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Nash Equilibria of Market Games: I. Existence and Convergence

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I. Introduction

In this paper, and its sequel [1], our aim is to establish existence, convergence and finiteness of Nash Equilibria of market games. This was suggested by parallel results for the competitive equilibria of markets [2, 3, 4, 5]. We leave the question of finiteness for [1]. Here we consider a sequence (Γ_n) of finite market games that "approach" a nonatomic market game Γ. If ${}_n p$ is a "Nash price" of Γ_n, and ${}_n p$ converges to $p > 0$, we show in Section 3 that p is a Nash price of Γ. This convergence result partially justifies the nonatomic market games considered in [6] and [7], and elsewhere, in that they are shown

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**I am grateful to Lloyd Shapley, Martin Shubik and Bob Weber for helpful discussions; Bob also went through the text, and found a flaw in the proof of Lemma 1, which has been retreived jointly with him in the Appendix. Bob perceives no other flaws, but will be on record as not being enchanted with numerous left subscripts that recur constantly.

to capture the Nash solutions of large (but finite) market games that have no singularly influential trader. Moreover the convergence result, combined with existence theorems for finite games, paves the way for proving the existence of Nash Equilibria in nonatomic games. This is carried out in Section 4. To be specific we work with the "commodity money, sell-all model" of [8], but the same techniques can be used to establish convergence and existence in the context of other models. The mathematical set up for the convergence problem is taken <u>en toto</u> from Hildenbrand [3], to whom we refer freely and frequently.

2. The Market Game

Let $\{T, \mathcal{C}, \mu\}$ be a measure space of traders, where $T \equiv$ the set of traders, $\mathcal{C} \equiv$ the σ-algebra of coalitions, $\mu \equiv$ a measure* on $\{T, \mathcal{C}\}$. Trade occurs in $\ell+1$ commodities. We will denote by $\Omega^{\ell+1}$ the non-negative orthant of the Euclidean space of dimension $\ell+1$. Vectors in $\Omega^{\ell+1}$ represent commodity bundles. For any $v \in \Omega^{\ell+1}$, v_j is the j^{th} component of v. 0 denotes the origin of $\Omega^{\ell+1}$, and also the number zero (there is no confusion). A set $S \in \mathcal{C}$ is called null if $\mu(S) = 0$; otherwise it is called non-null. When we talk of "all $t \in T$" we will mean all t except for a null set.

To describe the market let us first set up the space of traders' characteristics. Let $\mathcal{P}$ denote the set of all continuous, convex, monotonic preference relations on $\Omega^{\ell+1}$. An exchange market is a measurable** mapping $\mathcal{E} : (T, \mathcal{C}, \mu) \to \mathcal{P} \times \Omega^{\ell+1}$. It will be convenient to think of $\mathcal{E}$ as made up of two mappings, $\succ : (T, \mathcal{C}, \mu) \to \mathcal{P}$ and $a : (T, \mathcal{C}, \mu) \to \Omega^{\ell+1}$, where $\succ(t) \equiv \succ^t$ is the preference of t, and $a(t) \equiv a^t$ the initial endowment of t.

We will now recast the market $\mathcal{E}$ in the form of a noncooperative game $\Gamma(\mathcal{E})$ in strategic form. There are many ways of doing this [6, 7, 8, 9, 10], but we focus our attention on the "sell-all model with commodity money" of [8]. The $\ell+1^{st}$ commodity is singled out as money. There are ℓ trading posts, one for each of the other commodities. Traders are required to put up all of their first ℓ commodities for sale in these trading posts, and use the endowment of commodity money for bidding

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* μ may have finite support.

** $\mathcal{P} \times \Omega^{\ell+1}$ is a separable metric space as in [3] (see B.II and 1.2); the σ-algebra on it is the one generated by its open sets.

on them. The formal treatment is as follows. The strategy set S^t of t consists of bids in the ℓ trading posts, but he is constrained to bit within $a^t_{\ell+1}$:

$$S^t = \{b^t \in \Omega^\ell : \sum_{j=1}^{\ell} b^t_j \leq a^t_{\ell+1}\} .$$

Given a choice* of strategies $b : T \to \Omega^\ell$, $b^t \in S^t$, prices $p(b) \in \Omega^\ell$ are formed in the trading posts, and the markets cleared, with the final bundle $x^t(b)$ accruing to t, according to the rules:*

$$p_j(b) = \int_T b_j / \int_T a_j$$

$$x^t_j(b) = \begin{cases} b^t_j / p_j(b) & \text{if } p_j(b) > 0 \\ 0 & \text{if } p_j(b) = 0 \end{cases}$$

for $j = 1, \ldots, \ell$; and

$$x^t_{\ell+1}(b) = a^t_{\ell+1} - \sum_{j=1}^{\ell} b^t_j + \sum_{j=1}^{\ell} p_j(b) a^t_j .$$

A <u>Nash Equilibrium</u> (N.E.) of the game $\Gamma(\mathcal{E})$ is a measurable mapping** ${}^*b : T \to \Omega^\ell$, ${}^*b^t \in S^t$, such that for all $t \in T$

$$x^t({}^*b) \succsim_t x^t({}^*b|b^t) , \quad b^t \in S^t ,$$

where $({}^*b|b^t)$ is the same as ${}^*b$, but with ${}^*b^t$ replaced by b^t. An <u>N.E. price</u> is a price produced at an N.E.

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*When μ is nonatomic we encounter the problem of justifying that b, which arises from independent decisions made by the players, is nevertheless measurable. This has been discussed in [6]. Here we will always take b to be measurable.

**The * in *b does not refer to the footnotes here or in the sequel.

3. Convergence

The notion of a sequence of markets (with finite and increasing numbers of traders) that "approach" a nonatomic market is formalized by Hildenbrand as a "purely competitive sequence of simple markets" in [3]. We recapitulate the definition from [3]. Let $(\mathcal{E}_n)_{n=1}^{\infty}$ be a sequence of (simple) markets, $\mathcal{E}_n : (T_n, \mathcal{C}_n, \alpha_n) \to \mathcal{P} \times \Omega^{\ell+1}$, where $|T_n|$ is finite and α_n is the counting measure. The preference-endowment distribution of $\mathcal{E}_n$ is a measure μ_n on $\mathcal{P} \times \Omega^{\ell+1}$ defined by: $\mu_n(S) = \alpha_n(\mathcal{E}_n^{-1}(S))/|T_n|$ for any measurable set $S \subset \mathcal{P} \times \Omega^{\ell+1}$. The sequence $(\mathcal{E}_n)$ is called purely competitive (see Def. 4 in 2.1 of [3]) if:*

(i) $|T_n| \to \infty$

(ii) μ_n converges weakly on $\mathcal{P} \times \Omega^{\ell+1}$ to some measure μ

(iii) $\lim \int_{\Omega^{\ell+1}} a d\tilde{\mu}_n = \int_{\Omega^{\ell+1}} a d\tilde{\mu}$

(iv) $\int_{\Omega^{\ell+1}} a d\tilde{\mu} > 0$.

If $(\mathcal{E}_n)$ is purely competitive then, by Proposition 2 in Section 2.1 of [3], there is a continuous representation of it defined by a nonatomic economy $\mathcal{E}: (T, \mathcal{C}, \nu) \to \mathcal{P} \times \Omega^{\ell+1}$, and a sequence (β_n) of mappings $\beta_n : T \to T_n$ which have the properties:

(i) $\nu(\beta_n^{-1}(S)) = |S|/|T_n|$ for every $S \subset T_n$

(ii) $\tilde{\mathcal{E}}_n = \mathcal{E}_n \circ \beta_n \to \mathcal{E}$ for all t in T .

It will be much more convenient for us to work with a continuous representation of a purely competitive sequence. Intuitively $t \in T_n$ can be identified now with the set $\beta_n^{-1}(t) \subset T$, and t's endowment with

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* $\tilde{\mu}_n$ $(\tilde{\mu})$ is the marginal distribution of μ_n (μ) on $\Omega^{\ell+1}$.

$\nu(\beta_n^{-1}(t))\cdot {}_n a^t$. ${}_n a$ (${}_n \succ$) will denote the endowments (preferences) in $\mathcal{E}_n$.

Our aim in this section is

Theorem 1. Let $(\mathcal{E}_n)$ be a purely competitive sequence, and $\{\mathcal{E}, (\beta_n)\}$ a continuous representation of it. Let ${}_n p$ be an N.E.-price of $\Gamma(\mathcal{E}_n)$ and suppose ${}_n p \to p > 0$. Then p is an N.E. price of $\Gamma(\mathcal{E})$.

Before the proof we need some notation. Given any mapping $f : T_n \to Y$ (where Y is an arbitrary set), we will also think of it as a mapping from T to Y given by: $f \circ \beta_n : T \to Y$. (Thus $\tilde{\mathcal{E}}_n \equiv \mathcal{E}_n$.)

For $t \in T_n$, $\underset{\sim}{b} \in \Omega^\ell$, $b : T_n \to \Omega^\ell$ and $1 \le j \le \ell$ define*

$${}_n S^t = \{d \in \Omega^\ell : \sum_{j=1}^{\ell} d_j \le {}_n a^t_{\ell+1}\}$$

$$b_j^{-t} = \sum_{t' \in T_n \setminus \{t\}} b_j^{t'}$$

$$p_j^{-t}(b) = b_j^{-t} / {}_n\bar{a}_j$$

$${}_n x_j^t(b|\underset{\sim}{b}) = \underset{\sim}{b}_j \, {}_n\bar{a}_j / (\underset{\sim}{b}_j + b_j^{-t})$$

$${}_n x^t_{\ell+1}(b|\underset{\sim}{b}) = {}_n a^t_{\ell+1} - \sum_{j=1}^{\ell} \underset{\sim}{b}_j + \sum_{j=1}^{\ell} {}_n a^t_j (b_j^{-t} + \underset{\sim}{b}_j) / {}_n\bar{a}_j$$

$${}_n B^t(b) = \{{}_n x^t(b|\underset{\sim}{b}) \in \Omega^{\ell+1} : \underset{\sim}{b} \in {}_n S^t\}$$

$${}_n \hat{B}^t(b) = \{x \in {}_n B^t(b) : x \text{ is maximal under } {}_n\succ_t\} .$$

*Note (i) these are all mappings with domain T_n (also T , therefore); (ii) $\bar{b}_j$ denotes $\sum_{t \in T_n} b_j^t$ (and not $\int_T b_j^t$); similarly ${}_n\bar{a}_j$ denotes $\sum_{t \in T_n} {}_n a_j^t$, etc.

Let us assume that $b^{-t} > 0$. Then both ${}_nB^t$ and ${}_n\hat{B}^t$ are clearly compact. [It may be of interest to further note that ${}_nB^t(b)$ has the following property: if z is in the interior of* $Co[x,y]$, where $x \in {}_nB^t(b)$ and $y \in {}_nB^t(b)$, then there exists a $\hat{z} \in {}_nB^t(b)$ such that $\hat{z} > z$. (See Figure 1 below for $\ell+1 = 2$.)

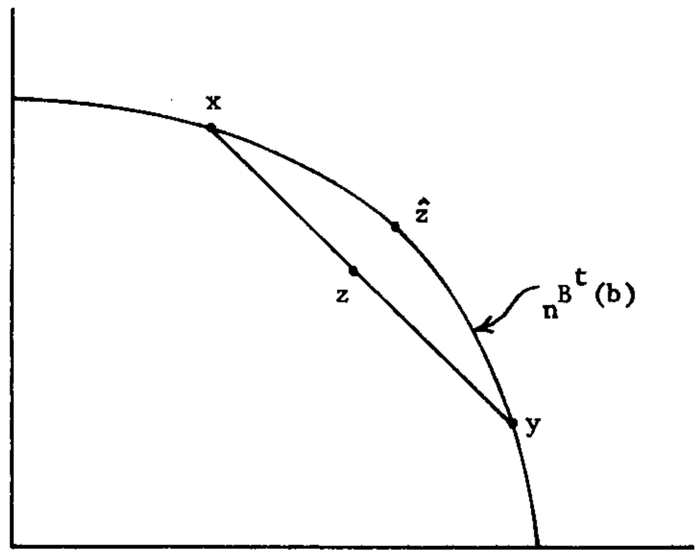

FIGURE 1

Hence ${}_n\hat{B}^t(b)$ is a singleton set, remembering that ${}_n\succ^t$ is monotonic.]

Also let us define, for any $t \in T$, $p \in \Omega^\ell$ $(p > 0)$, and $\underset{\sim}{b} \in \Omega^\ell$,

$$S^t = \{d \in \Omega^\ell : \sum_{j=1}^{\ell} d_j \leq a^t_{\ell+1}\} ;$$

$$x^t_j(p|\underset{\sim}{b}) = \underset{\sim}{b}_j/p_j , \quad j = 1, \ldots, \ell ;$$

$$x^t_{\ell+1}(p|\underset{\sim}{b}) = a^t_{\ell+1} - \sum_{j=1}^{\ell} \underset{\sim}{b}_j + \sum_{j=1}^{\ell} p_j a^t_j , \quad j = 1, \ldots, \ell ;$$

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* Co[x,y] is the convex hull of [x,y] .

$$B^t(p) = \{x^t(p|\underset{\sim}{b}) \in \Omega^{\ell+1} : \underset{\sim}{b} \in S^t\}$$

$$\hat{B}^t(p) = \{x \in B^t(p) : x \text{ is maximal under } >^t\} .$$

Clearly $B^t(p)$ and $\hat{B}^t(p)$ are both compact and convex.

<u>Proof of Theorem 1</u>. Let ${}_n\overset{*}{b} : T_n \to \Omega^\ell$ be a N.E. of $\Gamma(\mathcal{E}_n)$ which produces the price ${}_np$, $n = 1, 2, \ldots$. Think of ${}_n\overset{*}{b}$, ${}_n\overset{*}{b}^-$, ${}_na$, ${}_n\bar{a}$, etc. as mappings from T to Ω^ℓ (or $\Omega^{\ell+1}$), as described earlier. It will help to note certain limits that hold for all t. Since ${}_n\bar{a}_j/{}_n\bar{a}_h \to \int_T a_j/\int_T a_h > 0$, ${}_np \to p > 0$, ${}_na_j^t/{}_n\bar{a}_j \to 0$, we have:*

$${}_na^t_{\ell+1}\cdot{}_n\bar{a}_{\ell+1}/{}_np_j\cdot{}_n\bar{a}_{\ell+1}\cdot{}_n\bar{a}_j \to 0$$

$$\Longrightarrow {}_na^t_{\ell+1}/{}_np_j\cdot{}_n\bar{a}_j \to 0$$

$$\Longrightarrow {}_na^t_{\ell+1}/{}_n\overset{*}{\bar{b}}_j \to 0$$

$$\Longrightarrow {}_n\overset{*}{b}{}^t_j/{}_n\overset{*}{\bar{b}}_j \to 0$$

$$\Longrightarrow {}_np^{-t}({}_n\overset{*}{b}) \to p .$$

First we show that $\limsup {}_nB^t({}_n\overset{*}{b}) = B^t(p)$. (We will omit the arguments ${}_n\overset{*}{b}$ and p for a while, and t will be kept fixed throughout.) Suppose ${}_nx^t \in {}_nB^t$ and ${}_nx^t \to x$. Let

$${}_nx^t_j = \frac{{}_nb^t_j}{{}_n\overset{*}{b}{}^{-t}_j + {}_nb^t_j}\cdot{}_n\bar{a}_j , \quad j = 1, \ldots, \ell$$

$${}_nx^t_{\ell+1} = {}_na^t_{\ell+1} - \sum_{j=1}^{\ell}{}_nb^t_j + \sum_{j=1}^{\ell}{}_na^t_j\cdot\frac{{}_n\overset{*}{b}{}^{-t}_j + {}_nb^t_j}{{}_n\bar{a}_j} .$$

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*This ensures that for all t , there is an $n(t)$ such that $n > n(t)$ implies ${}_n\overset{*}{b}{}^{-t} > 0$.

Since

$$0 \leq \frac{{}_n b_j^t}{{}_n \bar{a}_j} \leq \frac{{}_n a_{\ell+1}^t}{{}_n \bar{a}_j} = \frac{{}_n a_{\ell+1}^t}{{}_n \bar{a}_{\ell+1}} \cdot \frac{{}_n \bar{a}_{\ell+1}}{{}_n \bar{a}_j} \to 0 ,$$

and, as shown,

$$\frac{{}^*_n b_j^{-t}}{{}_n \bar{a}_j} \to p_j ,$$

we have

$${}_n b_j^t \to p_j x_j \text{ , and } {}_n x_{\ell+1}^t \to a_{\ell+1}^t - \sum_{j=1}^{\ell} p_j x_j + \sum_{j=1}^{\ell} a_j^t p_j \quad .$$

Also $\sum_{j=1}^{\ell} {}_n b_j^t \leq {}_n a_{\ell+1}^t$, and ${}_n a_{\ell+1}^t \to a_{\ell+1}^t$, hence $\sum_{j=1}^{\ell} p_j x_j \leq a_{\ell+1}^t$. This proves that $x \in B^t(p)$, i.e. $\limsup {}_n B^t \subset B^t$. Now take any $x \in B^t$. Put $b_j = p_j x_j$, and note: $\sum_{j=1}^{\ell} b_j \leq a_{\ell+1}^t$, $x_{\ell+1} = a_{\ell+1}^t - \sum_{j=1}^{\ell} b_j + \sum_{j=1}^{\ell} a_j^t p_j$. Let C and D be positive numbers such that $C < (p_j)^2$ and $D > p_j$ for $j = 1,\ldots,\ell$. For any $\varepsilon > 0$, choose $n(\varepsilon)$ large enough to guarantee that, whenever* $n > n(\varepsilon)$,

$$||{}_n a^t - a^t|| < \varepsilon$$

$$C < p_j \left(\frac{{}^*_n b_j^{-t} + g}{{}_n \bar{a}_j} \right)$$

---

* $|| \ ||$ denotes the max norm.

$$D > \frac{{}_n^*b^{-t} + g}{{}_n\bar{a}_j}$$

$$\left|\left|\frac{{}_n^*b^{-t} + g}{{}_n\bar{a}_j} - p_j\right|\right| < \varepsilon$$

The last 3 conditions are imposed for $j = 1, \ldots, \ell$, and <u>uniformly</u> for $0 \leq g \leq {}_na^t_{\ell+1}$. (We can do this because ${}_n^*b_j^{-t}/{}_n\bar{a}_j \to p_j > 0$, and ${}_na^t_{\ell+1}/{}_n\bar{a}_j \to 0$.) Put

$${}_nb_j = \max[0,\ p_jx_j - \varepsilon/\ell]\ ,$$

$${}_nx_j = \frac{{}_nb_j}{{}_n^*b_j^{-t} + {}_nb_j} \cdot {}_n\bar{a}_j\ ,$$

$${}_nx_{\ell+1} = {}_na^t_{\ell+1} - \sum_{j=1}^{\ell} {}_nb_j + \sum_{j=1}^{\ell} {}_na^t_j \cdot \frac{{}_nb_j + {}_n^*b_j^{-t}}{{}_n\bar{a}_j}\ .$$

Clearly $\sum_{j=1}^{\ell} {}_nb_j \leq {}_na^t_{\ell+1}$, thus ${}_nx \in {}_nB^t$, for $n > n(\varepsilon)$. Also, as is straightforwardly verified, $||{}_nx - x|| < \max\left\{\max_{1\leq j\leq \ell}\left\{\frac{\varepsilon[p_j x_j + D]}{C}\right\}, 2\varepsilon + \ell\varepsilon D\right\}$.

We conclude that $\limsup {}_nB^t \supset B^t$, and thus

$$\limsup {}_n B^t = B^t \qquad (*)$$

Next we claim that $\limsup {}_n\hat{B}^t({}_n^*b) \subset \hat{B}^t(p)$, for all t . Let ${}_nx \to x$ where ${}_nx \in {}_n\hat{B}^t$. Then also ${}_nx \in {}_nB^t$, and by (*) we get $x \in B^t(p)$. Suppose there is a $z \in B^t$ such that $z \succ_t x$. Then, by (*), we can find a sequence $\{{}_nz\}, {}_nz \in {}_nB^t$, such that ${}_nz \longrightarrow z$. But then we must have ${}_nz \ {}_n{\not\succ}^t \ {}_nx$ because ${}_nx \in {}_n\hat{B}^t$. Hence by Theorem 1c in Section 1.2 of [3], $z \not\succ^t x$, a contradiction. We have verified our claim.

Now $\int_T {}_na^t \in \int_T {}_n\hat{B}^t({}_n^*b)$ for every n because ${}_n^*b$ is an N.E. of $\Gamma(\mathcal{E}_n)$. Therefore

$$\int_T a^t = \lim \int_{T^n} a^t$$

$$\in \limsup \int_{T^n} \hat{B}^t({}_n^*b)$$

$$\subset \int_T \limsup {}_n\hat{B}^t({}_n^*b)$$

$$\subset \int_T \hat{B}^t(p) .$$

The $=$ is an assumption. The first $\subset$ follows from Theorem 6 in Section DII in [3]; the second follows from the claim.

Let $x : T \to \Omega^{m+1}$ be a measurable mapping such that $x^t \in \hat{B}^t(p)$, and $\int_T x^t = \int_T a^t$. Put, for $t \in T$,

$$b_j^t = p_j x_j^t , \quad j = 1, \ldots, m .$$

It is readily checked that $b : T \to \Omega^m$ is an N.E. of $\Gamma(\mathcal{E})$ which produces the prices p.

Q.E.D.

4. Existence

A trivial N.E. of $\Gamma(\mathcal{E})$ is obtained by setting $b_j^t = 0$ for $t \in T$, $1 \leq j \leq \ell$. Other semi-trivial N.E. can be obtained by setting the bids at some subset of the trading posts to be 0. We are interested in establishing the existence of active Nash Equilibria, namely those at which all prices are positive. (This was done for markets with finite player sets in [9].)

The idea behind the proof is to construct a sequence $\{\mathcal{E}_n\}$ of finite markets, which is purely competitive, and which has the given nonatomic market $\mathcal{E}$ as its limiting continuous representation. By the existence result in [9], every $\Gamma(\mathcal{E}_n)$ has a Nash price ${}_np > 0$. If we could show that $\{{}_np\}$ has a cluster point $p > 0$ then, by Theorem 1, p would be a Nash price of $\Gamma(\mathcal{E})$. Much of the argument is thus centered around showing that the sequence $\{{}_np\}$ may be picked to lie in a compact set (in Ω^ℓ) that is bounded away from the faces of Ω^ℓ.

We begin with some more notations and definitions. Let $\mathcal{E}: \{T, \mathcal{C}, \mu\} \longrightarrow \mathcal{P} \times \Omega^{\ell+1}$ be a nonatomic market. We will in fact take T to be a separable metric space, endowed with a finite nonatomic measure μ, where w.l.o.g. $\mu(T) = 1$. Let us assume that $\succ^t$ can be represented by a continuous, nondecreasing, concave function $u^t : \Omega^{\ell+1} \to \Omega^1$ for every $t \in T$. (To express this we will replace $\mathcal{P}$ by U henceforth.) We will say that trader t desires commodity j if u^t is a strictly increasing function of the variable x_j , holding the other variables fixed; and that t is <u>moneyed</u> if $a^t_{\ell+1} > 0$. We wish to prove the following theorem.

<u>*THEOREM 2*</u>. *In the nonatomic economy* $\mathcal{E}: (T, \mathcal{C}, \mu) \longrightarrow U \times \Omega^{\ell+1}$ *, assume that for every* $j = 1,\ldots,\ell$ *, there is a nonnull set* S_j *of traders who are moneyed and who desire* j . *Then an active N.E. exists.*

For the proof of Theorem 2 we build up two lemmas.

<u>Lemma 1</u>. There is an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that (a) T can be partitioned into sets $\{S_1^{n_k},\ldots,S_{n_k}^{n_k}\}$ of equal μ-measure (i.e., $1/n_k$), and (b) $\exists$ a choice $t_m^{n_k} \in S_m^{n_k}$ (for any n_k and $1 \leq m \leq n_k$) which ensures that if we construct the simple economy $\mathcal{E}_{n_k}$ by alloting the characteristics $(u^{t_m^{n_k}}, a^{t_m^{n_k}})$ to all $t \in S_m^{n_k}$, $1 \leq m \leq n_k$, then $\mathcal{E}_{n_k} \longrightarrow \mathcal{E}$ for all t . **(A proof of this is given in the appendix.)**

We shall work with the sequence $\{\mathcal{E}_{n_k}\}_{k=1}^{\infty}$ provided by Lemma 1, but for brevity we will drop the subscript k henceforth. $\{\mathcal{E}_{n_k}\}_{k=1}^{\infty}$ will be rewritten $\{\mathcal{E}_n\}_{n=1}^{\infty}$, and so forth. $\mathcal{E}_n$ will also be thought of as a finite economy with n traders possessing the characteristics $(u^{t_1^n}, a^{t_1^n}), \ldots, (u^{t_n^n}, a^{t_n^n})$. The mapping $\beta_n : T \longrightarrow T_n \equiv \{1,\ldots,n\}$ which identifies $t \in S_m^n$ with m relates the simple and the finite economies. It is obvious that $(\mathcal{E}, \{\beta_n\}_{n=1}^{\infty})$ is a continuous representation of the purely competitive sequence $\{\mathcal{E}_n\}_{n=1}^{\infty}$.

For any $\alpha \in \mathbb{R}$, let $\langle\alpha\rangle^+$ denote the largest integer less than or equal to α . Put $\eta = \max_{1\leq j\leq \ell+1} \{\int_T a_j d_\mu\}$ and $\gamma = \min_{1\leq j\leq \ell} \{\mu(S_j)\}$. Choose L large enough to ensure that $L\gamma/2(\ell + 1) > u\eta$. Let ${}_nx$ be any allocation in $\mathcal{E}_n$, i.e., $\sum_{i=1}^{n} {}_nx^i = \sum_{i=1}^{n} {}_na^i$. Denote by $W({}_nx)$ the set $\{t \in T : ||{}_nx^{\beta_n(t)}|| > L\}$. We claim that there is an N^1 such that, if $n > N^1$, then for <u>any allocation</u> ${}_nx$ in $\mathcal{E}_n$, $\mu(W({}_nx)) < \gamma/2$. Observe:

$$\frac{1}{n}\sum_{i=1}^{n} {}_na^i \longrightarrow \int_T a d\mu \ , \quad \left(\left\langle\frac{\gamma/2(\ell + 1)}{n}\right\rangle^+ - 2\right) / \ n \longrightarrow \frac{\gamma}{2(\ell + 1)}$$

Pick N^1 large enough so that whenever $n > N^1$, $\frac{1}{n}\sum_{i=1}^{n} {}_na_j^i < 2\eta$ $(j = 1,\ldots,\ell+1)$ and $\left(\left\langle\frac{\gamma/2(\ell + 1)}{n}\right\rangle^+ - 2 \ / \ n\right) > \gamma/4(\ell + 1)$.

Suppose that for some $n > N^1$, and some allocation ${}_nx$ in $\mathcal{E}_n$, we have $\mu(W({}_nx)) > \gamma/2$. Then there must be a k , $1 \leq k \leq \ell+1$ such that $\mu(W_k({}_nx)) > \gamma/2(\ell+1)$, where $W_k({}_nx) = \{t \in T : {}_nx_k^{\beta_n(t)} > L\}$. Defining, for $S \in \mathcal{C}$, $\hat{\beta}_n(S) = \{t \in T_n : \beta_n^{-1}(t) \subset S\}$ it is clear that

$\left|\frac{\hat{\beta}_n(W_k({}_n x))}{n}\right| \geq \frac{\langle \gamma/2(\ell + 1)n\rangle^+ - 2}{n} \geq \gamma/4(\ell + 1)$. But then

$$2\eta > \frac{1}{n}\sum_{i=1}^{n} {}_n a_k^i \geq \frac{1}{n} \sum_{i\in\hat{\beta}_n(W_k({}_n x))} {}_n x^i$$

$$\geq \frac{L(\langle \gamma/2(\ell + 1)n\rangle^+ - 2}{n}$$

$$\geq \frac{L\gamma}{4(\ell+1)}$$

$$> 2\eta ,$$

a contradiction. Thus N^1 has the property asserted.

For $1 \leq j \leq \ell$, define* ${}_j\xi : S_j \longrightarrow \mathbb{R}$ by:

$${}_j\xi(t) = \min \{u^t(y + e^j) - u^t(y) : y \in Q(L+1)\},$$

where $Q(\omega) = \{x \in \Omega^{\ell+1} : ||x|| \leq \omega\}$. By our assumptions, ${}_j\xi > 0$; also**, by Proposition 3 in Section D.II.3 of [3], ${}_j\xi$ is measurable.

Next, for $0 < \delta < 1$, define $G_\delta : T\times Q(L) \longrightarrow \mathbb{R}$ and, then, $F_\delta : T \longrightarrow \mathbb{R}$ as follows:

$$G_\delta(t,x) = \max \{|u^t(x) - u^t(y)| : y \in \Omega^{\ell+1} , ||x - y|| < \delta\}$$

$$F_\delta(t) = \max \{G_\delta(t,x) : x \in Q(L)\}$$

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* e^j is the vector in $\Omega^{\ell+1}$ whose j^{th} component is 1 , and all others are 0 .

**It is in the application of this proposition that we use the fact that T itself is a separable metric space.

Again, by using Proposition 3 **of** [3], **it can be shown that** G_δ and (hence) F_δ are measurable.

It is clear that $F_{1/m} \downarrow 0$ on T. To see this, fix t. Note that by the uniform continuity of u^t on $Q(L+1)$, for any $\varepsilon > 0$, there is a $1 > \delta^t(\varepsilon) > 0$, such that: $x \in Q(L) \subset Q(L+1)$, $y \in Q(L+1)$, $\|x - y\| < \delta^t(\varepsilon) \Longrightarrow |u(y) - u(x)| < \varepsilon$. This shows $F_{1/m}(t) \downarrow 0$, therefore $F_{1/m} \downarrow 0$.

Put ${}_jZ_m = \{t \in S_j : F_{1/m}(t) < {}_j\xi(t)\}$. Then ${}_jZ_1 \subset {}_jZ_2 \subset \ldots$, and $\bigcup_{m=1}^{\infty} {}_jZ_m = S_j$. Hence, $\mu({}_jZ_m) \uparrow \mu(S_j)$. Also observe that:

$t \in {}_jZ_m$, $x \in Q(L)$, $||y - x|| < \frac{1}{m} \Longrightarrow u^t(y + e^j) > u^t(x)$.

Finally, consider $a_{\ell+1} : S_j \longrightarrow \Omega^1$. Put ${}_jV_m = \{t \in S_j : a^t_{\ell+1} > \frac{1}{m}\}$. Since $a_{\ell+1} > 0$ on S_j, and ${}_jV_1 \subset {}_jV_2 \subset \ldots$, we have $\mu({}_jV_m) \uparrow \mu(S_j)$. Recalling that $\mu({}_jZ_m) \uparrow \mu(S_j)$, we can choose M large enough to ensure that (for $j = 1,\ldots,\ell$) $\mu({}_jV_M) > 5\mu(S_j)/6$, $\mu({}_jZ_M) > 5\mu(S_j)/6$, i.e., $\mu({}_jV_M \cap {}_jZ_M) > 2\mu(S_j)/3$.

Piecing this together with the previous result we get that, for $n > N^1$, and for any $j = 1,\ldots,\ell+1$,

$$(*) \qquad \mu({}_jV_M \cap {}_jZ_M \cap \{S_j \setminus W({}_nx)\}) > \frac{1}{6}\mu(S_j).$$

But $|\hat{\beta}_n(R)| \geq \left\langle \frac{\mu(R)}{n} \right\rangle^+ - 2$ for any $R \in \mathcal{C}$. Denoting

$\left\langle\frac{\mu(S_j)}{6n}\right\rangle^+ - 2$ by $\phi_j(n)$, we restate (*): *for* $n > N^1$, *any* $j = 1,\ldots,\ell + 1$, *and any allocation* ${}_n x$ *in* $\mathcal{E}_n$, *there exist at least* $\phi_j(n)$ *traders in* T_n *such that*

(A) $\quad \left|\left|{}_n x^{t_i}\right|\right| < L$;

(B) $\quad {}_n a_{\ell+1}^{t_i} > 1/M$; *and*

(C) $\quad x \in Q(L)$, $||y - x|| < 1/M \Longrightarrow u^{t_i}(y + e^j) > u^{t_i}(x)$;

where t_i *is any one of the* $\phi_j(n)$ *traders.*

(The presence of these traders, intuitively speaking, ensures enough competition in the j-th trading post to prevent the Nash price from falling to 0 . We will make this precise in Lemma 2.)

Now $\phi_j(n)/n \longrightarrow \mu(S_j)/6$, and $\langle(\phi_j(n) - 1)/\ell\rangle^+/n \longrightarrow \mu(S_j)/6\ell$, for $1 \leq j \leq \ell$; also ${}_n\bar{a}_j/n \longrightarrow \int a_j$ for $1 \leq j \leq \ell + 1$. Thus there is an N^2 such that, if $n > N^2$,

$$\frac{\int a_j}{2} < \frac{{}_n\bar{a}_j}{n} < \frac{3}{2}\,\frac{\int a_j}{2}\ , \quad 1 \leq j \leq \ell + 1 \quad ;$$

and

$$\frac{\langle\phi_j(n) - 1)/\ell\rangle^+}{n} > \frac{\mu(S_j)}{12\ell}\ , \quad 1 \leq j \leq \ell \quad .$$

At the heart of the proof of Theorem 2 is the following lemma.

Lemma 2. There exist positive constants C_j and D_j $(j = 1,\ldots,\ell)$ and Nash prices ${}_np$ in $\Gamma(\mathcal{E}_n)$ for $n \geq N = \max\{N^1, N^2\}$ such that

$$C_j \leq {}_np_j \leq D_j$$

for $1 \leq j \leq \ell$ and for $n > N$.

Proof. To exhibit the appropriate Nash price ${}_np$ of $\Gamma(\mathcal{E}_n)$, it will be fruitful to consider "ε-perturbed" games ${}^{\varepsilon}\Gamma(\mathcal{E}_n)$, in which an external agency is imagined to have placed a bid of ε in every trading post. This does not change the strategy sets of the players, but it does of course change the outcomes and the payoffs. We will find a Nash price ${}^{\varepsilon}_{n}p$ of ${}^{\varepsilon}\Gamma(\mathcal{E}_n)$ first, and then, letter $\varepsilon \longrightarrow 0$, obtain* ${}_np$ as a cluster point of $\{{}^{\varepsilon}_{n}p\}$. The sequence $\{{}_np\}_{n>N}$ will be shown to have the appropriate properties.

Let us introduce some abbreviations. Let h stand for $1/M$, A for $1/M(\ell + 1)$, B_j for $[\mu(S_j)A]/[18\ell \, ||\int a||]$, D_j for $3\int a_{\ell+1}/4\int a_j$.

First, thereis absolutely no problem in proving the existence of an N.E. for ${}^{\varepsilon}\Gamma(\mathcal{E}_n)$. It is accomplished by a straightforward use

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*The idea of obtaining a Nash price as a limit of Nash prices of ε-perturbed games is due to L.S. Shapley. See [9] where this technique is used to establish the existence of active Nash equilibria of **finite market games.**

of the Kakutani fixed point theorem. (See [9] or [10] for details.) Our concern is in finding bounds for ${}^{\varepsilon}_{n}p$.

Let $({}^{\varepsilon}_{n}p, {}_{n}x)$ be the prices and allocation at an N.E. of ${}^{\varepsilon}\Gamma(\mathfrak{E}_n)$. We proceed to drop the subscript n since only ε will be varied for a while. W.l.o.g. we will focus our attention on bounds for ${}^{\varepsilon}p_{\ell}$. Denote by $(b^1,\ldots,b^n)$ the bids that produce $({}^{\varepsilon}p,x)$.

Put $\delta = {}^{\varepsilon}p_{\ell} = \dfrac{\bar{b}_{\ell}+\varepsilon}{\bar{a}_{\ell}}$. Suppose first that the condition

$$(1) \qquad b^t_{\ell} \leq \frac{\bar{b}_{\ell}}{2} \quad \text{and} \quad a^t_{\ell+1} - \sum_{j=1}^{\ell} b^t_j \geq A$$

holds for at least one of the $\phi_{\ell}(n)$ traders;* say for $t = 1$. Then an increase Δ in 1's bid for ℓ would be feasible if Δ is sufficiently small, say $0 < \Delta \leq \min(\varepsilon, A)$, and would have the following incremental effect on his final holding:

$$x^1_j(\Delta) - x^1_j = 0 \quad \text{for} \quad j = 1,\ldots,\ell-1 ;$$

$$(2) \qquad x^1_{\ell}(\Delta) - x^1_{\ell} = \frac{\bar{a}_{\ell}(b^1_{\ell}+\Delta)}{\bar{b}_{\ell}+\Delta+\varepsilon} - \frac{\bar{a}_{\ell}\, b^1_{\ell}}{\bar{b}_{\ell}+\varepsilon}$$

$$= \bar{a}_{\ell}\Delta\left(\frac{\bar{b}_{\ell}+\varepsilon-b^1_{\ell}}{(\bar{b}_{\ell}+\varepsilon)(\bar{b}_{\ell}+\varepsilon+\Delta)}\right)$$

$$> \bar{a}_{\ell}\Delta\left(\frac{\bar{b}_{\ell}/2+\varepsilon/2+\Delta/2}{(\bar{b}_{\ell}+\varepsilon)(\bar{b}_{\ell}+\varepsilon+\Delta)}\right)$$

$$= \frac{\bar{a}_{\ell}\Delta}{2(\bar{b}_{\ell}+\varepsilon)} = \frac{\Delta}{2\delta}$$

*I.e. traders for whom (A), (B) and (C) hold, given the allocation ${}_{n}x$.

(The "$\geqq$" above follows from: $\bar{b}_\ell + \varepsilon - b^1_\ell \geqq \frac{\bar{b}_\ell}{2} + \varepsilon \geqq \frac{\bar{b}_\ell}{2} + \frac{\varepsilon}{2} + \frac{\Delta}{2}$) ;

$$x^1_{\ell+1}(\Delta) - x^1_{\ell+1} = \left(\frac{a^1_\ell}{a_\ell} - 1\right)\Delta \geqq -\Delta \quad .$$

Define

$$z = -2\delta e^{\ell+1}$$

and note that we have the vector inequality

$$(3) \qquad x^1(\Delta) \geqq x^1 + \frac{\Delta}{2\delta}(z + e^\ell) \quad .$$

We are now in a position to **apply** (C). We have $x^1 \in \Omega^{\ell+1}$ **and** $x^1 \in Q(L)$. So if both $x^1 + z \geqq 0$ and $||z|| \leqq h$, **then**

$$u^1(x^1 + z + e^\ell) > u^1(x^1) \quad .$$

Since u^1 is concave, this implies that

$$u^1(x^1 + \frac{\Delta}{2\delta}(z + e^\ell)) > u^1(x^1)$$

holds sufficiently small Δ , and hence, by (3) and the monotonicity of u^1 , that

$$u^1(x^1(\Delta)) > u^1(x^1)$$

for such Δ . But this means that trader 1 could have improved, contradicting that $(b^1,\dots,b^n)$ constitute an N.E. Hence either $x^1 + z \ngeq 0$, or $||z|| > h$. If the former, we have

$$x^1_{\ell+1} - 2\delta < 0 .$$

But $x^1_{\ell+1} \geq A$; hence

(4) $$\boxed{2\delta > A} .$$

If the latter, we have

(5) $$\boxed{2\delta > h} .$$

We now consider the case where (1) fails for all of the $\phi_\ell(n)$ traders. Not more than one t could have $b^t_\ell > \bar{b}_\ell/2$. W.l.o.g. assume that $b^t_\ell \leqq \bar{b}_\ell/2$ for $1 \leqq t \leqq \phi_\ell(n) - 1$. Therefore for $1 \leqq t \leqq \phi_\ell(n) - 1$, we have

$$\sum_{j=1}^{\ell} b^t_j > a^t_{\ell+1} - A \geq \ell A .$$

Hence $b^t_{j_t} > A$ for at least one j_t , $1 \leqq j_t \leqq \ell$, for such t .

Thus for at least one commodity j, $\bar{b}_j \geqq [\phi_\ell(n) - 1)/\ell]^+ A$. If $j = \ell$, then afortiori

$$(6) \qquad \delta \geqq \frac{[(\phi_\ell(n) - 1)/\ell]^+ A}{n^{\bar{a}_\ell}}$$

$$\geqq \frac{2}{3\int a_\ell} \frac{\mu(S_j)}{12\ell} \cdot A \geqq B_\ell$$

If $j \neq \ell$, then $\ell \geqq 2$ and so w.l.o.g. assume $j = 1$. Trader 1 could then decrease b_1^1 by a small $\Delta > 0$ and increase b_ℓ^1 by the same amount, with the incremental effect:

$$x_1^1(\Delta) - x_1^1 = \frac{\bar{a}_1(b_1^1 - \Delta)}{\bar{b}_1 - \Delta + \varepsilon} - \frac{\bar{a}_1 b_1^1}{\bar{b}_1 + \varepsilon}$$

$$\geqq \frac{-\bar{a}_1 \Delta}{\bar{b}_1 + \varepsilon} \quad ;$$

$$x_j^1(\Delta) - x_j^1 = 0 \quad \text{for} \quad j = 2,\ldots,\ell - 1 \quad ;$$

$$x_\ell^1(\Delta) - x_\ell^1 \geqq \frac{\Delta}{2\delta}$$

(by the same calculation as at (2)), and

$$x_{\ell+1}^1(\Delta) - x_{\ell+1}^1 = \left(\frac{a_\ell^1}{\bar{a}_\ell} - \frac{a_1^1}{\bar{a}_1}\right)\Delta - \left(\frac{a_1^1}{\bar{a}_1}\right)\Delta \quad .$$

If we define

$$(7)\qquad z = \left(-2\delta \frac{\bar{a}_1}{(\bar{b}_1+\varepsilon)}\right)e^1 - \left(\frac{2\delta a_1^1}{\bar{a}_1}\right)e^{\ell+1}$$

then (3) is satisfied as before; so, arguing as before, we find that either $x^1 + z < 0$ or $||z|| > h$. If the former, then either $|z_1| > x_1^1$ or $|z_{\ell+1}| > x_{\ell+1}^1$. But we have

$$x_1^1 \geqq \frac{\bar{a}_1 b_1^1}{\bar{b}_1+\varepsilon} \geqq \frac{\bar{a}_1 A}{\bar{b}_1+\varepsilon}$$

and

$$x_{\ell+1}^1 \geqq \frac{b_1^1 a_1^1}{\bar{a}_1} \geqq \frac{A a_1^1}{\bar{a}_1} \quad .$$

Thus, referring to (7), we see that in both cases

$$(8)\qquad \boxed{2\delta > A}$$

If the latter, then either $|z_1| > h$ or $|z_{\ell+1}| > h$. In the first case, the inequality $\bar{b}_1 + \varepsilon > [(\phi_\ell(n) - 1)/\ell]^+ A$ yields

$$(9)\qquad \boxed{2\delta > hB_\ell}$$

by the same calculation as at (6); in the second, $a_1^1 \leqq \bar{a}_1$ yields

$$(10)\qquad \boxed{2\delta > h}$$

This exhausts all cases. Combining (4)-(6) and (8)-(10), we see that

$$\delta \equiv {}^{\varepsilon}_{n}p_{\ell} > C_{\ell} \equiv \min\ [A/2, h/2, B_{\ell}, hB_{\ell}/2]$$

Defining C_j for $1 \leq j \leq \ell-1$ as above, with β_{ℓ} replaced by B_j, we can show, in exactly the same fashion, that:

$$ {}^{\varepsilon}_{n}p_j \geq C_j \quad , \quad 1 \leq j \leq \ell \quad .$$

But C_j *is independent of* ε. Therefore by letting $\varepsilon \longrightarrow 0$, and holding n fixed, we obtain a limit point ${}_np$ of $\{{}^{\varepsilon}_{n}p\}_{\varepsilon\to 0}$, where ${}_np_j \geq C_j$, $1 \leq j \leq \ell$. Since ${}_np$ is a point of continuity of the payoff functions, ${}_np$ is clearly a Nash price of $\Gamma(\mathcal{E}_n)$. *Also* C_j *is independent of* n. Thus we have a sequence $\{{}_np\}_{n>N}$ of Nash prices of $\{\Gamma(\mathcal{E}_n)\}_{n>N}$ such that ${}_np_j > C_j$ for all $1 \leq j \leq \ell$, and for all $n > N$.

As for upper bounds, note that, if $n > N$,

$${}_np_j \leq \frac{{}_n\bar{a}_{\ell+1}}{{}_n\bar{a}_j} = \frac{{}_n\bar{a}_{\ell+1}/n}{{}_n\bar{a}_j/n} \leq \frac{3\int a_{\ell+1}}{4\int a_j} = D_j$$

Q.E.D.

Proof of Theorem 2. The sequence $\{{}_n p\}_{n>N}$ of Lemma 2 lies in the compact set $\bigtimes_{j=1}^{\ell} [C_j, D_j]$. Hence it has a cluster point $p \in \bigtimes_{j=1}^{\ell} [C_j, D_j]$. Obviously $p > 0$. By Theorem 1 p is then a Nash price of $\Gamma(\mathcal{E})$. Q.E.D.

Remarks:

(1) Note that, in the process of proving Lemma 2, we have also established the existence theorem in [9]:

> Consider a finite economy $\hat{\mathcal{E}}$, and suppose that for each $1 \leq j \leq \ell$ there are at least two moneyed traders who desire j . Then there exists an active N.E. of $\Gamma(\hat{\mathcal{E}})$.

Unfortunately, there was no way of invoking this result directly in our proof. The search for bounds on ${}_n p$ (independent of n) required that we actually show the existence of "appropriate" active N.E. of the finite games $\Gamma(\mathcal{E}_n)$.

(2) The active N.E. of $\Gamma(\mathcal{E})$ need not yield prices and allocations that are competitive for the economy $\mathcal{E}$, without special restrictions on $\mathcal{E}$. For further discussion of this point, see [6].

APPENDIX

Lemma 1 will follow from the following result (recalling that $U \times \Omega^{\ell+1}$ is a separable metric space).

Claim. Let X be a separable metric space endowed with the σ-algebra generated by its open sets, and let $f : T \to X$ be measurable. Then there exists an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that (a) T can be partitioned into measurable sets $\{S_1^{n_k}, \ldots, S_{n_k}^{n_k}\}$ of equal μ-measure (i.e. $1/n_k$), and (b) $\exists$ a choice $t_m^{n_k} \in S_m^{n_k}$ (for any n_k and $1 \le m \le n_k$) which ensures that the sequence $\{f_{n_k}\}_{k=1}^{\infty}$, $f_{n_k} : T \to X$, converges almost everywhere to f, where $f_{n_k}(t) = f(t_m^{n_k})$ for $t \in S_m^{n_k}$.

Proof. Denote the metric on X by d. For any $x \in X$ and $r > 0$, let $S(x,r) = \{y \in X : d(x,y) < r\}$. Since X is separable, for any integer n we can find a sequence $\{x_i^n\}_{i=1}^{\infty} \subset X$ such that $\bigcup_{i=1}^{\infty} S(x_i^n, 1/n) = X$. Define $\{A_i^n\}_{i=1}^{\infty} \subset \mathcal{C}$ by $A_i^n = f^{-1}(S(x_i^n, 1/n)) \setminus \bigcup_{j<i} A_i^n$. Then $\bigcup_{i=1}^{\infty} A_i^n = T$ and all the A_i^n are mutually disjoint. Hence $\mu(\bigcup_{i=1}^{k} A_i^n) \uparrow_k 1$. Pick $i(n)$ large enough to have $\mu(\bigcup_{i=i(n)+1}^{\infty} A_i^n) < 1/n^2$. Partition each A_i^n for $1 \le i \le i(n)$ into measurable sets $A_{i,1}^n, \ldots, A_{i,k(i)}^n, \tilde{A}_i^n$ such that $\mu(A_{i,j}^n) = 1/i(n)\cdot n^2$ for $1 \le j \le k(i)$ and $\mu(\tilde{A}_i^n) < 1/[i(n)\cdot n^2]$. (This can be done since μ is nonatomic.) Put $\tilde{A}^n = (\bigcup_{i=1}^{i(n)} \tilde{A}_i^n) \cup (\bigcup_{i=i(n)+1} A_i^n)$. Clearly $\mu(\tilde{A}^n) = m(n)/i(n)\cdot n^2$, for some integer $m(n)$, because $\mu(\tilde{A}^n) = 1 - \mu(\bigcup_{i=1}^{i(n)} (\bigcup_{j=1}^{k(i)} A_{ij}^n))$ $= 1 - \sum_{i=1}^{i(n)} \sum_{j=1}^{k(i)} \mu(A_{ij}^n)$. Also $\mu(\tilde{A}^n) \leq 2/n^2$. Partition $\tilde{A}^n$ into measurable sets $A_{i(n)+1,1}^n, \ldots, A_{i(n)+1,k(i(n)+1)}^n$ such that $\mu(A_{i(n)+1,j}^n) = 1/i(n)\cdot n^2$ for $1 \leq j \leq k(i(n)+1)$. [Of course, $m(n) = k(i(n)+1)$.] Pick $t_{i,j}^n \in A_{i,j}^n$ arbitrarily, for $1 \leq i \leq i(n)+1$, and $1 \leq j \leq k(i)$. Put $f_n(t) = f(t_{i,j}^n)$ where $t \in A_{i,j}^n$. Then

$$\mu\{t \in T : |f(t) - f_n(t)| > 1/n\} < 2/n^2 .$$

This implies that for any positive integer n'

$$\mu\{t \in T : |f(t) - f_n(t)| > 1/n' \text{ for some } n \geq n'\} < 2 \sum_{n \geq n'} \frac{1}{n^2} .$$

The right hand bound converges to 0 as $n' \to \infty$. Hence $f_n \to f$ almost everywhere.

Q.E.D.

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