,$$

whence

$$v(\underline{\alpha}) = v(\underline{1} - \underline{\alpha}) = 1/2 v(\underline{1}) = v(1/2, \dots, 1/2) .$$

Thus $(\underline{x}, \underline{x})$ and $(\underline{y}, \underline{y})$ are also equilibrium pairs. We claim that $\underline{x} = \underline{y} = \underline{w}^*$. For this, it is enough to show that $\underline{x} = \underline{w}^*$.

First, for any shares α , consider the dual program to the program (7) which A uses to get his payoff function:

$$\min \sum \alpha_i w_i$$

$$(12) \quad \text{subject to } M\bar{w} \geq (1, \dots, 1)$$

$$\bar{w} \geq 0 .$$

This is essentially the same program as (6) if all α_i are equal. In fact, if all $\alpha_i = 1/2$, then \bar{w}^*/a^* is the unique optimal solution to (12), because by hypothesis \bar{w}^* is the unique optimal solution to (6). The payoff to A is then $v = (1/a^*) \sum w_i^*/2$. Further, by a well-known result in linear programming, (12) has a unique optimal solution at $\alpha = (1/2, \dots, 1/2)$ implies the same solution is also the unique optimal solution for all sufficiently small perturbations of the cost function α .

Suppose now that $\bar{x} \neq \bar{w}^*$. Let $z = \epsilon w^* + (1 - \epsilon)\bar{x}^*$ for some small $\epsilon > 0$, and let α' be the resulting shares: $\alpha'_i = z_i / (z_i + x_i)$. Then α'_i is close to $1/2$ for all i , so \bar{w}^* is the unique optimal solution to (12).[†] Thus, the payoff to A under (z, \bar{x}) is

$$v'_\epsilon = (1/a^*) \sum z_i w_i^* / (z_i + x_i) .$$

We claim that $v'_\epsilon > v$, contradicting the fact that (\bar{x}, \bar{x}) is an equilibrium. The following quite general inequality will be used ([5], p.39):

[†]Unless perhaps $x_i = 0$ and $w_i^* > 0$ for some i . Let T be the set of all such i and consider the modified shares $\alpha''(\epsilon)$ defined by $\alpha'_i(\epsilon) = z_i / (z_i + x_i)$ for $i \notin T$, $\alpha_i(\epsilon) = 1/2 + \delta$ for $i \in T$ and some fixed choice of δ , $1/2 \geq \delta > 0$. If δ is sufficiently small, \bar{w}^* will be the optimal solution to (12) when $\alpha = \alpha''(\epsilon)$ for all sufficiently small $\epsilon > 0$. Since $v''_\epsilon = v(\alpha''(\epsilon)) \rightarrow v + \delta \sum_{i \in T} w_i^*$ as $\epsilon \rightarrow 0$, it follows that either $v''_\epsilon > v$ for sufficiently small ϵ (but then $v'_\epsilon \geq v''_\epsilon > v$ so (\bar{x}, \bar{x}) was not an equilibrium), or $w_i^* = 0$ whenever $x_i = 0$, contrary to hypothesis.

for any real numbers p and q ,

$$(13) \quad 2p(p - q) \geq p^2 - q^2 \quad \text{with } = \text{ iff } p = q .$$

This is verified by dividing by $p - q$ and considering the two cases $p - q > 0$, $p - q < 0$.

$$\text{Now } a^*(v'_\epsilon - v) = \sum z_i w_i^* / (z_i + x_i) - \sum w_i^* / 2 .$$

$$\text{Since } \sum w_i^* = \sum x_i = \sum z_i = 1 ,$$

$$\begin{aligned} a^*(v'_\epsilon - v) &= \sum z_i w_i^* / (z_i + x_i) - \sum z_i / 2 , \\ &= \sum z_i [x_i + (z_i - x_i) / \epsilon] / (z_i + x_i) - \sum z_i / 2 . \end{aligned}$$

For each i we have

$$\begin{aligned} z_i [x_i + (z_i - x_i) / \epsilon] / (z_i + x_i) - z_i / 2 &= \frac{z_i (z_i - x_i) / \epsilon + z_i x_i / 2 - z_i^2 / 2}{z_i + x_i} \\ &= (1/\epsilon - 1/2) [z_i (z_i - x_i)] / (z_i + x_i) \\ (14) \quad &\geq \frac{(1/\epsilon - 1/2) (z_i^2 - x_i^2)}{2(z_i + x_i)} = \frac{(1/\epsilon - 1/2) (z_i - x_i)}{2} , \end{aligned}$$

the inequality by (13).

Summing (14) over i , we obtain

$$(15) \quad a^*(v'_\epsilon - v) \geq (1/2\epsilon - 1/4) \sum_i (z_i - x_i) = 0 ,$$

with equality only if $z_i = x_i$ for all i . Since $\underline{x} \neq \underline{w}^*$, $z_i \neq x_i$ for some i and so $v'_\epsilon - v > 0$, a contradiction to equilibrium.

Therefore $\underline{x} = \underline{w}^*$ and similarly $\underline{y} = \underline{w}^*$, so any equilibrium must have form $(\underline{w}^*, \underline{w}^*)$. It remains only to verify that $(\underline{w}^*, \underline{w}^*)$ is actually an equilibrium. For any $\underline{x} \geq 0$, such that $\sum x_i = 1$, the payoff to A when B plays \underline{w}^* is given by v''/a^* where

$$\begin{aligned}
 v'' &= \min \sum x_i w_i / (x_i + w_i^*) \\
 (16) \quad &\text{subject to } \underline{Mw} \geq (1, 1, \dots, 1) \\
 &\underline{w} \geq \underline{0} .
 \end{aligned}$$

Since \underline{w}^*/a^* is feasible for (16),

$$v'' \leq (1/a^*) \sum x_i w_i^* / (x_i + w_i^*) .$$

But, again by inequality (13),

$$w_i^*/2 - x_i w_i^* / (x_i + w_i^*) \geq 1/4 (w_i^* - x_i) \quad \text{with } = \text{ only if } x_i = w_i^* .$$

Summing over i , it follows that

$$v - v'' \geq \sum_i \left[w_i^*/2 - x_i w_i^* / (x_i + w_i^*) \right] \geq \sum_i (w_i^* - x_i) / 4a^* = 0 ,$$

with equality only if $\underline{x} = \underline{w}^*$. Hence $(\underline{w}^*, \underline{w}^*)$ is the unique equilibrium pair, proving the theorem. \square

A variation of the model is to define the session lobbying game as a zero-sum game. In this case each of the two lobbyists is assumed to be as interested in *preventing* his opponent from being able to form coalitions as to be able to form them himself. Specifically, if A and B bid \underline{x} and \underline{y} respectively, so that A's shares of the players are $\alpha_i = x_i / (x_i + y_i)$ and B's shares are $\beta_i = 1 - \alpha_i$, then A will be able to form $v(\underline{\alpha})$ coalitions whereas B will be able to form $v(\underline{1} - \underline{\alpha})$ coalitions. We define the *zero-sum payoff* to A as

$$v(\underline{\alpha}) - v(\underline{1} - \underline{\alpha}) .$$

Similarly, the *zero-sum payoff* to B is

$$v(\underline{1} - \underline{\alpha}) - v(\underline{\alpha}) .$$

Then the same type of argument used above shows that

- (17) *if Γ is nondegenerate with nucleolus \underline{w}^* , then $(\underline{w}^*, \underline{w}^*)$ is a local equilibrium solution to the zero-sum game.*

This follows by the observation that, if lobbyist A changes to a strategy $\underline{x} \neq \underline{w}^*$ when B is playing \underline{w}^* , then A's payoff will be *at most*

$$(18) \quad u = (1/a^*) \left[\sum x_i w_i^* / (x_i + w_i^*) - \sum w_i^* w_i^* / (x_i + w_i^*) \right] ,$$

assuming that the coefficients $w_i^* / (x_i + w_i^*)$ are sufficiently close to $1/2$ that \underline{w}^* / a^* is still the optimal solution to (12) when $\alpha_i = w_i^* / (x_i + w_i^*)$.

Using inequality (13) again, we see that for each i

$$\frac{(x_i - w_i^*) w_i^*}{(x_i + w_i^*)} \leq \frac{x_i - w_i^*}{2} \quad \text{with } = \text{ only if } x_i = w_i^* ,$$

so, since $\sum x_i = \sum w_i^* = 1$, and $\underline{x} \neq \underline{w}^*$,

$$u < \sum \frac{x_i - w_i^*}{2} = 0 ,$$

showing that A is worse off than he was under $(\underline{w}^*, \underline{w}^*)$.

We conjecture that the nucleolus is in fact a global equilibrium for the zero-sum game, even when the game is degenerate. The situation for nondecisive games -- where the roles of the lobbyists are not symmetric -- is under investigation.

We further note that a class of models closely related to these provide a natural extension of the Colomblot class of games to a more general and tractable class of problems. This application will be considered elsewhere.

References

- [1] Banzhaf, John F., One Man, 3.312 Votes: a Mathematical Analysis of the Electoral College, *Villanova Law Review*, 13 (1968), 304-332.
- [2] Brams, Steven J., *Game Theory and Politics*, The Free Press, New York, 1975.
- [3] Gross, O., *The Symmetric Colonel Blotto Game*, RAND Memorandum 424, Santa Monica, California, 1951.
- [4] Gross, O., and R. Wagner, *A Continuous Colonel Blotto Game*, RAND Memorandum 408, Santa Monica, California, 1950.
- [5] Hardy, G.W., J.E. Littlewood, and G. Polya, *Inequalities* (2nd ed.), Cambridge University Press, Cambridge, 1952.
- [6] Kopelowitz, A., *Computation of the Kernels of Simple Games and the Nucleolus of N-Person Games*, RM No. 31, Research Program in Game Theory and Mathematical Economics, Department of Mathematics, Hebrew University of Jerusalem, September 1967.
- [7] McDonald, John, and John W. Tukey, Colonel Blotto: A Problem of Military Strategy, in M. Shubik, ed., *Game Theory and Related Approaches to Social Behavior*, John Wiley & Sons, New York, 1964.
- [8] Schmeidler, D., The Nucleolus of a Characteristic Function Game, *SIAM Journal of Applied Mathematics*, 17(6), (1969) 1163-1170.
- [9] Shapley, Lloyd S., A Value for n-Person Games, in H.W. Kuhn and A.W. Tucker, eds., *Annals of Mathematics Studies*, No. 28, *Contributions to the Theory of Games*, Volume II, Princeton University Press, Princeton, N.J., 307-317, 1953.
- [10] Shapley, Lloyd S., and Martin Shubik, A Method for Evaluating the Distribution of Power in a Committee System, *American Political Science Review*, 48 (1954), 787-792.
- [11] Shapley, Lloyd S., and Martin Shubik, *Game Theory in Economics - Chapter 6: Characteristic Function, Core, and Stable Set*, RAND R-904-NSF/6, July, 1973.
- [12] von Neumann, John, and Oskar Morgenstern, *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, N.J., 1953.
- [13] Young, H. Peyton, *Power, Prices, and Incomes in Voting Systems*, RR-77-5, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1977.

- [14] Young, H. Peyton, The Allocation of Funds in Lobbying and Campaigning, to appear in *Behavioral Science* 1978.
- [15] Young, H. Peyton, *A Tactical Lobbying Game*, Conference on Game Theory and Political Science, Hyannis, Mass., July, 1977.