A Light in the Dark: Ultralight Dark Matter Phenomenology in Simulations

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Of the outstanding problems in astronomy, the nature of dark matter is certainly one of the most mysterious. Containing five times more energy density than its luminous counterpart, dark matter has been shaping the large-scale structure of our Universe for billions of years. The expansion of accessible and accurate cosmological simulations has revolutionized how we visualize the imprint of dark matter in the structure of our Universe. In my Ph.D., I contributed to this revolution through the development and implementation of a new code, chplUltra: a parallel, portable, and efficient tool for HPC simulations of a promising dark matter candidate, Fuzzy or UltraLight Dark Matter (ULDM). ULDM is a well-motivated axion-like dark matter candidate whose incredibly small mass results in naturally cored profiles, thus ameliorating many of the small-scale problems of cold dark matter (CDM) while maintaining the same robust large-scale results. When unperturbed, the lowest energy solution of the ULDM system is a spherical “soliton” structure with a known mass density profile. A ULDM dark matter halo is formed through collisions of these solitons and has two characteristic parts: a central soliton core, and an NFW “skirt” surrounding it. In order to investigate ULDM dynamics, I calculated the full spectrum of eigenstates for ULDM systems with approximately stationary potentials, thus allowing me to 1) link qualitative behavior of soliton cores in ULDM simulations with superpositions of specific modes and 2) decompose CHPLULTRA simulations of ULDM halos into individual eigenstates. Us-
ing this formalism, I investigated the formation of halos through soliton collisions and the dependence of the final halo product on initial parameters. Crucially, this allowed me to explore how halo cores form and explain discrepancies in the literature surrounding the core-halo mass relation: a key prediction of ULDM. I was also able to comment on the composition of the halos’ skirts, including their qualitative behavior and eigenstate makeup, as a function of initial binary parameters. Finally, I sketched out some of the exciting future directions for understanding ULDM through the language of its eigenstates; these include combining my work on ULDM with my previous work on primordial black holes, which is included as part of this dissertation.
A Light in the Dark: Ultralight Dark Matter Phenomenology in Simulations

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Jovana Žagorac

Dissertation Directors: Nikhil Padmanabhan & Richard Easther

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Некада давно, ђака првака је из школе пратио деда.
“Деко”, мрштећи упита првачић, “колико још има школе?”
“Па, овако,” одговорио је деда, “има осам година обавезне школе, па онда још четири године обавезне школе...”
(узгред буди речено, лаж— средња школа није обавезна у Србији)
“... и онда има још четири године које нису обавезне.”
Првачић се још више намршти. “Ја нећу те последње четири године да идем у школу.”
“E, хоћеш, хоћеш,” насмејао се деда.
“Нећу, нећу!” Запе првачић. “Ко ће још да намерно иде дуже у школу?!"

ДЕСЕТ НЕОБАВЕЗНИХ ГОДИНЕ ШКОЛЕ КАСНИЈЕ: ЗА ТЕБЕ, ДЕДИЦЕ.

Once upon a time, a first grader was being walked home from school by her grand-
father.
“Grandpa,” the little girl said, frowning, “how much school is there?”
“Well,” her grandfather responded, “there’s eight mandatory years, and then there’s another four mandatory years...”
(a lie, by the way—high school is not legally required in Serbia)
“...and then there’s four optional years.”
The first grader frowned harder. “I’m not doing the four optional years of school.”
“Oh, yes you are,” her grandpa chuckled.
“I’m not, I’m not!” The first grader insisted. “Who in the world would do more school than they have to?!"

TEN OPTIONAL YEARS OF SCHOOL LATER: FOR YOU, GRANDPA.
Acknowledgments

On March 14th, 2022, my “local” advisor—Professor Nikhil Padmanabhan, current Dean of Undergraduate Studies of the Physics Department at Yale University and dog dad to Reilley—advised me to redact my acknowledgements from this submission until such a time as I receive comments from my committee on the contents of this dissertation. In an uncharacteristic act of obedience, I have decided to follow his advice (but had to embarrass him a little anyway).

Happy reading!

On April 7th, 2022, in the midst of my ongoing post-defense daze, I sat down at a rickety table in Koffee? on Audubon Street to write my real acknowledgements. Sitting next to my spiced oat chai latte*, I found myself no less full of gratitude for the people and communities that had fostered me thus far in life, but at a significant loss for words. You see, once upon a time, I was advised by someone older and wiser than me to leave off writing the acknowledgements section of my thesis for my darkest moments, when I most needed a reminder of what and who I was doing this for. I had apparently taken this warning to heart—too closely, even, since I now find myself with a written dissertation and only a semi-joke on page x of my manuscript. As it turns out, the hardest part of writing this dissertation is the acknowledgements; I find myself so full of pent-up gratitude and love and humility that I’m afraid my heart will burst all over the (virtual) paper if I try to properly thank everyone who deserves to be thanked. This is an effort—not unlike coming up with theories of dark matter!—preemptively doomed to failure.

Let me try anyway.

Firstly, let me tell you the tale of Nikhil Padmanabhan and Richard Easther. Back in 2016, I was weathering the storm that is the first year of graduate school while looking for the kind of research question I yearned to tackle. I was feeling frustrated one day when I sat down to write what I want from a Ph.D. thesis in a document I called “Designing advisors.” There were five bullet points:

i. in my interest field:
   cosology (inflation), gravity, early Universe, particles, theory, dark matter/energy;

*(and opposite Emily’s “the same”)
2. jive with advisor (within reason);†

3. project lets/makes me travel;

4. funding is plausible & there are jobs after grad;

5. has the possibility of making a significant impact on physics.‡

Now, I am told that Nikhil and Richard have existed in this world for longer than I have and have had rich and independent lives outside of advising me, and I am sure they will hence; nevertheless, I am still not convinced that they didn’t spring directly from the contents of my writing. Since the day I emailed Richard and introduced myself as a Yale first year interested in exploring the early Universe, he has been nothing but invested and encouraging. Since the afternoon I first entered into Nikhil’s office to say some version of “Hey, BAOs are cool and all, but how about we research primordial black holes for a little bit instead?,” he has never faltered in both supporting my learning of new physics and learning alongside me. Together, they have formed an absolute advising dream team at the intersections of particle theory and cosmology; they’ve made me a better writer, better researcher, and better physicist; and being mentored by them has made me a better mentor in turn. I want to thank them for navigating a collaboration which, in Nikhil’s words, “comes close to breaking causality;” for seeing me through a global pandemic (including my too brief and slightly ill-fated trip to Auckland); for dealing with all my nonsense with a smile and with grace; for the dog and cat pictures; for the occasional academic gossip; and for the chats on how a little bit of anthropology can make one a better quantitative scientist than a whole lot of math. I am forever grateful to them for molding me into the scientist I am today, and for setting the stage for the scientist I will continue to become.

My academic corner would not be complete without the other members of my committee: Priya Natarajan, Frank van den Bosch, and my external reader, Chanda Prescod-Weinstein. Priya was always ready both to proffer advice on observational parallels to my work and bring the conversation back to black holes, while also supporting my interdisciplinary interests with both words and actions. It is in no small part because of her taking an interest in me that I was able to progress my side project on Egyptian archaeoastronomy as far as I have, and it is because of her leadership of the Franke Program in Science and the Humanities that I am in a position to continue that work in my next step. My conversations with Frank always felt like they were meant to be slightly intimidating—he did, after all, literally write the book on galaxy formation!—but I could never quite bring myself to be scared by him. I always left our conversations feeling emboldened of my own understanding of the subject, and with new and interesting ideas; even more so, I was ever appreciative of the small Serbian phrases he would sneak into our emails and chats (“odlično!”) which always made me feel like I had an extra slice of home that day. Finally, I want to thank Chanda, who always carved a piece out of her

†I’ve always had high expectations.
‡I guess I used to be an optimist.
incredibly busy days when I needed it most, and who was the first and only person to ever describe the possibility of my leaving physics behind as a “tremendous loss for the field.” Like so many lines from The Disordered Cosmos, those words have stuck with me ever since, and I hope they always do.

The peaks and valleys of my Ph.D. experience would have all taken a downward bend without the support of my greater research environment. I would particularly like to thank Naim Gökşel Karaçaylı, Xinyi Chen, and Sasha Safonova for being a great research group to work with, talk with, eat with, and overwhelm the office with plants with. My experience was molded in no small part by Isabel Sands and Claire Recamier, the wonderful undergraduates I had the opportunity to collaborate with and mentor. Thank you to Geriana van Atta who always made sure my flights were booked and my meals paid for; to Teena Griggs for all the spare office keys; to Hannah Carroll for feeding my obsession with office decor; to Stacey Watts who answered my many (many) questions about the particulars of graduation and defense requirements; to Meg Urry for giving me Richard; to Andy Szymkowiak for being a friendly office neighbor always in possession of the right dongle. A big thanks to the entire Auckland team—including Emily Kendall, Mateja Gosenca, Nathan Musoke, Shaun Hotchkiss, Lillian Guo, Frank Wang, and Sue Western—for the collaboration and for their kindness during my whirlwind stay in New Zealand as the COVID pandemic crept to a crescendo around the world. Thank you to NASA’s FINESST for funding two years of my work. Finally, I would like to thank everyone at Cray Inc. for their technical support with CHPLULTRA, and in particular Elliot Ronaghan and Brad Chamberlain—thank you so much for enabling me to produce and play with more than 120 TB of data in the last year of my Ph.D. alone, and for letting me hog all the nodes.

Graduate school has taught me that there are precious few things in this Universe I am sure of, but sure I am that my experience would have been worse without my many communities. Thank you to the Franke Program and the NELC people who gave me a second home and always made sure to shine a light on the other side of my heart. Thank you to my fellow McDougal people for all the Fellows’ pours and the shenanigans before and after FFF. To AstroBites: the best decision I made in grad school was joining this collaboration of wonderful, kind, caring, creative people, and I consider each and every one of them my friend. Thank you to WIP+ and QuARK: I was really bad at showing up but was always happy you all were there. Thank you to the Yale DHLab and all its wonderful denizens—Peter, Cathy, and Doug in particular—you all immediately made me feel like I belonged, and I’m immensely grateful for the creativity and zest with which you approach every new problem and every new grain of knowledge.

More than anything, I cherish the people I have met and the relationships we have built throughout graduate school. Thanks to my cohort—Connor, Emma, Emily, Jack, Joe, Kelly, London, Oskari, Sam, Sohan, Sumita; for the nights doing problem sets, for the days playing Avalon, for bringing offerings to Gibbs, for karaoke nights, for letting me drag you to roller derby, for camping, for beaches.

§PS: Sorry I tried to put it all in my home directory.
and snowy hills, for Lenin, for East Rock walks and backyard sits, for crying, for laughing, and for everything exciting and mundane in between. Thank you to my first year friends (and in particular my roommate Rita) for reminding me there’s more to life than physics. Thank you to the forever second years—Chris, Danny, Duncan, Jeremy, Mariel, Paul, Will—not just for recruiting me to come here but for making it that much more fun to be here. Thank you to all the lower years who had to sit through Classical Mechanics with me and still decided I was cool enough to hang. Thanks to Sanah for the gym mornings, sheesha evenings, and all the love and giggles. Thank you to GRARR, which quickly became my most meaningful weekly activity and the thing that I looked forward to every week in the pandemic. Thank you to astro girls’ night: for getting me through job applications and thesis writing, and also for Rasputin. Thank you to the entire Nicoll Street gang—but thank you to Will Tyndall in particular. For all the lasag(n)a, and for everything that goes with it.

I am incredibly lucky to have amassed the support of amazing people in my life even before graduate school. Thank you first and foremost to the Astronomy/Physics Department at Colgate University; I surely could not have gotten this far without the solid and loving foundation the department instilled in me. Thanks in particular to Thomas Balonek, Jeff Bary, Patrick Crotty, Jonathan Levine, and Ken Segall for showing me that I am not an observationalist, broadening my cosmological horizons, preparing me for Yale, and dealing with my antics in general. Thank you to my wonderful physics cohort—including Brendan, Gary, Jon, Lillie, Lindsay, Nathan, Nick, Sean, and Warren—for the late nights in the Ho Science Center, for freezing together outside Foggy Bottom Observatory, for the physics phormals, for all the memorable quotes that I still mutter to myself, and for surprising me with your Zoomical presence at my defense. A special thanks to Lindsay, who orchestrated that amazing Zoom bomb, for things that I don’t even know how to convey—you are the other part of my soul and always will be. Thank you to all of the non-physics programs and people that housed and raised me: the MIST Department, the SOAN Department, and the AMS program in particular. Thank you to Noori Baji and Ustaaz Nady and Ifty for being my family. Thank you to Mary Moran and Jordan Kerber for still thinking of me as an anthropologist and archaeologist (I swear, I am!). Thank you to Anthony Aveni for giving me the best archaeoastronomical start I could ask for.

An enormous thank you to Analiza, Yusra, Vincent, Ieva, Natalija, and Sara: not just for being the best of friends, but also for providing safe refuge from my Ph.D. Thank you to all my friends from high school—American and Serbian alike—and from even before for always celebrating my wins even when I didn't want to. No matter how hard I try, I can never sufficiently thank you all for reminding me of who I am when I forgot. I’m running out of words to say how much you all mean to me, but I promise to tell each of you often and loudly to make up for it (ideally over a meal in Cairo/Belgrade/London/Paris/Barcelona/somewhere new).

---

4 Insert “the Bary face” here.
5 a.k.a., Not Claire 1. Signed, Not Claire 2.
Finally, thank you with my whole chest to my family, starting with my mom, my dad, and my sister. You've had to hear me cry over nonsensical phrases like “qualifying exams” and “research funding” and “finding a postdoc” so many times, and I'm in awe of both the love you've showed me in return and how well you've learned to speak academic gibberish. Thank you for telling me over and over again that it was ok to quit—it's what gave me the strength to persevere. Thank you to my aunt Vanja, my constant Mira, my brothers/cousins Luka, Aleksa, Andrija, my great-uncle Boda, and everybody else rooting for me behind the scenes. To say that I love you is a grievous understatement.
In the beginning there was nothing, which exploded.

Terry Pratchett, Lords and Ladies

Introduction

In a Universe filled with wonderful things, it is the parts we cannot see that are perhaps the most tantalizing. This is true not only because of the mystery inherent in that which we do not know; but also because that which we do not know is both vast and expanding, far dwarfing the density of luminous matter we are able to directly access. Thus it is we, the luminous ones, who are special and rare in this Universe we call home.¹ The following text, then, is no more and no less than a product of luminous matter attempting to understand its translucent counterparts.
1.1 WHAT WE CAN’T SEE IN THE UNIVERSE

I will highlight three aspects of the Universe we don’t understand: the apparent acceleration of the universe (dark energy), the large scale behavior of matter (dark matter), and the initial conditions of the overall cosmos (inflation, and the very early universe). This dissertation will focus on the latter two; the first is introduced due to its importance to cosmology and the present evolution of spacetime, but will not form a large part of subsequent chapters.

1.1.1 DARK ENERGY

Dark energy is responsible for the current accelerating expansion of the Universe. Containing approximately 70% of the total mass-energy of the Universe, dark energy acts as almost the direct opposite to gravity, causing massive objects to retreat from one another instead of collapsing and forming structure. While cosmologists have a pretty clear idea of what dark energy *does*, what it *is* remains less clear. Regardless of its exact details, however, a description of dark energy as some cosmological constant $\Lambda$ has been incredibly useful in recreating observations of the Universe in miniature—through computer simulations—and the exact value and nature of $\Lambda$ is continuously probed by dedicated teams observing the evolution of spacetime.

1.1.2 DARK MATTER

While dark energy might be the enemy of gravity’s pull, dark matter might be its biggest conduit. Making up approximately 25 percent of the mass-energy of the Universe—5 times more than luminous matter!—dark matter behaves like mass that cannot interact with light. Said another way, dark matter is the name physicists and astronomers gave to the missing mass in the Universe. Observational
constraints show that whatever dark matter is made up from can't be completely akin to the Standard Model of particle physics describing luminous matter, or baryons, that we've directly observed; it must be something completely new to us. An often-simulated, generic candidate particle is called Cold Dark Matter (CDM). Together with the cosmological constant of dark matter it forms $\Lambda$CDM, the Standard Model of cosmology, which does very well at re-producing in simulations most of the picture of the Universe we see from observations. The bulk of this text will go beyond both the Standard Models of particle physics and cosmology to consider one compelling alternative to CDM.

1.1.3 The Early Universe

Finally, let us consider not a fundamental building block of the Universe, but rather a period of its evolution that lies beyond our ability to observe it. After our Universe was born, approximately 14 billion years go, it was incredibly hot and dense —so hot that atoms couldn't form, and a primordial soup of subatomic particles was flying wildly, bouncing against the photons that existed. As the Universe grew, it cooled, and atoms were slowly allowed to form in an event knows as Recombination, eventually allowing the packets of light to stream outwards instead of being bounced around like popcorn. This happened approximately 300,000 years after the Big Bang—a mere blip of time in the age of the cosmos—and the light that streamed out is the oldest light in the Universe, known as the Cosmic Microwave Background (CMB). Before this event, however, the Universe remains opaque, and we couldn't see what it looked like before that time even in principle. In part of this text, I will discuss how we might try to take a peek behind the CMB regardless—not to see the Universe, but to feel it.
1.2 Thesis Overview

With the exception of the Introduction, Background, and Conclusion, this thesis is primarily constructed as a series of papers, with a central theme of a dark matter candidate called UltraLight Dark Matter (ULDM), known for forming haloes made up of central “cores” with surrounding “skirts”. Chapter 2 will, therefore, serve as sinew giving the appropriate background and tying the topics covered together in a single narrative. Unlike this Introduction, the Background will go into deeper scientific details on topics covered, including a literature review and details on the code used for the simulations presented in this thesis. The papers corresponding to each chapter are also detailed in Table 1.1.

Chapter 3 describes the evolution and signatures of a population of primordial black holes which could have dominated the very early Universe. I open at the very beginning by working on Grand Unified Theory (GUT)-scale primordial black holes, which could have formed quickly after the end of inflation, rapidly merged, and evaporated before Big Bang Nucleosynthesis (BBN). Their evaporation precludes these black holes from being dark matter candidates; nevertheless, this phase of the Universe can be described by the same set of equations which govern my chosen dark matter candidate, thus tying the topic into the overarching theme.

The equations governing UltraLight Dark Matter also make it convenient to calculate the eigenstates of certain ULDM systems. This approach is further described in Chapter 4, where I verify dynamics of perturbed ULDM ground states using perturbation theory—first in a spherically symmetric system, then in an axisymmetric one. Finally, I use the same approach to decompose a ULDM halo into its eigenstates and draw conclusions about its evolution.

Building on this approach, in Chapter 5 I use eigenstates to precisely measure
the relationship between the core masses and halo masses of individual ULDM halos. I then compare that relationship—the so-called core-halo mass relation—to scalings derived in the literature. By using my eigenstate approach, I am able to bridge the gap between different results in the literature, showing that the core-halo mass relation is dependent on more parameters than initially thought.

In Chapter 6, I again use eigenstates to characterize halos formed through soliton mergers. I restrict myself to only binary soliton mergers, but expand the number of eigenstates I consider while varying initial parameters of the binary. Particular attention is paid to characterizing the resulting halos’ qualitative behavior and eigenstate makeup, as well as the oscillation timescales of the resulting core.

Finally, I summarize my conclusions and outline next steps in Chapter 7.

### Table 1.1: This table lists the individual papers or pre-prints corresponding to dissertation chapters.

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Light thinks it travels faster than anything but it is wrong. No matter how fast light travels, it finds the darkness has always got there first, and is waiting for it.

Terry Pratchett, Reaper Man

2

Background

FOCUSING NOW ON THE MISSING MASS in the Universe scientists have taken to calling dark matter, let us attempt to lend it mathematical language in this chapter. To embark on such a quest, we must first consider the long history of humans noticing this cosmic disbalance.

2.1 A (VERY) BRIEF HISTORY OF DARK MATTER

The first mention of the possibility of gravitational detection of massive bodies dates to John Mitchell’s 1783 work on “dark stars”, which we would know today as
black holes. Before physicists had even shed the idea of the luminiferous aether around the turn of the 20th century, terms like “dark bodies” and “matière obscure” were being tossed around by names such as Lord Kelvin and Henri Poincaré. By the 30’s, Jan Oort and Fritz Zwicky were finding evidence of excess mass in the galactic plane and galaxy clusters. In the 70’s and 80’s, Vera Rubin and Kent Ford used galaxy rotation curves to show that the excess, mysterious mass totaled around six times more than the present luminous mass (Fig. 2.1).

Since Rubin’s groundbreaking work, the field of dark matter research has been not so much one of strict consensus as it has been one of creativity and ideas. Possible solutions to the mysterious mass span new and exciting particles, massive-compact halo objects (MACHOs) such as primordial black holes, and even changes to how the force of gravity functions. The flames of cosmological invention have been further fanned by persistent null results on some of the more famous solutions. The dark matter candidate which this work will focus on is an offshoot of a more recognized candidate that sprang back from obscurity into such an ecosystem, where it has been steadily growing in popularity: a potential light in the dark.

2.2 ULTRALIGHT DARK MATTER: THE WHY'S

2.2.1 AXIONS AND ALPS

One popular family of dark matter solutions began as solutions to a completely different problem: that of charge parity (CP) symmetry. In the Universe we observe, we have found no evidence to suggest that the presence of a left-handed positively charged particle is meaningfully different from that of a right-handed negatively charged counterpart. Yet in quantum chromodynamics (QCD), the theory of the strong interactions between quarks mediated by gluons, there is nothing explicitly enforcing this. This presents a fine-tuning problem—the Universe could have cho-
Figure 2.1: Fig. 3 from the paradigm-shifting work Rubin et al. (1978).\textsuperscript{10} Shown are the rotational velocities for seven galaxies as a function of distance from nucleus. Note that rotational velocities (which are proportional to mass) don't fall off with radius as would be expected for galaxy without additional mass. 

sen to be this way, but it does seem awfully convenient for our current theories.

* 

In 1977, Helen Quinn and Roberto Peccei first postulated a more “natural” solution to CP in by introducing a hypothetical pseudo scalar boson known as the axion\textsuperscript{13}. The presence of a very light spin-zero field $\phi$ introduces an extra symmetry in the action\textsuperscript{14}

$$I = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi.$$  \hspace{1cm} (2.1)

\textsuperscript{13}For more on the power and limits of fine-tuning and naturalness arguments, see the works of Dr. Nima Arkani-Hamed; e.g., his “Why is there a Macroscopic Universe?” lecture or his contribution in “A Fortunate Universe: Life in a Finely Tuned Cosmos\textsuperscript{15}.”
This is a shift symmetry $\varphi \rightarrow \varphi + C$, where $C$ is some constant; the form of the Lagrangian ensures that $C$ vanished from the derivative terms. The symmetry is exact when the mass and spin-couplings of other particles to $\varphi$ are precisely zero; for very small (but not vanishingly small!) particle mass $m$, the action still maintains an approximate symmetry. Very light spin-zero fields like $\varphi$ arise in models where the potential function $V(\varphi)$ is a periodic function of $\varphi$. This is important to note for the sake of our “naturalness” argument because such fields are plentiful in different models, not just ones invented for the sake of slotting in an approximate symmetry where useful.\textsuperscript{14}

These fields are often fittingly called “axion-like fields” and their resulting particles “axion-like particles” (ALPs). Axion-like is a collective term encapsulating a whole family of particles, including Peccei and Quinn’s QCD axion. Our Universe could contain zero, one, or many such fields and particles with a variety of different behaviors.\textsuperscript{15,16} Depending on the particle mass, these fields have been proposed as

1. contributors to dark energy;
2. QCD axions;
3. candidates for UltraLight Dark Matter; and
4. candidates for the inflaton field.

Note that this relatively simple extension of the Standard Model carries potential implications for each of the phenomena introduced in Section 1.1: What We Can’t See in the Universe. Going forward, this text will primarily focus on the third possibility listed above, along with limited commentary on item number four.
2.2.2 SMALL-SCALE CRISIS IN CDM

An ALP with a mass on the order of \( m \sim 10^{-22} \text{ eV} \) is a dark matter candidate with many names. Known sometimes as scalar-field dark matter, \( \Psi \) dark matter, BEC dark matter, or fuzzy dark matter, in this text I will refer to it exclusively as Ultra-Light Dark Matter, or ULDM. Its name refers to its key characteristic: its diminutive mass. As with any particle, its mass will have a corresponding de Broglie wavelength

\[
\frac{\lambda}{2\pi} = \frac{h}{mv} = 1.92 \text{kpc} \left( \frac{10^{-22} \text{eV}}{m} \right) \left( \frac{10 \text{ km s}^{-1}}{v} \right),
\]

where \( v \) represents velocity on the order of those measured around dark matter halos. Indeed, the ultralight mass of the particle is motivated by a desiring a particle with a de Broglie wavelength of about \( 1 \) kpc: the approximate scale where the successful but generic CDM begins having issues reproducing observations of the Universe. This phenomenon is sometimes known as the “Small-Scale Crisis” in CDM, and consists of three different effects found in simulations of CDM: the core-cusp problem, the missing satellites problem, and the too-big-to-fail problem.\(^{17}\)

Each of these will be described in the coming sections, along with descriptions of how ULDM ameliorates these problems in simulations.

2.2.2.1 CORE-CUSP PROBLEM

The core-cusp problem is a classic case of simulation versus observation.\(^{18-20}\) CDM simulations tend to produce the Navarro-Frenk-White (NFW) radial profile\(^{21}\):

\[
\rho(r) = \frac{\rho_0}{\frac{r}{R_s} \left( 1 + \frac{r}{R_s} \right)^2}.
\]
where $\rho_0$ and the scale radius $R_s$ vary from halo to halo. Note that in the limit $r \to 0$, the halo profile asymptotes to $\rho(r) \propto 1/r$, which is divergent. As $r \to \infty$, however, the profile does a fantastic job at modelling dark matter halos with the introduction of a cutoff radius, $r_{\text{vir}}$; consequently, the NFW profile has remained popular. On the other hand, observations of dwarf galaxies favor a “cored” central profile where as $r \to 0$, $\rho(r) \propto r^0 \sim \text{const}$. This discrepancy is illustrated in Fig. 2.2.

Unlike CDM, ULDM naturally does not predict a central cusp, but does predict NFW-like behavior at large $r$. In fact, UDLM halos are often modelled as composite with

$$\rho(r) = \begin{cases} 
\rho_{\text{sol}}(r), & 0 \leq r \leq r_\alpha \\
\rho_{\text{NFW}}(r), & r_\alpha \leq r \leq r_{\text{vir}} 
\end{cases}$$

(2.4)

where $\rho_{\text{sol}}$ describes a central core known as the “soliton” $r_\alpha$ is a transition radius between the soliton core and NFW “skirt,” and $r_{\text{vir}}$ is the usual virial radius of the NFW halo.

Much of this thesis will focus on the soliton core of ULDM halos, and Chapters 4 and 5 will delve much more deeply into the physics behind the soliton. For the purposes of this section, let the ULDM core be simply the motivation behind ULDM and a convenient solution to the core-cusp problem.

### 2.2.2.2 Missing satellites problem

Another discrepancy between CDM simulations and astronomical observations is the problem of missing dwarf galaxies, or satellites. At redshift $z = 0$, it’s rare that a large halo doesn’t come with substructure in the form of smaller halos. Fur-
Figure 2.2: Figure 7 from Oh et al. (2011). The discrepancy between NFW profile “cusps” (black dotted) and pseudo-isothermal profile “cores” (red dashed) for several dwarf galaxies is evident.
thermore, the smaller the mass of the individual components, the more of them we expect to see. While we have only observed on the order of 10 dwarf galaxies orbiting our Milky way, CDM simulations predict number around an order of magnitude larger.\textsuperscript{26–28} Since this issue was first flagged in the late 90’s, the discrepancy has been somewhat ameliorated through the detection of eight ultra-faint dwarfs around the Milky Way, some of whom evaded detection for so long by being comprised of $\sim 99\%$ dark matter.\textsuperscript{29} Nevertheless, this still leaves a factor of 4 difference between the observed and simulated numbers, suggesting that there are other processes at play.

Unlike CDM, ULDM has a natural cutoff in its mass function arising due to its astrophysical de Broglie wavelength.\textsuperscript{30} This effect is illustrated in Fig. 2.3 for the case where 10\% of dark matter is in ULDM and compared to 100\% CDM. Even when ULDM constitutes a small percentage of the total dark matter density, the halo-mass function is changed significantly: not only does it not predict more dwarf haloes, it essentially suppresses predictions of halos of masses smaller than $10^9 h^{-1} M_\odot$ at all redshifts. This significantly lowers the number of satellites predicted for a halo such as the Milky Way’s, thus reducing the tension between the observed and simulated numbers.

2.2.2.3 Too big to fail problem

In the previous section, the presence of ultrafaint dwarf galaxies which might have had their stars stripped through tidal interactions presented a partial solution to the missing satellites problem. However, this same non-ULDM solution poses a new problem. Dubbed the “too big to fail” problem, this latest small-scale issue is rooted in the idea that the predicted satellites of the Milky Way are simply too massive \textit{not} to host stars.\textsuperscript{31} Put differently: the observed satellites of the Milky Way are not massive enough to be consistent with CDM predictions.
The Halo-Mass Function is shown for 100% CDM (dashed) and 10% ULDM / 90% CDM (full). Note that ULDM essentially does not predict halos below $10^{9.5} h^{-1} \text{M}_\odot$. The ULDM mass is set to $m = 10^{-22} \text{eV}$.

This discrepancy can be lessened by tracking baryonic and tidal effects which could alter the central profile of the satellites, thus changing predictions of how CDM satellites would evolve.\(^3\)\(^2\) The key to reconciling simulation and observation lies in creating shallower central profiles of satellites. While this is possible to achieve with CDM when tracking tidal effects, it is even easier to achieve with ULDM's naturally cored profiles without addenda.

2.2.2.4 A CRISIS IN CRISIS?

Before moving on, it is worth noting that the severity (or even the existence!) of the “Small-Scale Crisis” is not agreed-upon in the literature. There are a number of researchers who believe that the collective discrepancies between simulations and observations on galactic scales can be attributed to the lack of baryons in simulations.\(^3\)\(^3\) Some arguments have been sketched out in the section on too big to fail.\(^3\)\(^4\),\(^3\)\(^5\) Other options include introducing parent-satellite interactions among ha-
los, supernovae, and generally modelling the subhalos’ environment.

Recent ideas also include investigating multi-component dark matter and how it affects halos on small scales. Furthermore, strides are being made in using machine learning to predict baryon effects on subhalo populations—an approach that could possibly simplify simulations of ULDM + baryons in the future. Finally, some astronomers are looking to JWST to constrain baryonic physics on these scales, though degeneracies between different DM models may persist.

Finally, it is important to emphasize that, though ULDM might have initially gained traction by providing solutions to the above-stated problems, it remains a compelling dark matter candidate in their absence. Its fascinating phenomenology—which includes not only the central core, but also turbulent behavior in its “skirt”—remains of interest to the field of dark matter research on its own merit. Moreover, any strides made in understanding ULDM dynamics can be applied to describing other ALP-fields in a variety of cosmological environments.

2.2.3 Mass constraints

Since the field of ULDM astrophysics was brought to life by the exciting results of Schive et al. in 2014, there have been numerous papers attempting to put mass constraints on the ULDM particle. A fantastic summary of the majority of these methods and their results can be found in Ferreira (2020), a recent detailed review article. This appropriate figure is reproduced in Fig. 2.4, with short explanations and citations for each of the constraints.

The information presented in Fig. 2.4 is thorough but not exhaustive. For instance, strong lensing has been used to put a conservative lower bound of \( m > 10^{-24} \text{eV} \) along with necessitating the presence of the NFW skirt. One study of the 2018 EDGES measurement of a global 21cm spectrum found that, in order to match that spectrum, ULDM is expected to have a mass in the range
Figure 2.4: Constraints on the ULDM particle’s mass, as presented in review article Ferreira (2020). Shaded regions correspond to masses excluded under the assumption that ULDM makes up most of the dark matter content of the Universe. Each exclusion is briefly summarized and cited below.

**CMB + LSS**: Planck (2015) TT CMB and galaxy-galaxy auto-power spectra.

**Lyman-α**: Lyman-α constrains from several analyses in the literature arranged from the darker to lighter blue.

**Eridanus II**: Both constraints come from the survival of its star cluster.

**BHSR**: Constraints from black hole superradiance (BHSR), including bounds on M87 spin by the Event Horizon Telescope and bounds from BHSR due to stellar and super-massive black holes (SMBH).

**21-cm (EDGES)**: The global 21-cm signal from the EDGES team can be used to put bounds on the mass of ULDM.

**SHMF**: Bounds from the suppression of the sub-halo mass function (SHMF).

**Dyn. friction**: Constraints from dynamical friction of ULDM applied to the Fornax globular cluster.

**Heating**: Limits on Milky Way disk heating due to the velocity dispersion of stars in the solar neighborhood.

**dSphs**: Constraints on the ULDM mass assuming measured central density of dwarf spheroidal galaxies (dSphs) Draco and Sextants should match maximum ULDM core size. Two additional analyses of the Milky Way’s dSphs are included in the lighter and darker coffee-colored regions.

On the other hand, the kinematics of the ultra-diffuse galaxy Dragonfly 44 have been used to argue for a mass of around $3 \times 10^{-22} \text{ eV}$. Some constraints are more pessimistic, such as one Lyman-α forest analysis which excludes $m < 10^{-21} \text{ eV}$, and one paper identifying supermassive black holes with ultra-compact soliton cores which even excludes $m < 10^{-18} \text{ eV}$.

Even without these added constraints, Fig. 2.4 paints a picture of a dark matter
particle in peril. Certainly the region of allowed masses is narrowing as more and more predictions of allowed mass come into tension with one another. Nevertheless, there is yet more ULDM parameter space to explore, most notably by including a self-interacting term to its Lagrangian\textsuperscript{66,67}. While this dissertation will strictly deal with non-self-interacting ULDM, it will delve into assumptions from numerical models fueling the above constraints. Many mass constraints rely in particular on the core-halo mass relation of ULDM derived from (other) numerical simulations. The veracity of this relation is the central topic explored in Chapter 5 of this dissertation; until we get to exploring that topic further, let us consider such constraints model-dependent.

2.3 UltraLight Dark Matter: the How’s

Having motivated why ULDM is an interesting particle, let us now develop its underlying mathematical description.

2.3.1 Time Evolution

2.3.1.1 Ancient Field Relics

In Section 2.2.1, I stated that ALPs are a natural consequence of some scalar field $\varphi$ with some small mass $m$. We can model the field’s particle as a pseudo-Goldstone boson with a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi) \quad (2.5)$$

with a choice of potential $V(\varphi)$, for instance

$$V(\varphi) = \Lambda^4 (1 - \cos(\varphi / f)) \quad (2.6)$$
Here, $f$ and $\Lambda$ are two constants,\textsuperscript{‡} and the additive constant is chosen as a convenience so that $V(0) = 0$.\textsuperscript{14,68} Thus, expanding the cosine in $V(\phi)$ around $\phi = 0$, we find that the mass of the particle scales as

$$m = \frac{\Lambda^2}{f}.$$  \hspace{1cm} (2.7)

Typically, $f$ is close to Planck-mass scale $\sim 10^{17}$ GeV, while $\Lambda$ is exponentially suppressed\textsuperscript{16} to around $\sim 100$ eV, yielding an astrophysically-motivated mass of $m \sim 10^{-22}$ eV.\textsuperscript{§}

Following Eq. 2.5, the question of the evolution of $\phi$ naturally arises. Under the assumption that $\phi$ is created early in our Universe’s history, its evolution will naturally depend on the Hubble rate $H(t)$ as

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0.$$ \hspace{1cm} (2.8)

While $H(t)$ is large (as in the early Universe), its friction will keep $\phi$ slowly rolling, and subsequently $V(\phi)$ will play the role of dark energy. For the purposes of this thesis, however, we are interested in the limit where $H(t)$ has slowed down enough for structure formation to begin, after the Hubble rate has reached become comparable to the mass of the particle, $H \sim m$. With no more extreme $H$ value to “slow the roll,” $\phi$ will oscillate around the minimum of the potential. As the Universe expands, the oscillations damp, effectively making the approximation

$$V(\phi) \sim \frac{1}{2}m^2\phi^2.$$ \hspace{1cm} (2.9)

\textsuperscript{‡}The constant $f$ is an energy scale often referred to as the “axion decay constant;” this might give a false impression of particle instability for the parameters I will be considering, and the reader is encouraged not to put too much stock in this particular nomenclature.

\textsuperscript{§}If we were interested in the QCD axion, this derivation would follow quite similarly (if not exactly), and a choice of $f \sim 10^{13}$ GeV would yield a mass $m \sim 10^{-6}$ eV.
more and more appropriate. Consequently, the energy in these oscillations is

\[ \rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2. \]  

(2.10)

It follows from Eq. 2.8 that \( \rho \propto a^{-3} \), where \( a \) is the scale factor of the Universe. This implies that the oscillations redshift like non-relativistic matter, and that they could be interpreted as a set of particles constituting dark matter.

### 2.3.1.2 Suitability of Dark Matter Candidate

Three more ingredients are needed to verify the appropriateness of said particles as dark matter: they must be cold, they must be weakly interacting, and they must make up \( \Omega_{m,0} \sim 0.3 \) of the present Universe. The particles' coldness is implicit in their formulation, as they start as a scalar field in the early Universe. The weakness of their interactions they owe to the large axion decay constant \( f \), which suppresses their self-interactions, as well as their interactions with photons (\( \nu \)) and fermions (\( \Psi \)):

\[
\begin{align*}
\mathcal{L}_{\text{self}} & \sim \frac{m^2}{f^2} \phi^4 \\
\mathcal{L}_{\text{int}}^{\gamma} & \sim \frac{\phi}{f} F^{\mu\nu} \tilde{F}_{\mu\nu} \\
\mathcal{L}_{\text{int}}^{\Psi} & \sim \frac{\partial_{\mu} \phi}{f} \bar{\Psi} \gamma^\mu \gamma_5 \Psi.
\end{align*}
\]

Finally, their relic density depends on both \( f \) and \( m \) (though much more strongly on the former):

\[
\Omega_{\text{axion}} \sim 0.1 \left( \frac{f}{10^{17} \text{GeV}} \right)^2 \left( \frac{m}{10^{-22} \text{eV}} \right)^{1/2}.
\]

(2.11)

Encouragingly, the right order of relic density \( \Omega_{\text{axion}} \sim \Omega_{m,0} \sim 0.3 \) pops out for the above-motivated choices of \( f \) and \( m \).
2.3.2 Spatial Fluctuations

2.3.2.1 The Schrödinger-Poisson System

In the previous subsection, we considered the effects of \( \varphi(t) \) using Eq. 2.8. Now, let us consider how \( \varphi(r) \) evolves by starting with the Klein-Gordon equation

\[
-\Box \varphi + m^2 \varphi = 0, 
\]  

(2.12)

which follows from Eq. 2.5 using \( V(\varphi) = \frac{1}{2} m^2 \varphi^2 \). We are interested in the evolution of \( \varphi \) during structure formation, in the non-relativistic regime; in this limit, it is useful to define a complex scalar field \( \psi \) as

\[
\varphi = \frac{1}{\sqrt{2m}} (\psi e^{-imt} + \psi^* e^{imt}). 
\]  

(2.13)

Thus, the fast time variations of \( \varphi \) with frequency \( m \) have been factored out into exponentials, and we can assume \( \psi \) is slowly varying such that \( |\ddot{\psi}| \ll m |\dot{\psi}| \). Plugging Eq. 2.13 into Eq. 2.12, we have

\[
\imath \partial_t \psi = -\frac{\nabla^2}{2m} \psi + m \Phi \psi, 
\]  

(2.14)

also known as: the Schrödinger equation!

Here, I will refer to \( \psi \) as the complex (albeit classical) wavefunction of ULDM; \( \psi \) traces dark matter density \( \rho \) as \( \rho = m|\psi|^2 \). The last piece of the puzzle, then, is the evolution of the dark matter’s corresponding gravitational potential \( \Phi \). For this, we invoke the trusty Poisson equation

\[
\nabla^2 \Phi = 4\pi G \rho. 
\]  

(2.15)

Together, Eqs. 2.14 and 2.15 make up the Schrödinger-Poisson (SP) system of
equations which govern the dynamics of ULDM.

2.3.2.2 Fluid Dynamics and the Madelung Representation

An alternative, fluid description of ULDM mechanics is known as the Madelung representation.\(^6^9\) The mass density of DM remains \(\rho = m|\psi|^2\), but the complex \(\psi\) is now represented as

\[
\psi = \sqrt{\frac{\rho}{m}} e^{i\theta}.
\]  

(2.16)

This representation is particularly useful when calculating the fluid velocity

\[
\tilde{v} = \frac{1}{m} \tilde{\nabla} \theta
\]

\[
= \frac{i}{2m|\psi|^2} \left( \psi \tilde{\nabla} \psi^* - \psi^* \tilde{\nabla} \psi \right),
\]  

(2.17)

(2.18)

which resembles the representation of a superfluid. With that understanding, we can write down the continuity and Euler equations

\[
\partial_t \rho + \tilde{\nabla} \cdot (\rho \tilde{v}) = 0,
\]

(2.19)

\[
\partial_t \tilde{v} + (\tilde{v} \cdot \tilde{\nabla}) \tilde{v} = -\tilde{\nabla} \Phi + \frac{1}{2m^2} \tilde{\nabla} \left( \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right).
\]

(2.20)

Together, these two equations are an alternative formulation of the SP system. The last term in the Euler equation is particularly interesting. Dubbed the “quantum pressure term”\(^*\), this quantity has an analogous “quantum energy” which stops the ULDM soliton core from collapsing. Finally, note that in the limit of large \(m\), the Euler equation reduces to describing a pressureless fluid—an appropriate formulation of particle dark matter.\(^6^8\)

\(^*\)Despite being neither “quantum” nor “pressure,” strictly speaking.
2.3.3 An Aside: The How’s and Why’s of the Early Universe

The preceding discussion served to motivate Chapters 4—6; Chapter 3, however, deals with primordial black holes in the early Universe immediately post-inflation. Though this project was completed before the author was aware of ULDM and the SP system, the physics underlying the two phenomena is incredibly similar (as hinted in this chapter). This section will serve to explicitly draw those parallels, retroactively situating Chapter 3 in the larger context of this opus.

Where before we were using $\phi$ to denote the ALP scalar field, let it now denote the inflaton field. As before, it obeys the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

while spacetime obeys the Friedmann equation

$$H^2 = \frac{1}{3M_{Pl}^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi)\right)$$

where $M_{Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass. After slow-roll inflation, the inflaton condensate oscillates around its potential minimum, which can once again be described as

$$V(\phi) = \frac{1}{2} m^2 \phi^2,$$

causing the Universe to expand as $a^3$. Thus, the early Universe enters a (temporary) matter-dominated phase during which sub-horizon density perturbations grow linearly with $a$, eventually collapse into gravitationally bound structures.

Due to high occupation numbers and in the absence of significant interactions with other fields, the evolution of these structures can be tracked with the (comoving) SP equations, exactly as formulated in Eqs. 2.14 & 2.15. These structures are
named “axion halos,” with central soliton-like “axion stars.” If the universe thermalizes at the TeV scale, typical axion star overdensities could reach $\delta \sim 10^{32}$, leading the most massive cores to collapse into primordial black holes.\textsuperscript{74}

Chapter 3 describes gravitational signatures from such a population of black holes, under the assumption of a monochromatic mass function. Future directions for using our code to motivate a more realistic mass function and relax that assumption are sketched in Section 7.2.

2.4 chpl\textsuperscript{ULTRA} code

The simulations use a pseudo-spectral Schrödinger-Poisson solver, chpl\textsuperscript{ULTRA}, described in Padmanabhan et al.\textsuperscript{75}. The algorithm mirrors that of PyULTRALIGHT\textsuperscript{76}, with the added capability to compute the gravitational potential with isolated boundary conditions.

2.4.1 Overview of the Algorithm

To illustrate the algorithm, we begin with the dimensionless Schrödinger-Poisson system,

\begin{align}
    i\partial_t \psi &= -\frac{1}{2} \nabla^2 \psi + \Phi \psi \tag{2.24}
    \\
    \nabla^2 \Phi &= 4\pi |\psi|^2 . \tag{2.25}
\end{align}

In order to evolve the system through time, we first write out the unitary evolution of the Schrödinger equation

\begin{equation}
    \psi(\vec{x}, t + h) = \mathcal{U} \exp \left[ -i \int_{t}^{t+h} dt' \left\{ -\frac{1}{2} \nabla^2 + \Phi (\vec{x}, t') \right\} \right] \psi(\vec{x}, t) \tag{2.26}
\end{equation}
where $\mathcal{T}$ is the time-ordering operator. Using the trapezoidal rule to approximate the integral of $\Phi$ for a sufficiently small timestep $h$, we are left with

$$\dot{\psi}(\vec{x}, t + h) \approx \exp \left[ i \frac{h}{2} (\nabla^2 - \Phi(\vec{x}, t + h) - \Phi(\vec{x}, t)) \right] \psi(\vec{x}, t).$$

(2.27)

The two potential terms above are easy enough to evaluate in position space, while the kinetic term is significantly simpler in Fourier space. Thus, we split the exponential terms (again assuming the timestep $h$ is small), yielding an equation of the form

$$\psi(\vec{x}, t + h) \approx \exp \left[ -i h \frac{\Phi(\vec{x}, t + h)}{2} \right] \exp \left[ i h \nabla^2 \right] \exp \left[ -i h \frac{\Phi(\vec{x}, t)}{2} \right] \psi(\vec{x}, t).$$

(2.28)

This form lends itself to the so-called symmetrized split-step Fourier method. The equation is evaluated right-to-left: first, a half step is taken in the position domain. The result is Fourier transformed, a full step is taken in the Fourier domain, and the potential is updated via the Poisson equation. Finally, the result is inverse Fourier transformed, and a final half step is taken. The schematic of this algorithm can be presented as

$$\psi(\vec{x}, t + h) = \exp \left[ -i h \frac{\Phi(\vec{x}, t + h)}{2} \right] \mathcal{F}^{-1} \exp \left[ -i h \frac{k^2}{2} \right] \mathcal{F} \exp \left[ -i h \frac{\Phi(\vec{x}, t)}{2} \right] \psi(\vec{x}, t)$$

(2.29)

$$\Phi(\vec{x}, t + h) = \mathcal{F}^{-1} \left( -\frac{1}{k^2} \right) \mathcal{F} 4\pi |\psi(\vec{x}, t)|^2,$$

(2.30)

where $\mathcal{F}$ represents a Fourier transform, and $\mathcal{F}^{-1}$ represents its inverse.

### 2.4.2 Isolated Boundary Conditions

Until this point, the algorithm has been identical to that described in Edwards et al.\textsuperscript{76} for PyULTRA\text{\textsc{light}}; however, Eq. 2.30 is where chPLUGTRA diverges.
potential calculation as presented above requires periodic boundary conditions for
Φ, which speed up the potential calculation but cause issues when the simulated
mass approaches the edge of the box. For example, in the case of two bodies orbit-
ing, we would expect them to circle around their center of mass in the middle of
the box. However, if they approach the edge too closely, they will “feel” the other
body more strongly through the boundary and may even begin orbiting through
the edge of the box.

A simple—but costly—way to minimize this effect would be to double the size of
the box and the number of grid points to maintain resolution. We chose the alter-
native: implementing isolated boundary conditions using the Hockney-Eastwood-
Brownrigg algorithm. Instead of evaluating the Laplacian as in Eq. 2.30, we
evaluate it using a Green’s function. The Green’s function depends only on the
gridsize n, and so needs to be calculated only once when the code is initialized.

We use a Discrete Cosine Transform (DCT) to calculate a (n, n, n)-sized array. In-
side the innermost loop, we grab individual (n, n) planes which are doubled to
(2n, 2n) before being convolved with the padded y − z planes. This way, we save
on memory by never storing a full (2n, 2n, 2n) array. See Fig. 2.5 for an illustra-
tion of the results.

We implement this in CHAPEL, a next-generation programming language
being developed by Cray/HPE. CHAPEL’s native features allow for productive par-
allel programming, and (relatively) seamlessly targets systems from traditional su-
percomputers to commodity clusters to personal computers. We have successfully
scaled CHPLULTRA out to 512 nodes, running with grids up to 81923, although most
of the results presented in this paper use 5123 to 10243 grids. In addition to CH-
PLULTRA, we also developed a spherically symmetric code for the ℓ = 0 results. In-
stead of operator splitting, this directly computes the exponential of a discretized

\[ B = \int d^3 \rho \nabla \times \left( \frac{\rho}{\nabla^2} \nabla \times \rho \right) \]
version of the Hamiltonian to implement the symplectic time stepping. We find good agreement between runs done with both codes.

2.4.3 Code Units

All of our results (save Chapter 3, which doesn’t rely on CHPLULTRA) are presented in “code” units. To convert these to more astrophysically recognizable values, we start by recalling that the SP system remains invariant when scaled by a parameter $\lambda$ as follows:\textsuperscript{81}:

$$\{t, x, V, \psi, \rho\} \rightarrow \left\{ \lambda^{-2} t, \lambda^{-1} \dot{x}, \lambda^2 \dot{V}, \lambda^2 \dot{\psi}, \lambda^4 \rho \right\} \quad (2.31)$$

From the above, we can calculate how the total mass, energy, and angular momentum scale with $\lambda$:

$$\{M, E, L\} \rightarrow \left\{ \lambda M, \lambda^3 E, \lambda L \right\} \quad (2.32)$$

Furthermore, the SP system can also be transformed through scaling the ULDM
particle mass $m \rightarrow \alpha m$ as:

$$\{t, x, V, \psi, \rho\} \rightarrow \left\{\alpha^{-1/2}\hat{t}, \alpha^{-3/2}\hat{x}, \alpha^{-1}\hat{V}, \alpha^{-3/2}\hat{\psi}, \alpha^{-3/2}\hat{\rho}\right\} \quad (2.33)$$

with the total mass, energy, and angular momentum then scaling as

$$\{M, E, L\} \rightarrow \left\{\alpha^{-3/2}\hat{M}, \alpha^{-5/2}\hat{E}, \alpha^{-2}\hat{L}\right\}. \quad (2.34)$$

We adopt a fiducial value of $m = m_{22} \times 10^{-22}$ eV, where the scaling of our results with the axion mass is captured by $m_{22}$. Finally, we can introduce appropriate length, time, and mass scales as in Edwards et al. (2018) as a function of the parameters $\lambda$ and $m_{22}$:

\begin{align*}
L &= \left(\frac{8\pi\hbar^2}{3m^2H_0^2\Omega_{m_0}}\right)^{1/4} \lambda^{-1} \approx 38.3 \text{kpc} \times \lambda^{-1} m_{22}^{-1/2}, \quad (2.35) \\
T &= \left(\frac{8\pi}{3H_0^2\Omega_{m_0}}\right)^{1/4} \lambda^{-2} \approx 75.5 \text{Gyr} \times \lambda^{-2}, \quad (2.36) \\
M &= \left(\frac{8\pi G^4}{3H_0^2\Omega_{m_0}}\right)^{-1/4} \left(\frac{\hbar}{m}\right)^{3/4} \lambda \approx 2.2 \times 10^6 M_{\odot} \times \lambda m_{22}^{-3/2}. \quad (2.37)
\end{align*}

Each of these scales is equal to one code unit of length, time, and mass, respectively. I have presented a few choices of $\lambda$ for different astrophysical systems in Table 2.1 for the convenience of the reader.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$t$ [Gyr]</th>
<th>$x$ [kpc]</th>
<th>$M$ [$M_{\odot}$]</th>
</tr>
</thead>
</table>
| Units in Ref.\textsuperscript{76} | 1.0 | 75 | 38.3 | 2.2 $\times 10^6$
| One gigayear time unit | 8.7 | 1 | 4.4 | 1.9 $\times 10^7$
| Hubble time unit | 2.3 | 14 | 16.6 | 5.1 $\times 10^6$
| Dwarf galaxy halo core | 1.8 | 23 | 21.2 | 4.0 $\times 10^6$
| Very massive halo core | 4500 | $4 \times 10^{-6}$ | 0.008 | 1.0 $\times 10^{10}$

Table 2.1: This table lists the physical values corresponding to a single code unit of time, length, and mass. We consider different physical situations with their corresponding $\lambda$ values. We also highlight the scaling of these values with $\lambda$ and the mass of the axion on the top, but have only considered $m_{22} = 1$ in the construction of this table.
Coming back to where you started is not the same as never leaving.

Terry Pratchett, A Hat Full of Sky

3

GUT-Scale Primordial Black Holes: Mergers and Gravitational Waves

3.1 Introduction

This paper examines the dynamics of a possible transient matter dominated phase in the very early Universe in which the Universe is dominated by primordial black holes $^{82-84}$, analysing the binary formation and coalescence and consequent gravitational wave production during this era. Primordial black holes (PBH) can form immediately after the Big Bang through the gravitational collapse of large pri-
mordial density fluctuations in the early Universe. We infer from observations that the early Universe was smooth on comoving scales corresponding to present-day astrophysical distances, but this does not constrain the portion of the primordial power spectrum relevant to PBH formation. Given that the measured portion of the primordial power spectrum is featureless and mildly red, naively extrapolating astrophysical results to smaller scales suggests that no PBH will form via direct collapse in the early Universe.\(^*\) From the opposite perspective, however, limits on PBH abundance constrain the primordial power spectrum at small length scales and the extent to which the scale-invariance of the power spectrum can be broken. These limits have implications for any physical mechanism responsible for generating the primordial perturbations, such as inflation.\(^87\text{–}90\)

Black holes have finite lifetimes due to the emission of Hawking radiation. The Hawking temperature is inversely proportional to the mass and small black holes lose mass more rapidly than large ones. PBH can, in principle, range in size from the Planck mass to hundreds of solar masses and beyond; a PBH with an initial mass exceeding \(10^{15}\) g survives until the present and is thus potentially detectable today.\(^†\) Decaying black holes may disrupt nucleosynthesis, recombination, and reionization, from which limits on populations of lighter PBH can be derived. However, a PBH population that decays well before nucleosynthesis has little impact on the canonical thermal history of the hot Big Bang and is thus largely unconstrained by observations.\(^95\)

The absence of limits on populations of very short-lived PBH creates a range of phenomenological possibilities for the very early Universe, as discussed in Anantua et al.\(^84\). These include a transient matter dominated phase in which PBH are

\(^*\)Mechanisms such as parametric resonance following inflation may dynamically magnify primary perturbations, providing alternative routes to PBH formation in scenarios for which primordial perturbations at all scales are small as they leave the horizon.\(^86\) In what follows we focus on PBH formed via the direct collapse of a near horizon-sized volume.

\(^†\)The possibility that at least some of the LIGO detections reflect mergers of primordial rather than stellar black holes has attracted considerable attention.\(^91,92\)
the dominant constituent of the Universe, as well as a stochastic gravitational wave background associated with Hawking radiated gravitons. This matter dominated stage would be part of the “primordial dark age”\textsuperscript{96}, and could have an observable impact on the matching between primordial and present-day perturbation amplitudes\textsuperscript{97–99} but the dynamics and consequences of this matter dominated phase are largely unexamined. In particular, the possible interactions between PBH in this epoch are entirely unexplored.

In this paper we focus on black hole mergers and their consequences during a PBH dominated phase, in the primordial Universe to sharpen our understanding of the possible implications of such a scenario. Firstly, merged PBH can live up to eight times longer than their progenitors (since lifetime goes as the cube of the mass), creating the possibility that mergers extend the length of the matter dominated phase. Furthermore, mergers provide a second source of gravitational waves. Individual mergers will typically convert a larger fraction of progenitor restmass, about 5%, into gravitational waves than Hawking decay, which only converts 2% of evaporated mass into gravitons.\textsuperscript{100}

This paper is arranged as follows. In Section 3.2 we survey the parameters governing the lifetime of PBH. In Section 3.3 we present an estimate of the population and merger statistics of binary black holes in a Universe where PBH are uniformly and randomly distributed at formation, computing the binary capture rate induced by gravitational radiation during near-hyperbolic close encounters. These binaries can undergo mergers and the resulting gravitational wave spectrum is computed in Section 3.4; we discuss our results in Section 3.5. We work in natural units where $\hbar = c = k_B = 1$ and the gravitational constant $G = M_P^{-2}$, where $M_P$ is the unreduced Planck mass.
3.2 Ultralight PBH in the Early Universe

The formation of primordial black hole formation in the early Universe has been well-studied topic\textsuperscript{82,83,101–106}. In what follows we make a number of simplifying assumptions to expose the underlying physics of this scenario, as we are interested in obtaining an estimate of the overall gravitational wave background, rather than its detailed form. Following the analysis of Anantua et al.\textsuperscript{84}, we begin by assuming that the ultralight PBH have a mass equal to the energy inside a Hubble volume at the moment of collapse, and that these black holes form at the same time. This implies the black holes all have an initial mass of

\[ M_i = \frac{4}{3\pi} \left( \frac{1}{H} \right)^3 \rho = \sqrt{\frac{3}{32\pi}} \frac{M_P^3}{E_i}, \]  

(3.1)

where \( \rho = E_i^4 \) is the density in a radiation dominated Universe and \( H^2 = \frac{8\pi\rho}{3M_P^2} \).

We also assume that the black holes are formed at rest and randomly distributed with an initial mass fraction of \( \Omega_{PBH,i} = \beta \). Our model is thus completely specified by two parameters, \( E_i \) and \( \beta \). These black holes have a temperature

\[ T = \frac{M_P^2}{8\pi M} \]  

(3.2)

and their mass decreases via Hawking radiation at a rate

\[ \frac{dM}{dt} \bigg|_H = -\frac{g}{30720\pi} \frac{M_P^4}{M^2}. \]  

(3.3)

The number of degrees of freedom into which the black hole can radiate is denoted by \( g \). The resulting lifetime is then

\[ t_H = \frac{10240\pi M^3}{g M_P^4} = \frac{240}{g} \sqrt{\frac{3}{2\pi}} \frac{M_P^5}{E_i^6}. \]  

(3.4)
Since the temperature of the black hole increases as its mass decreases, \( g \) is a dynamical quantity and should generically increase with time, but we assume it is constant for simplicity. This is a good approximation since the mass, and therefore the temperature of the black hole does not significantly change over most of its lifetime. Because Hawking radiation is not strictly thermal and “grey body” corrections are more dramatic for higher-spin particles, \( g \) is an effective number of degrees of freedom. In general, the decay rate depends on the mix of spin-statistics in the particle spectrum, but we absorb this ambiguity into the definition of \( g \). For the PBH discussed here the temperature is always at the TeV scale or above, so Standard Model species may form a small subset of all relevant particle states. Furthermore, apart from the gravitational wave background described in \(^8^4\), we assume that all other radiated species equilibrate, leaving the early Universe in a thermal state after the PBH evaporate. In our numerical examples, we set \( g = 1000 \) for definiteness, but none of our conclusions depend strongly on this choice.

Since radiation density scales as \( a^{-4} \) while the matter scales like \( a^{-3} \), the Universe can pass through an effective matter-dominated phase, with matter-radiation equality occurring when \( a \sim 1/\beta \). Comparing the time to PBH-radiation equality with the PBH lifetime, the condition for the existence of a matter-dominated phase is \(^8^4\)

\[
\beta \geq \frac{1}{8} \sqrt{\frac{g}{15}} \frac{E_i^2}{M_P^2}.
\]  \( (3.5) \)

Matter-domination ends when the black holes fully evaporate, returning the Universe to radiation domination. Requiring this to occur before nucleosynthesis gives the constraint \( t_H \leq 100 \text{ s} \). Taking \( E_i \geq 10^{12} \text{ GeV} \) results in \( t_H = 29/g \) seconds, and an initial Hawking temperature of \( T = 18.8 \text{ TeV} \). This is actually an effective lower bound on \( E_i \); the lifetime depends on \( E_i^5 \) so even a small softening of this limit allows PBH to survive into nucleosynthesis. \(^{10^7}\) Finally, if we assume an initial inflationary phase, bounds on the primordial gravitational wave background de-
derived from microwave background data imply that $E_i \lesssim 10^{16} \text{ GeV}$, which is roughly $10^{-3} M_P^{85}$, so we take this as a (tentative) upper bound. The resulting parameter space is summarized in figure 3.1.

Immediately after formation, the black holes are colder than the surrounding radiation bath, which has a temperature

$$T_{\text{univ}} = \left( \frac{30 \rho_{\text{rad}}}{\pi^2 g} \right)^{\frac{1}{4}}$$

(3.6)

so the accretion of radiation across the horizon can initially dominate the Hawking emission.‡ While $T_{\text{univ}} > T$, the mass of the black hole is increasing at the rate of

$$\left. \frac{dM}{dt} \right|_{\text{acc}} = f \times A \times \rho_{\text{rad}} = 16\pi f \frac{M^2}{M_P^2} \rho_{\text{rad}},$$

(3.7)

where $f$ is the accretion efficiency, and $A$ is the area of the black hole$^{108}$. We assume that when $T_{\text{univ}} = T$ the Hawking radiation and accretion rates are roughly equal, $\dot{M}_H = \dot{M}_{\text{acc}}$ which implies that $f = 1/(4\pi)$, and in general $\dot{M} = \dot{M}_H + \dot{M}_{\text{acc}}$. Since the early Universe cools quickly, the overall mass-gain is not huge – the maximal increase within the parameter range we consider is 14%, but we include this term – which was not accounted for in Anantua et al.$^{84}$ – for completeness.

3.3 Mergers in the Early Universe

Analyses of PBH mergers have recently enjoyed a renaissance thanks to suggestions that observed gravitational wave signals might be drawn from such a population rather than stellar remnants$^{91,109-113}$. We can map this directly to our prob-

‡Note that the $g$ is in this equation is not fully consistent with the effective number of degrees of freedom in Hawking radiation expression, which includes the grey body terms; this difference can be ignored at the level of precision needed here.
Figure 3.1: The $\beta - E_i$ parameter space. The gray dashed area on the right ($E_i > 10^{-3}M_{Pl}$) is excluded by CMB constraints if we assume a prior inflationary phase; the left hand area is excluded by requiring that PBH do not survive into BBN ($E_i < 10^{-7}M_{Pl}$). The green portion never undergoes a matter dominated phase. For regions above the dashed line the typical black hole separation yields a nominal merger timescale less than $t_H$; in the lower region only those black holes that form in relatively close proximity undergo mergers.

lem\textsuperscript{8} and we follow the simplified formalism developed in Nakamura et al.\textsuperscript{114}. We assume PBH are formed at rest and randomly distributed: in the absence of significant primordial non-Gaussianity, PBH will coalesce at random, uncorrelated locations. The first mergers occur between pairs that happen to form in close proximity; the next-nearest neighbour is likely to be at a substantially greater distance and supplies a small perturbation preventing a head-on collision. However, grav-

\textsuperscript{8}We ignore any clustering of PBH; the overall validity of this approximation is evaluated in Section 3.5.
itational radiation can lead to mutual capture during what would otherwise be a hyperbolic encounter in a Newtonian context, leaving the two PBH in a highly elliptical orbit. The larger accelerations experienced at periastron in an elliptical orbit enhance energy loss due to gravitational radiation relative to a circular binary with the same semimajor axis, shortening the coalescence time.

If PBH are randomly and uniformly distributed the expected number of PBH in any region depends only on its volume. When the condition in equation 3.5 is satisfied there will be a transition between radiation and matter dominated growth in the primordial Universe, at a time $t = t_{eq}$. At this moment $\Omega_{\text{rad}} = \Omega_{\text{PBH}} = 1/2$, the total density is $\rho_{eq}$, and the characteristic comoving distance between PBH with mass $M$ is thus approximately

$$\bar{x}_{eq} = \left( \frac{2M}{\rho_{eq}} \right)^{\frac{1}{3}}. \tag{3.8}$$

Given their creation mechanism, the PBH are initially at rest with respect to the background; they start falling toward each other when their contribution to the density of a sphere with diameter equal to their separation $x$ is greater than the average density of the Universe, or

$$\rho_{\text{PBH}} = \rho_{eq} \left( \frac{x}{xa(t)} \right)^{3} \tag{3.9}$$

$$a > a_{m} \equiv \left( \frac{x}{\bar{x}} \right)^{3}. \tag{3.10}$$

Here $a_{m}$ is the ($x$-dependent) value of the scale factor at which the “density” of PBH in a sphere of diameter $x$ containing two black holes exceeds the radiation density, at which point they decouple from the overall expansion of the Universe, and begin to fall toward each other.
Before considering the coalescence rate of PBH binaries in detail, let us consider the key time scales that appear in the problem. The coalescence time for circular binaries is

\[ t_c = \frac{5}{16} \left( \frac{\pi}{6} \right)^{5/6} M_P \frac{1}{E_i^{3/16}} \beta^{3/16} , \tag{3.11} \]

if the initial diameter is \( \bar{x} \) at \( t_{\text{eq}}^{115,116} \). Comparing this to the Hawking lifetime (3.4),

\[ \frac{t_c}{t_H} = \frac{g}{2304} \left( \frac{\pi}{6} \right)^{4/3} E_i^4 M_P^{-1/3} \beta^{16/3} \approx 1.1 \times 10^{28} \left( \frac{E_i}{10^{-5} M_P} \right)^4 \left( \frac{10^{-9}}{\beta} \right)^{16/3} , \tag{3.12} \]

divides our parameter space into two regions. For parameter choices with \( t_H < t_c \) only those binaries that form in close proximity will merge before the black holes evaporate; conversely if \( t_H > t_c \) there is the potential for a nontrivial number of second (and higher) generation mergers. These regions are illustrated in figure 3.1.

In addition, we can compare the circular coalescence timescale \( t_c \) with the Hubble time \((1/H)\) at matter-radiation equality

\[ t_c H_{\text{eq}} = \frac{5}{24} \left( \frac{\pi^4}{6} \right)^{3/2} \frac{1}{\beta^{10/3}} \tag{3.13} \]

showing that the merger time will always be long compared to the initial Hubble time unless \( \beta \) is very close to unity. This means that matter dominated phase will be roughly \( 1/\beta^{10/3} \) times longer than the Hubble time at equality, from which we deduce that the matter dominated Universe grows by a factor of \( t^{2/3} = 1/\beta^{20/9} \). We can then imagine a second generation of mergers beginning at an effective separation of \( \bar{x}/\beta^{20/9} \), with their initial number reduced by a factor of 2. This extends the naive timescale for all mergers to complete by \( 2^{4/3} / \beta^{80/9} \). Even for a high value of \( \beta = 0.1 \), this increase is on the order of \( 10^9 \), which swamps the roughly eightfold increase in lifetime gained by second generation black holes. Consequently, it seems that even if \( t_H > t_c \), a “cascade” of mergers is unlikely. However, this ar-
gument assumes an initially uniform, random distribution of PBH and the second

generation mergers could be facilitated by clustering and N-body interactions, a

treatment of which is beyond the scope of this analysis. However, the authors

of Raidal et al.\textsuperscript{117} explored the N-body merger dynamics of younger, more massive

black holes, and found that the merger rate is suppressed through the disruption

of early-formed binaries by nearest neighbors, making subsequent rounds of merg-

ers even more unlikely.

To estimate the merger rate in more detail we follow the approach of Nakamura

et al.\textsuperscript{114}. Consider a pair of black holes with comoving separation $x$, $0 < x < \bar{x}$, which breaks free from the Hubble expansion when $a = a_m$. Their initial separa-
tion sets the major axis of their orbit, $\alpha_m$:

$$\alpha_m = x a_m = \frac{x^4}{\bar{x}^3}. \quad (3.14)$$

We estimate the minor axis $\beta_m$ as the (tidal acceleration) $\times$ (free fall time)$^2$, giving

$$\beta_m = \frac{GMx a_m}{(ya_m)^3} \left( \frac{x a_m}{y} \right)^3 = \left( \frac{x}{y} \right)^3 \alpha_m, \quad (3.15)$$

where $y$ is the separation of the next-to-nearest neighbour to the center of mass of

the binary.$^6$ From here, it is straightforward to work out the ellipticity,

$$e = \sqrt{1 - (x/y)^6}. \quad (3.16)$$

We must now decide on the form of the distribution of $x$ and $y$.\textsuperscript{114} assume that $x$

\textsuperscript{6}We follow the presentation of Nakamura et al.\textsuperscript{114}. One could add a factor of $\frac{1}{2}$ to equation 3.14, since this is technically the major (not semimajor) axis and this would induce a factor of $\frac{1}{2}$ in

\textsuperscript{equation 3.15 (i.e. so that reads $\frac{1}{2}at^2$ instead of $at^2$). These factors, along with a proper normalization of 3.17 leave the final probability distribution unchanged.
and \( y \) are uniformly distributed in 3D, resulting in

\[
f(\alpha_m, e) d\alpha_m de = 18 \frac{x^2 y^2}{x^6} dx dy = \frac{3}{2} \frac{\alpha_m e^{\frac{1}{2}}}{x^{\frac{7}{2}}(1 - e^2)^{\frac{1}{2}}} d\alpha_m de.
\] (3.17)

While this assumption cannot be strictly correct, Ioka et al.\(^{118}\) show that it is accurate to within a factor of 2. Given our other assumptions, we adopt the simpler formulation in Nakamura et al.\(^{114}\). As we’ll see below, using the more accurate formulation will not change our conclusions.

Elliptical binaries coalesce in a time\(^{115,116}\)

\[
t = t_c \left( \frac{\alpha_m}{\bar{x}} \right)^4 (1 - e^2)^{\frac{7}{2}}
\] (3.18)

where \( t_c \) is the circular coalescence time, equation 3.11. Substituting into equation 3.17 and integrating yields the distribution of binary lifetimes:

\[
f(t) dt = \frac{3}{29} \left[ \left( \frac{t}{t_c} \right)^{\frac{7}{2}} - \left( \frac{t}{t_c} \right)^{\frac{1}{2}} \right] dt = \frac{t}{t_c}.
\] (3.19)

This suggests that all black holes have merged when \( t = t_c \); however this analysis is only strictly applicable in the tail of the distribution for pairs with very small initial separations, given that it is based upon the lifetime of a tightly bound binary that is far from its nearest neighbours. However, as noted above, \( t_c \) sets the timescale over which we would expect a significant fraction of the population to undergo mergers. Note that we have not accounted for nontrivial three-body interactions, which could leave two participants tightly bound while the third is accelerated to a large (relative) velocity. A full treatment of this such interactions would require a full N-body simulation.
3.4 Gravitational Wave Spectrum

To calculate the density of gravitational waves, we set up the following system of equations. The two parameters which scale as matter are denoted \( \rho_M \) and \( \rho_{2M} \) representing the density of “first generation” black holes with original mass \( M \), and \( \rho_{2M} \) accounts for the post-merger population, respectively. The two parameters which scale as radiation are denoted as \( \rho_{\text{rad}} \), representing the density of the Universe in radiation, and \( \rho_{GW} \), which represents the gravitational radiation produced by mergers. Accounting for the expansion of the Universe and the source/sink terms, and recalling that \( \rho_M = n(t)M(t) \), we have the network of equations

\[
\dot{\rho}_M = \dot{n}(t)M(t) + n(t)\dot{M}(t)
= -3\frac{\dot{a}}{a}\rho_M - \sigma(t)\rho_M + \rho_M\frac{\dot{M}}{M} \tag{3.20}
\]

\[
\dot{\rho}_{2M} = -3\frac{\dot{a}}{a}\rho_{2M} + (1 - \varepsilon)\sigma(t)\rho_M \tag{3.21}
\]

\[
\dot{\rho}_{\text{rad}} = -4\frac{\dot{a}}{a}\rho_{\text{rad}} - \rho_M\frac{\dot{M}}{M} \tag{3.22}
\]

\[
\dot{\rho}_{GW} = -4\frac{\dot{a}}{a}\rho_{GW} + \varepsilon\sigma(t)\rho_M \tag{3.23}
\]

\[
\frac{\dot{a}}{a} = \left[ \frac{8\pi}{3M_p^2}(\rho_M + \rho_{2M} + \rho_{\text{rad}} + \rho_{GW}) \right]^{\frac{1}{2}} \tag{3.24}
\]

\[
\dot{M} = -\frac{g}{30720\pi} \frac{M_p^4}{M^2} + 4\frac{M^2}{M_p^4}\rho_{\text{rad}}. \tag{3.25}
\]

The last term in equation 3.20 accounts for both Hawking radiation and accretion; the sign convention ensures that \( \dot{M}(t) \) is positive during accretion and negative during evaporation (see equation 3.25). The first two terms in equation 3.20 arise from the changing number density of PBH: the first indicates the density of black holes scales as matter in the expanding Universe, and the second accounts
Figure 3.2: The evolution of $\Omega_{\text{rad}}, \Omega_M, \Omega_{2M}$, and the merger-generated gravitational wave background, $\Omega_{\text{GW}}$. On the upper left, we plot the behavior of the system for $E_i = 10^{-5} M_P$ and $\beta = 10^{-8}$, for which $t_H < t_c$. On the right, we plot the behavior for $E_i = 10^{-5} M_P$ and $\beta = 10^{-3}$, for which $t_c < t_H$, and so most binaries coalesce. For this case, we stop the integration when the first generation of mergers has completed. The turnover in the density of gravitational waves is due to their redshifting during the matter dominated era.

for black hole mergers, where

$$
\sigma(t) = f(t) \left[ 1 - \int_0^t f(t) \right]^{-1}
$$

(3.26)

is the normalized fractional change in number density per unit time\footnote{The lower bound of the integral of $f(t)$ here should technically be PBH formation time $t_0$; compared to any other timescale in the problem, this is essentially zero.} derived from the merger time probability distribution of equation 3.19. We assume that a fixed fraction $\epsilon$ of the rest mass is converted into gravitational waves by the merger while the remaining $1 - \epsilon$ comprises the newly formed black hole; in what follows we take $\epsilon = 0.05$\textsuperscript{119,120}. The first terms in equations 3.22 and 3.23 accounts for the usual dilution of radiation in an expanding Universe. Finally, the evolution of the
scale factor is given by equation 3.24.

The initial conditions are $M(t_0) = M_i$, $\rho_M(t_0) = \beta E_i^4$, $\rho_{\text{rad}}(t_0) = (1 - \beta)E_i^4$, $a(t_0) = 1$, and $\rho_{2M}(t_0) = \rho_{\text{GW}}(t_0) = 0$. We evolve the equations until $M(t)$ becomes $10^{-5}M_i$, with $M_i$ given by equation 3.1; we cannot use the analytic expression for $t_H$ (equation 3.4) as a stopping condition since it does not account for radiation accretion. Illustrative examples for two sets of parameters are shown in figure 3.2.

We have made a number of approximations here. Firstly, we are ignoring Hawking radiation from the more massive black holes; the practical advantage of this is that because we are assuming the initial PBH population has identical masses, the Hawking emission per unit mass is only a function of time. However, the masses of the black holes produced during mergers will depend on when the mergers take place, which would complicate the computation of the emission rate. As we are only looking at classically produced gravitational waves and since secondary mergers are likely to be rare, this approximation is not unreasonable. We also ignore the impact of decreasing black hole masses on merger timescales; since the mass initially changes slowly, this is a reasonable approximation and will lead to an overestimate of gravitational wave production during the last few Hubble times. However, since the Universe grows by a very large factor during any PBH dominated phase this simplification is also relatively benign.

To compute the spectrum of classically produced gravitational waves, we assume mergers produce a monochromatic gravitational wave signal. This very strong approximation is workable given that the Universe typically expands by many orders of magnitude during the black hole dominated phase, “smearing” the spectrum of the resulting stochastic background. We build the overall present-day spectrum by redshifting the produced gravitational waves after they are produced. The peak frequency (at the time of merger) can be deduced from Phinney$^{119}$ and Abbott
Figure 3.3: Present day spectra of gravitational waves from coalescing black holes for parameter choices where most binaries do not coalesce before evaporating, $t_H < t_c$. The top panel has $E_i = 10^{-3}M_P$, while the lower panel has $E_i = 10^{-7}M_P$, and the values of $\beta$ are shown in the legends. The fractional mass $M(t)/M_i$ of the initial black holes at the moment the corresponding gravitational waves are produced is shown in gray and labeled on the right axis, and the fraction of the initial Hawking time (which including the impact of early coalescence) is plotted on the top axis. Adjusting $E_i$ changes the frequency range, while $\beta$ controls the amplitude of the spectrum.
The frequency of the emitted GW thus scales as

\[ \nu(t) = \frac{a(t)}{a(t_e)} \nu_{\text{rest}}, \tag{3.28} \]

where \( t_e > t_H \) is the time of PBH evaporation when accounting for accretion. Solving equations 3.20 - 3.25 yields an expression for the total \( \Omega_{GW}(t) \); to obtain the gravitational radiation at a given frequency we follow Easther and Lim\textsuperscript{121} and Price and Siemens\textsuperscript{122}:

\[ \Omega_{GW}(\nu) = \frac{1}{\rho_c(t_e)} \frac{d\rho_{GW}}{d \ln(\nu)}. \tag{3.29} \]

We transform this into a more convenient form

\[ \Omega_{GW}(\nu) = \frac{\dot{\rho}_{GW}}{\rho_c(t_e)} \frac{\nu}{\nu'}, \tag{3.30} \]

where \( \dot{\rho}_{GW} = \varepsilon \sigma(t) \rho_M(t) \) is the contribution to the gravitational wave background at time \( t \), \( \rho_c(t_e) \) is the critical density at PBH evaporation, and \( \nu/\nu' \) can be calculated from equation 3.28. To calculate the present-day \( \Omega_{GW} \) from mergers we assume that the Universe is radiation dominated from the moment the primary black hole population decays through to the moment matter-radiation equality prior to recombination. We redshift this background to present day values \( \Omega_r \) via the relationship\textsuperscript{121}

\[ \Omega_{GW}^{\text{today}} = \left( \frac{g_0}{g_*} \right)^{4/3} \frac{\Omega_{\text{rad}}^{\text{today}}}{\Omega_{\text{rad}}^{\text{evap}}} \Omega_{GW}^{\text{evap}}. \tag{3.31} \]

We take \( g_* = 1000, g_0 = 3.91 \textsuperscript{123}, \Omega_{\text{rad}}^{\text{today}} = 8.24 \times 10^{-5}, \) and \( \Omega_{\text{rad}}^{\text{evap}} = 1 \) and assume
that the temperature of the Universe immediately after evaporation is

\[ T_e = \frac{30 \rho_{\text{rad}}(t_e)}{(\pi^2 g_*)^{\frac{1}{4}}} \tag{3.32} \]

which yields the redshifted frequencies

\[ \nu_{\text{today}} = \left( \frac{g_*}{g_0} \right)^{\frac{3}{2}} \left( \frac{T_{\text{CMB}}}{T_e} \right) \nu. \tag{3.33} \]

Representative results for the region of parameter space where \( t_H < t_c \) are shown in figure 3.3. The rising part of the spectrum is generated when \( t \leq 0.5 t_H \); in this regime, after accounting for accretion, the PBH masses are relatively unchanged from their initial values. This phase can see substantial gravitational wave production, but the gravitational waves produced by earlier mergers undergo a greater redshift during the primordial matter dominated phase, reducing their amplitude in the present-day Universe. Conversely, the falling portion of the spectrum is generated when \( t > 0.5 t_H \), and drops because the masses of coalescing black holes are now significantly reduced by evaporation. It is evident that \( E_i \) controls the frequency range, while \( \beta \) controls the amplitude of the spectrum.

For parameter choices for which \( t_H > t_c \), we evolve our coupled differential equations until \( t = t_c \), where the majority of mergers will have concluded. Then, we redshift from \( a(t_c) \) to \( a(8t_H) \), making the approximation that the Universe was matter dominated during this time. We then redshift from \( t_c \) to today as previously, using

\[ \rho_{\text{rad}}(8t_H) = \left( \frac{a(8t_H)}{a(t_c)} \right)^{-\frac{7}{3}} \rho_{2M}(t_c), \tag{3.34} \]

since at \( t_c \) all the density of the Universe (save the produced gravity waves) is contained in \( \rho_{2M} \), which Hawking evaporates into radiation at \( 8t_H \). The resulting spectra are shown in figure 3.4. The additional redshift from the longer matter domi-
Figure 3.4: Gravitational wave spectra from coalescing black holes for regions of parameter space when most binaries coalesce, $t_c < t_H$. Each color corresponds to a single value of $E_i$ as identified in figure 3.3, with the addition of orange for $E_i = 10^{-4}M_P$, green for $E_i = 10^{-5}M_P$, and blue for $E_i = 10^{-6}M_P$. The highest $\beta$-values for each $E_i$ are $\beta = 10^{-1}$, and each darker hue moving to the right represents a factor of 10 decrease in $\beta$. In order to produce a spectrum with LIGO-range frequencies (10 Hz - 1 kHz), about 10% of the Universe must start out in the form of black holes and the resulting amplitude is undetectably small.

Narded phase can significantly reduce the frequency, but at the cost of also lowering the density of gravitational waves. This makes it possible to produce spectra in the LIGO frequency band through mergers, but the corresponding $\Omega_{GW}$ is many orders of magnitude below even the most optimistic sensitivities, and requires $\beta$ to be on the order of 0.1, or greater.

Figure 3.4 allows us to guess at the behavior when multiple generations of mergers take place. If we imagine that the first generation of mergers completes rapidly, we can treat the resulting population of black holes as being approximately coeval and simply iterate our calculation. As Figure 4 shows, the radiation from the prior generations is heavily redshifted and will contribute a very small fraction to the final radiation density. Ultimately, this process gets truncated by the expansion of the Universe during the matter dominated era, which dilutes the density of PBH (as described previously).
Finally, figure 3.5 shows the present-day value of $\Omega_{GW}$ as a function of $\beta$ and $E_i$. For scenarios where $t_H > t_c$ we can estimate $\Omega_{GW}$ with reasonable confidence; but as noted above when $t_H < t_c$ we make a guesstimate based on an assumption that mergers continue until $t = t_c$ and ignore secondary mergers. In these scenarios, our assumptions imply that a larger value of $\beta$ leads to a smaller present-day value of $\Omega_{GW}$; in these cases the total production of gravitational waves is similar, but when the mergers complete relatively early in the matter dominated phase, the subsequent dilution of the radiation component is greater. The integrated background of gravitational waves can, in principle, be detected as an additional radiation component. These are often parametrized as a fractional increase in the number of effective neutrino species, with $\Delta N_{\text{eff}} = 1$ corresponding to $\Omega_{GW} h^2 = 5.6 \times 10^{-6}$ today\textsuperscript{124}. However, as is clear from this figure, even the most optimistic scenarios would not be detectable today.

3.5 DISCUSSION

We have analyzed primordial black hole (PBH) mergers and resulting gravitational wave production during a transient matter dominated phase in the primordial Universe, when the matter content of the Universe predominantly consists of PBH. We estimate the merger rate during this phase, and for some parameter values find that there will be numerous interactions between the PBH. These mergers generate gravitational waves, which will survive through to the present day. However, the characteristic frequency of these gravitational waves is typically huge – well above $10^{10}$ Hz in the present epoch for most parameter choices. In fact, the only candidate gravitational wave background with a higher frequency are the Hawking radiated gravitons produced as these black holes decay, which was described in Anantua et al.\textsuperscript{84}. However, the Hawking radiated background achieves higher values of $\Omega_{GW}$ than the merger-generated gravitational waves, but peaks at higher
Figure 3.5: Maximum value of $\Omega_{GW}$ today, for parameter choices with which the Universe undergoes a matter dominated phase and are consistent with other broad physical constraints. After redshifting, the maximal amplitude is found for a parameter choice for which $t_c$ is approximately equal to $t_H$, around and below the dashed line. Although the upper region has a majority of black holes merging, it contributes less to $\Omega_{GW}$ because the additional redshifting from completion of merger to Hawking evaporation is significant.
frequencies and falls off much more quickly than the merger background to the left of its peak. The background from mergers can thus dominate at lower frequencies, but the present-day density of gravitational waves is very low in this regime.

Our treatment makes a number of strong assumptions, so our results should be seen as an order of magnitude estimate rather than a detailed prediction, but this is adequate for present purposes. In particular, we assume that the initial PBH population has a monochromatic mass function (where $M = M_{\text{Horizon}}$) at time $t_0$ and that the initial spatial distribution of black holes is uniform and randomly distributed. Raidal et al.\textsuperscript{125} maintains the assumption of uniformity while accounting for the possibility of a log-normal mass distribution for a younger population of PBH. Naively, this skews the average mass to a slightly higher number, corresponding to lower $E_i$ values and thus leading to somewhat lower $\Omega_{GW}$ values for our spectra. In addition, while we include the impact of radiation accretion on the black hole mass, we do not allow for the additional dissipation induced as rapidly moving black holes encounter a “headwind” of radiation. We also assume that the resulting gravitational waves are radiated monochromatically when computing the spectrum, but the continuous merger processes in combination with the large growth that occurs during the matter dominated softens the impact of this assumption.

The major omission in our analysis is the impact of clustering, both at the point of formation, which will depend on the detailed physical mechanism associated with PBH formation\textsuperscript{126–130}, and as a result of large scale clustering which grows during the matter dominated phase. In scenarios where only a small fraction of the total black hole population undergoes mergers, our treatment is likely to be sufficiently robust if the assumption of a uniform, random initial distribution is reasonable. However, in scenarios where the typical coalescence time is less than the Hawking time our formalism predicts that all black holes will undergo one merger.
but discounts the possibility of multiple mergers. It is worth emphasizing that our calculated backgrounds are a number of orders of magnitude away from the expected detection limits. Furthermore, when merging is efficient, we observe that the gravitational radiation from early mergers is redshifted away. This suggests that clustering is unlikely to qualitatively change this picture.

The key result of this paper is to demonstrate that any transient PBH dominated phase in the early Universe is likely to support mergers. In almost all cases, while these mergers further enrich the possible phenomenology of the “primordial dark age” they are unlikely to prolong this PBH dominated phase to the point that it would disrupt nucleosynthesis and thus cannot further constrain the PBH populations or the underlying mechanisms that gives rises them, with the caveat that the full N-body dynamics of this epoch are still to be explored. That said, these mergers would generate a stochastic background of gravitational waves that can presumably survive through to the present day. However, these gravitational waves typically have very high frequencies, and would be subdominant relative to the Hawking radiated gravitational wave background produced as the PBH decay.

3.6 Acknowledgments

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Schrödinger-Poisson Solitons: Perturbation Theory

4.1 Introduction

Standard Lambda Cold Dark Matter (ΛCDM) cosmology successfully describes structure formation on large scales; however, it does not necessarily account for observations on galactic and subgalactic scales. For example, CDM N-body simulations predict dark matter halos with a central “cusp” while many observed galaxy rotation curves are better described by “cored” profiles with roughly constant cen-
tral densities\textsuperscript{21,131,132}. Likewise, CDM simulations yield more subhalos than are expected from the observed numbers of dwarf galaxies, leading to the so-called “missing satellite” problem\textsuperscript{27,28,133}. Such discrepancies may be attributable to baryonic processes or even non-Newtonian dynamics\textsuperscript{134}, but may also be resolved by dark matter scenarios whose properties differ from those of simple CDM.

One such candidate is UltraLight Dark Matter (ULDM), also known as Fuzzy Dark Matter (FDM). Consisting of an axion-like boson with a mass between \(10^{-23}\) to \(10^{-20}\) eV, structure formation in ULDM scenarios is suppressed on scales smaller than the corresponding de Broglie wavelength of up to a few kiloparsecs\textsuperscript{135}. ULDM can coalesce into a Bose-Einstein condensate (BEC) whose behavior is described by a macroscopic wavefunction\textsuperscript{43,136–138} governed by the coupled equations of the Schrödinger-Poisson system. The ground state solution of this system is a soliton, but the astrophysical dynamics of halo formation lead to configurations with a solitonic core embedded in a Navarro-Frenk-White (NFW) “skirt”\textsuperscript{42}.

Structure formation with ULDM reproduces the successes of ΛCDM on large scales while producing cored halos and substructure that are potentially more consistent with observations on small scales\textsuperscript{43,68,139}. In addition to dark matter, the Schrödinger-Poisson system of equations governing ULDM dynamics emerges in other systems of interest, including boson stars\textsuperscript{140–143} and the very early universe\textsuperscript{70,144,145}. This motivated our study of the dynamics of the Schrödinger-Poisson system.

While the ground state of the Schrödinger-Poisson system is well studied, in most astrophysical systems, one would expect the excited states to be just as relevant as the ground state, given that the “NFW skirt” of a ULDM halo must be built up of excited states\textsuperscript{81,146–148}. However, the gravitational coupling makes the system nonlinear in the wavefunction making it challenging to explore the excited states of this system, and most analyses have relied on directly simulating the full
system.

As was pointed out in Li et al.\textsuperscript{149}, in the limit that the density of the system is approximately constant in time, one can avoid the complications of the full system and solve the Schrödinger equation alone, treating the fluctuations in the density as perturbations. This is further helped by the fact the mapping from density to gravitational potential is a smoothing operation, and therefore naturally reduces the impact of small scale fluctuations. This paper aims to develop this idea, primarily focusing on the perturbations to the soliton as a toy example. This work is a natural continuation of the results presented in Li et al.\textsuperscript{149}, although there have been a number of other explorations of perturbations in this system, eg.\textsuperscript{150–153}.

Throughout this paper, we present numerical results from a pseudo-spectral solver of the full Schroödinger-Poisson system, CHPLULTRA. We developed CHPLULTRA based on the algorithm of PYULTRALIGHT: a sibling code whose specifics are discussed in detail in Edwards et al.\textsuperscript{76}. One detail in which the PYULTRALIGHT and CHPLULTRA diverge is the algorithm used for computing the potential; while PYULTRALIGHT uses Fourier transforms and periodic boundary conditions, CHPLULTRA utilizes a Green’s function approach that allows for isolated boundary conditions. This difference, along with the implementation of CHPLULTRA is explained in detail in Padmanabhan\textsuperscript{154}. Additionally, details of CHPLULTRA and our code units are summarized in Appendix 2.4.3.

The rest of our paper is organized as follows. We review the construction of the relevant eigenstates in Section 4.2, paying attention to the impact of the boundary conditions on our results. Section 4.3 starts by demonstrating that perturbing a soliton by these eigenstates can qualitatively reproduce many of the results seen in full ULDM simulations. It then continues to show that the time evolution of these perturbations in the full system can be accurately captured by a simple perturbative calculation. In Section 4.4 we consider a more realistic case, and decompose
a ULDM halo into its eigenstates and track their evolution. Finally, we discuss our results in Section 4.5.

4.2 ULDM Eigenstates

4.2.1 Eigenfunction Expansion

We will be solving the Schrödinger-Poisson system,

\[-i\hbar \frac{\partial}{\partial t} \psi = \left[ -\frac{\hbar^2}{2m_a} \nabla^2 + m_a \Phi \right] \psi \quad (4.1)\]

\[\nabla^2 \Phi = 4\pi G m_a \rho \quad (4.2)\]

where \(\psi\) is the ULDM wavefunction, with \(\rho = |\psi|^2\) as the corresponding density and \(\Phi\) as the gravitational potential. In what follows, we work in units of \(m_a = \hbar = G = c = 1\), where \(m_a\) is the mass of the particle. The mapping from natural to physical units is given in Appendix 2.4.3.

The Schrödinger equation is linear but the gravitational interaction introduces a nonlinear dependence on \(\psi\), rendering the system substantially more challenging to solve. However, in many systems of interest the potential is approximately constant, especially when averaged in time and over small-scale fluctuations. This suggests the approximation

\[-i\frac{\partial}{\partial t} \psi = \left[ -\frac{1}{2} \nabla^2 + \langle \Phi \rangle \right] \psi \quad (4.3)\]

where \(\langle \Phi \rangle\) is an averaged gravitational potential that is assumed to be constant in time.
We expand the ULDM wavefunction at $t = 0$ as

$$\psi(t = 0) = \sum_{i=1}^{N} c_i \varphi_i$$  \hspace{1cm} (4.4)

where the $c_i$ are complex expansion coefficients, $\varphi_i$ are the system’s eigenstates, and $N$ is a finite truncation of the basis. If the $\varphi_i$ are assumed to be orthonormal we can project out their weights

$$c_i = \int d^3r \psi^*(r) \varphi_i^*(r)$$  \hspace{1cm} (4.5)

where the integral is over all space. If we ignore the backreaction on the potential, the wavefunction evolves via

$$\psi(t) = \sum_{i=1}^{N} c_i \exp(-iE_i t) \varphi_i$$  \hspace{1cm} (4.6)

where $E_i$ is the eigenenergy associated with state $i$.

4.2.2 CONSTRUCTION OF EIGENSTATES

There is substantial literature on solving the Schrödinger-Poisson (or Schrödinger-Newton) eigensystem. However, since we have assumed that $\Phi$ is constant we are effectively determining eigenstates of the Schrödinger equation, without the additional coupling to the Poisson equation. Furthermore, we restrict our attention to spherically symmetric potentials but allow the perturbations to break spherical symmetry.

With these assumptions we can separate variables so that the eigenstates are each products of a radial and an angular component: $\varphi_{n\ell m} = f_{n\ell}(r)Y_{\ell m}(\theta, \varphi)$. Re-
Figure 4.1: We illustrate the radial profiles $f_{n\ell}$ of the ULDM eigenstates for $n \leq 3$, $\ell \leq 2$. Recall that the $n$-index corresponds to the number of nodes in the state, with the energy of the eigenstate increasing with $n$. The $\ell$-index corresponds to the angular variation of the wavefunction (given the appropriate $Y_{\ell m}$); recall that $f(r) \sim r^\ell$ as $r \to 0$. The $n = 0$ states are colored blue, the $n = 1$ states are yellow, the $n = 2$ states are green, and the $n = 3$ states are red. We keep to this convention whenever possible throughout the paper for continuity and clarity. All data is shown in internal code units.

arranging Eq. 4.3 and dividing through by $Y_{\ell m}$, we arrive at

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f_{n\ell}}{\partial r} \right) + \frac{\ell (\ell + 1)}{r^2} f_{n\ell} = 2(\langle \Phi \rangle - E_{n\ell}) f_{n\ell}, \quad (4.7)$$

where $E_{n\ell}$ is the eigenvalue of eigenstate $f$. The substitution $u_{n\ell} = rf_{n\ell}$ transforms the above equation into

$$\frac{\partial^2 u_{n\ell}}{\partial r^2} + \frac{\ell (\ell + 1)}{r^2} u_{n\ell}(r) - 2 \langle \Phi(r) \rangle u_{n\ell}(r) = -2 E_{n\ell} u_{n\ell}(r). \quad (4.8)$$

We now have a formulation of the Schrödinger equation that can be solved for a given spherical static potential $\langle \Phi \rangle$. We discretize our variables into vectors of
length \( N \) and our operators into \( N \)-by-\( N \) matrices over a distance \( r < r_{\text{max}} \) with a grid spacing \( \Delta r = r_{\text{max}}/N \). The differential equation then becomes the matrix eigenvalue problem

\[
\mathcal{H} = \begin{bmatrix}
\chi(r_1) & \cdots & \cdots & \cdots & 0 \\
0 & \chi(r_2) & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \chi(r_{n-1}) & 0 \\
0 & \cdots & 0 & 0 & \chi(r_n)
\end{bmatrix} - \frac{1}{\Delta r^2} \begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & -2 & 1 \\
0 & \cdots & 0 & 1 & -2
\end{bmatrix}
\]

(4.9)

where \( \chi(r) \equiv 2\langle \Phi \rangle(r) - l(l+1)/r^2 \) is the gravitational potential and centrifugal barrier. This can be solved numerically, with \( f_{nl} = u_{nl}/r \) being the radial component of a given eigenstate and \( E_{nl} \) its eigenenergy.

The boundary conditions must be specified to ensure we have a unique solution. The definition of \( u_{nl} \) and the requirement that the wavefunction is finite at \( r = 0 \) implies that \( u_{nl} = 0 \) at \( r = 0 \). We also assume the \( u_{nl} = f_{nl} = 0 \) at \( r_{\text{max}} \). Physically, this corresponds to embedding the system in a spherically symmetric infinite well. We clarify the implications of this choice below. Both boundary conditions are built into the matrix equation above. This outer boundary condition is not the natural choice in a pseudo-spectral code with periodic boundary conditions on a cubic spatial lattice (such as CHPLULTRA), but it is easily implemented by setting the wavefunction to zero outside of \( r_{\text{max}} \).

We solve the matrix equation for a static potential \( \langle \Phi \rangle \) corresponding to an un-
perturbed soliton of mass $M = 50$ in code units. The radial $f_{n\ell}$ states that follow from this choice are illustrated in Fig. 4.1. The $n$-index matches the number of nodes: $n = 0$ states have no nodes, $n = 1$ states have one node, and so on. The $\ell$-index is recognizable in the behavior of the function as $r \to 0$: each state asymptotes to a slope of $r^\ell$, such that the $\ell = 0$ state has a central core and higher $\ell$-states fall off more quickly.

### 4.2.3 Parameter Dependence of Eigenstates

<table>
<thead>
<tr>
<th>$r_{\text{max}}$</th>
<th>$\langle \Phi(r_{\text{max}}) \rangle$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
<th>$n = 10$</th>
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<td>-56.17</td>
<td>-21.70</td>
<td>24.83</td>
<td>83.39</td>
<td>153.2</td>
<td>233.7</td>
<td>324.7</td>
<td>425.9</td>
</tr>
<tr>
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<td>-406.9</td>
<td>-175.5</td>
<td>-93.91</td>
<td>-58.05</td>
<td>-39.34</td>
<td>-27.83</td>
<td>-16.63</td>
<td>-2.185</td>
<td>15.56</td>
<td>36.36</td>
<td>60.04</td>
</tr>
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<td>-406.9</td>
<td>-175.5</td>
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<td>-16.77</td>
<td>-13.21</td>
<td>-9.43</td>
<td>-4.75</td>
</tr>
</tbody>
</table>

Table 4.1: Calculated eigenenergy values in code units for different values of $r_{\text{max}}$ and $\ell = 0$. The cells where eigenenergies begin exhibiting $O(1)$ differences from the higher $r_{\text{max}}$ values are **bolded**. Note that these also correspond to the appropriate values of the potential at $r_{\text{max}}, \langle \Phi(r_{\text{max}}) \rangle$. Thus, comparing the potential at $r_{\text{max}}$ and the derived eigenenergies is a relatively easy way to determine the eigenstates affected by the boundary condition at $r_{\text{max}}$.

When discretizing the Schrödinger equation (and subsequently our eigenstates) we made two independent choices: the grid spacing $\Delta r$, and the outer boundary, $r_{\text{max}}$. Provided $\Delta r$ is small enough to adequately resolve the full width at half maximum (FWHM) of the central soliton, $r_c$, its value does not affect the results of the calculation. We use $\Delta r \approx r_c/25$ throughout.

On the other hand, the value of $r_{\text{max}}$ qualitatively impacts the eigenstates. Requiring that the wavefunction vanishes beyond this radius is physically equivalent to putting the entire system into an infinite spherical well of radius $r_{\text{max}}$. So long as the radial extent of the eigenfunction is much smaller than $r_{\text{max}}$ the boundary does not affect our results, but the modes are affected when the scales overlap.

To gain some intuition, let us consider a state with $n$ nodes would fit comfortably into a sphere of some radius $r_{\text{max}}$. Higher-order states with more than $n$ nodes,
then, can only obey the boundary conditions of the same sphere if its nodes are pushed together further than would be the case without the barrier at $r_{\text{max}}$. The more nodes a state has, the more it is distorted by a boundary at $r_{\text{max}}$.†

Table 4.1 shows eigenenergies for spherically symmetric perturbations ($\ell = 0$) for $n \leq 10$ and $1 < r_{\text{max}} < 10$. For $n < 2$ these are identical; at $n = 3$, we see $O(1)$ differences when $r_{\text{max}} = 1$. With $n = 5$ we need $r_{\text{max}} > 2$ and at $n = 8$ we need $r_{\text{max}} > 4$ for the eigenenergies to be independent of $r_{\text{max}}$. Physically, eigenenergies are independent of $r_{\text{max}}$ when they do not exceed the (unperturbed) gravitational potential at $r_{\text{max}}$.

In realistic astrophysical systems, however, $r_{\text{max}} \to \infty$. We defer a detailed treatment to future work, but note that for large $r_{\text{max}}$, the eigenenergies scale as $E_n \sim -1/n^2$, as expected from a hydrogen-like system, until the effect of the spherical well becomes apparent. This implies a large number of states with relatively small energy splittings near $E \sim 0$. It is thus possible to excite many of these states as $r_{\text{max}} \to \infty$ to similar levels, which could have implications for the relaxation of perturbed solitons to the ground state.

### 4.3 Perturbed Solitons

We expand the wavefunction as $\psi = \sum_{n,\ell, m} f_{n,\ell}(r) Y_{\ell}^m(\theta, \phi)$ and now explore the time evolution of these states, focusing on perturbations to the gravitational potential that arise as the system evolves. In what follows we fix $m = 0$, preserving azimuthal symmetry (although our methods apply to the general case), and write the eigenvectors as $|n, \ell\rangle$. We focus on perturbing the soliton ground state, $\psi_{\text{sol}} = f_0(r) Y_0^0(\theta, \phi)$ (or $|n, \ell\rangle = |00\rangle$) with excited states. We construct the eigenstate basis using a gravitational potential with a mass $M = 50$ and normalize the eigenstates to unit mass.

†In this work, we consider idealized simulations of a single perturbed soliton or isolated halo in
Figure 4.2: An illustration of how the mass density in the plane is perturbed when combining the soliton ground state with excited states. The perturbations are as follows: row 1 has 30\% of its mass in the first $\ell = 0$ excited state, row 2 in the first $\ell = 1$ excited state, and row 3 in the first $\ell = 2$ excited state. Each column represents a time that is defined with respect to the state’s period, $T = 2\pi / \Delta E$. The contours are spaced logarithmically, from $10^{-4}$ to $10^{3}$ in code density units and are kept constant along each row.

4.3.1 QUALITATIVE BEHAVIOR

We begin with snapshots of three different systems in which a soliton is perturbed by $|10\rangle$, $|01\rangle$, and $|02\rangle$, shown in Fig. 4.2. In order to illustrate the qualitative behavior of the system, we apply substantial perturbations which induce visible oscillations. In each case the ground state contributes 70\% of the mass density, and the excited $\ell = 0, 1, 2$ states make up the remaining 30\%. Each system is shown at times $t = 0 T$, $0.15 T$, $0.30 T$, and $t = 0.45 T$ where $T = 2\pi / \Delta E$ is the period of oscillation set by the difference in eigenenergies of the ground state and each a box, beyond which space is empty, so the wavefunction $\psi$ is effectively zero beyond the boundary.
perturber.

The top row of Fig. 4.2 shows the consequence of adding an $\ell = 0$ excited state. This causes the soliton to contract and collapse, revealing the so-called “breathing mode” that has been noted in ULDM simulations. The $\ell = 1$ mode (middle row) results in the peak of the soliton moving back and forth, in line with Schive et al. and Dutta Chowdhury et al., who found that a soliton in a ULDM halo performs a random walk. Finally, an $\ell = 2$ term (bottom row) results in a quadrupole oscillation, where in the density is elongated first in one direction and then in the perpendicular direction. These examples illustrate how the phenomenology of ULDM systems overlaps with the eigenstate description, in agreement with Li et al.

4.3.2 Solitons with Spherically Symmetric Perturbations

We start by examining spherically symmetric systems ($\ell = 0$) whose initial wavefunction is given by

$$\psi(t = 0) = \sqrt{M} (|0\rangle + \varepsilon |n\rangle) ,$$

(4.11)

where we have suppressed the $\ell, m$ indices on the kets for brevity. The unperturbed mass of the system is $M$ while the perturbation increases the mass by $\varepsilon^2 M$ since we are perturbing the wave function and the density scales as $|\psi|^2$.

Fig. 4.3 shows the evolution of the first excited state ($n = 1$) with $\varepsilon = 0.05$, drawn from a solution of the full Schrödinger-Poisson system. We decompose the full wavefunction into the eigenstate basis $\psi(t) = \sum c_n(t) |n\rangle$ and plot the magnitudes of the $c_n$ with time. For small perturbations, the amplitude $c_0$ of the ground state will remain constant, and this is true in practice to better than 0.1% for this scenario. Mode-coupling in the full nonlinear system excites the $|2\rangle$ and $|3\rangle$ modes.
Figure 4.3: The evolution of a $M = 50$ soliton wavefunction, perturbed by the first $\ell = 0$ excited state $|1\rangle$ with amplitude $\varepsilon = 5\%$ and expanded into the eigenstate basis. The figure shows the magnitudes of these expansion coefficients (normalized by $\sqrt{M}$ for the excited states) as a function of time. The prominently displayed curves are the amplitudes of the $|1\rangle$, $|2\rangle$, and $|3\rangle$ states (from top to bottom), while the other curves show the next 21 eigenstates. The horizontal dashed line shows the initial amplitude $c_1(t = 0) = 0.05$. The inset shows the same system evolved to a later time, plotted with a lower time resolution. Also shown is the evolution of the amplitude of the ground state which remains at its initial value of 1 to better than 0.1%.

to significant amplitudes, relative to the original perturbation, as it evolves.

The eigenstate expansion does not account for the gravitational couplings between modes. To do so, we extend our expansion to the interaction picture,

$$\psi(t) = \sum_{n=1}^{N} c_n(t) \exp(-iE_n t) |n\rangle. \quad (4.12)$$

where our expansion coefficients $c_n$ (which are in general complex) are now time dependent. The evolving eigenstates will perturb the potential $\Phi \rightarrow \Phi_0 + \Delta \Phi(t)$, where $\Phi_0$ is the gravitational potential of the fiducial, ground state profile. The Schrödinger equation then reduces to a set of coupled differential equations for
Figure 4.4: The time evolution of the amplitudes (from top to bottom) of the \( |n = 1⟩, |n = 2⟩ \) and \( |n = 3⟩ \) eigenstates, compared to a perturbative calculation. The brighter color lines show the evolution of states with an initial perturbation proportional to \( |n = 1⟩ \), while the lighter lines show the \( |n = 2⟩ \) case. Perturbative predictions are dashed and dotted for \( |n = 1⟩ \) and \( |n = 2⟩ \) respectively. In the absence of nonlinear couplings due to gravity, the amplitudes would remain at their initial values of 0.05 and 0.

This equation is nominally exact, but also gives a framework with which to approximate the evolution of this system. To determine \( \Delta \Phi \) we first compute the perturbations to the density profile,

\[
\Delta \rho = |\psi|^2 - |\psi_o|^2 
\]

\[
\approx \sum_{p=1}^{N} 2 \Re \left[ c_0(t)c_p(t)\ast |0⟩ |p⟩ e^{i(E_p-E_0)t} \right] 
\]

(4.13)
where we drop terms below leading order in $|c_n|$ for $n > 0$. If we define $\Delta \Phi_{0p}$ as the gravitational potential that results from a density profile $2|0\rangle|p\rangle$, then Eq. 4.13 can be written as

$$\frac{dc_n}{dt} = -i \sum_{p=0}^{N} \sum_{k=0}^{N} \langle n|\Delta \Phi_{0p}|k\rangle \text{Re} \left[ c_0(t)c_p(t)^* e^{i(E_p-E_0)t} \right] \times c_k(t)e^{-i(E_k-E_0)t}.$$ 

(4.16)

This equation must be slightly modified for $p = 0$ to avoid double counting and including the unperturbed solution, but we elide this here for simplicity. We tested the evolving the perturbation equations holding $c_0$ fixed (i.e. ignoring the $p = 0$ term) and we find that this makes no difference to our results.

Fig. 4.4 shows the evolution following initial perturbations of $\varepsilon |1\rangle$ and $\varepsilon |2\rangle$, with $\varepsilon = 0.05$. In the absence of mode coupling $|1\rangle$, $|2\rangle$ and $|3\rangle$ would stay at their initial values. We find the perturbative treatment gives a close match to the weights extracted from solutions to the full equations of motion. The discrepancy between the approximation and the full solution grows (albeit slowly) with time.

We expect the match between the perturbative calculation and the full system to improve as the initial amplitude is decreased. Fig. 4.5 demonstrates the expected scaling, between the simulations; a 10% perturbation diverges relatively quickly from the full solution, but a 1% perturbation tracks relatively well through multiple oscillations. As we perturb the soliton with higher energy ($n$) states, we observe that the time-dependence of the resulting $c_n$ amplitude decreases. The amplitude of the 5th excited state is constant to within 0.3%, whilst the 15th excited state varies by 0.02%. It appears that the more rapid fluctuations in both space and time (higher eigenstates oscillate more rapidly as a function of radius and time) average

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‡ Since our Hamiltonian is real and symmetric, it is possible to choose our eigenstates to be completely real. We therefore do not need to consider the complex conjugate of the eigenstates. We also note the non-standard notation $|a\rangle\langle b| \equiv \psi_a^\dagger \psi_b$ for the simple product of eigenfunctions.

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out variations in the potential, reducing the coupling matrix elements and keeping $c_n$ constant in time. This suggests that even when density profiles are composed of many eigenstates, the lowest order modes dominate the resulting gravitational couplings and will drive deviations from the simple eigenstate evolution.

Examining Eq. 4.16, we see that the dominant corrections to a state $|n\rangle$ come from its coupling to the ground state through the potential perturbation, corresponding to the $k = 0$ terms. One might expect that these couplings to be further suppressed by the rapidly oscillating exponentials (due to the energy differences). Given this, the largest contribution to the change in $c_n$ comes from the $p = n$ terms. This qualitatively explains why the ground state does not see corrections of order $\varepsilon$, but the perturber does, as shown Fig. 4.5.

![Figure 4.5: The time evolution of the amplitudes of the $|n = 1\rangle$ eigenstates compared to a perturbative calculation for different initial amplitudes. The full yellow lines show the evolution of the states when the soliton is perturbed by $|n = 1\rangle$ with an initial amplitude of $\varepsilon = 0.01$, $\varepsilon = 0.05$, $\varepsilon = 0.1$, respectively. The lighter yellow lines show a scaled version of the $\varepsilon = 0.05$ simulation. The corresponding perturbative calculations are shown in dashed lines. Note that the $\varepsilon = 0.01$ figure has a small range in amplitude, so any divergence between the simulation and perturbation theory is visually amplified and that the perturbative calculation does not match the full system at late times for $\varepsilon \geq 0.1$.](image-url)
4.3.3 Solitons with Aspherical Perturbations

Next, we turn to full 3D simulations consisting of a single soliton with a nonzero $\ell$-perturbation. Similar to the spherically symmetric systems, we consider the case

$$\psi(t = 0) = \sqrt{M} (|00\rangle + \varepsilon |n \ell\rangle), \quad (4.17)$$

where we restore the $\ell$ indices to our kets.\(^5\) We use $|n 1\rangle$ and $|n 2\rangle$ as perturbers for the discussion below, but our conclusions hold for states with higher $\ell$. We decompose the resulting wavefunctions into eigenstates at each saved timestep. We start by plotting total $\ell$ mode coefficients $|C_\ell|^2 \equiv \sum_n |c_{n,\ell}|^2$ in Fig. 4.6. As with the radial perturbation in Fig. 4.4 above, the soliton amplitude remains the mostly constant dominant component, while each total $\ell$-mode oscillates about a constant amplitude. The figure shows the mixing between the $\ell$ modes and demonstrates that, to leading order, the $\ell$ modes remain independent of each other. We show that this follows directly from the perturbative treatment below.

In the case where $\ell = 1$ is the initial perturbation of $\sim 5\%$ in the wavefunction, its $|C_\ell|^2$ value oscillates around just above $(5\%)^2 = 0.25\%$, while each subsequent total $\ell$-mode is excited to a progressively smaller amplitude. When $\ell = 2$ is the initial perturbation, each subsequent even value of $\ell$ is excited to a smaller and smaller amplitude, while the odd $\ell$-coefficients are only excited at the level of noise in the simulation box.

We compare these findings with the case where we perturb solitons using the same modes, but at a larger amplitude of $\varepsilon = 0.25$. The dominant $\ell$ modes behave almost exactly the same as in the case of a 5% perturbation, except that they oscillate around higher amplitudes. On the other hand, by inspecting the higher $\ell$ behavior we see how the larger perturbation amplitude results in a more pronounced

\(^5\)We continue to set $m = 0$.\)
Figure 4.6: The figure illustrates $|C_\ell|^2$ evolution as a function of time in the case of a soliton perturbed by a single non-radially symmetric ($\ell > 0$) state. The upper panel illustrates this evolution in the case where the soliton is perturbed by $|0 1\rangle$ with initial amplitudes of $(5\%)^2$ (full lines) and $(25\%)^2$ (dashed lines). The lower panel illustrates the equivalent $|0 2\rangle$ case. The $\ell$-indices correspond to colors as indicated in the legend at the bottom of the lower panel. Note the $\ell = 1$ and $\ell = 3$ states in the case of an $\ell = 2$ perturbed (lower panel) are at the noise-floor in the numerical box.

coupling to the higher $\ell$ modes, raising these from noise floor.

Our perturbative treatment from the previous section can be extended to the nonspherical case. As before, we sum over states, except that these now run over
both $\ell$ and $n$, instead of just $n$. We then have
\[
\frac{dc_{n_1\ell_1}}{dt} = -i \sum_{\ell_2, \ell_3=0}^{L} \sum_{n_2, n_3=1}^{N} \langle n_1 \ell_1 | \Delta \Phi_{n_2 \ell_2} | n_3 \ell_3 \rangle \\
\times \Re \left[ c_{00}(t)c_{n_2 \ell_2}(t)^* e^{i(E_2 - E_0)t} \right] \\
\times c_{n_3 \ell_3}(t) e^{-i(E_3 - E_1)t}
\]
(4.18)

where $E_0$ is shorthand the eigenenergy of the unperturbed soliton, $E_1 = E_{n_1 \ell_1}$, $E_2 = E_{n_2 \ell_2}$, and $E_3 = E_{n_3 \ell_3}$ and $L, N$ are the highest $n$ and $\ell$-states we track. As before, we approximate the potential perturbations by considering density fluctuations that arise from the combination of the ground state with an excited state. While the above appears cumbersome, it is identical in structure to the $\ell = 0$ case we considered previously. The only new feature comes from the angular terms in the matrix element, arising from integrating over the product of three spherical harmonics. Appendix A presents the details of this calculation.

Even without solving these equations, we can recover the qualitative behavior seen in Fig. 4.6. If we work to the lowest nontrivial order in the perturbation, we see that $n_3$ and $\ell_3$ must both be zero, i.e. $|n_3 \ell_3\rangle$ is the ground state. Considering the product of the three spherical harmonics represented by the matrix element $\langle n_1 \ell_1 | \Delta \Phi_{n_2 \ell_2} | n_3 = 0 \ell_3 = 0 \rangle$, we see that $\ell_1 = \ell_2$ for a nonvanishing matrix element at lowest order. Physically, this means that perturbations mix radial eigenstates, but remain at the same angular eigenstate, which is exactly the behavior seen in the figure. However, this is only true at lowest order—with larger perturbations there is mixing across angular modes.

We now proceed by integrating the differential equations as in the previous subsection. The results for perturbing by $|01\rangle$ and $|02\rangle$ are shown in Fig. 4.7, and for perturbing by $|11\rangle$ and $|12\rangle$ are shown in Fig. 4.8. In each of the cases solv-

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\*\*The perturbative results for the figures in this manuscript were produced with $N = 25$, $L = 3$.\*\*
**Figure 4.7**: The time evolution of the amplitudes (from top to bottom) of the $|0 \ell \rangle$, $|1 \ell \rangle$, $|2 \ell \rangle$ and $|3 \ell \rangle$ eigenstates for $\ell = 1, 2$ compared to a perturbative calculation. The brighter color lines show the evolution of the states when the soliton is perturbed by $|0 1 \rangle$, while the lighter color lines show a perturbation by $|0 2 \rangle$. The perturbative calculations are shown in dashed (for a $|0 1 \rangle$ perturber) and dotted (for a $|0 2 \rangle$ perturber) lines.

ing Eq. 4.18 accurately matches the evolution of the full system. The perturbative calculation is most accurate for lowest-$n$ states, while at late times higher-$n$ state calculations begin to diverge from simulation data, as is particularly evident in the bottom row of Fig. 4.8. We have also verified that the behavior of the system is well captured in the case of a $|0 3 \rangle$ perturber, while $|0 4 \rangle$ and $|0 5 \rangle$ perturbers’ values remain constant to better that 0.1%, at which level our simulation is subject to noise.

In general, Figs. 4.7 and 4.8 show good agreement between the simulations and our perturbative calculations. However, one notable divergence is visible in the top row of Fig. 4.8 for the soliton perturbed by $|0 1 \rangle$. This highlights a subtlety with our perturbative approach for odd $\ell$ perturbations due to momentum conservation. The velocity is determined by $d\theta/dx_i$ where $\theta$ is the phase of the wavefunction.
and $x_i$ is a coordinate direction. Consider now a perturbed wavefunction of the form $|00\rangle + c |n, \text{odd } \ell \rangle$, where $c$ is the relative complex amplitude of the perturbation relative to the ground state. If $c$ has a non-zero imaginary component, the above wavefunction will have a spatially varying phase since the two eigenstates have different shapes. That, combined with the antisymmetric nature of the odd $\ell$ spherical harmonics, means that the system will have non-zero overall momentum. For even $\ell$ values, the phase will again be spatially varying, but the net momentum will be zero.

![Graph](image)

**Figure 4.8:** The same as Fig. 4.7, but with the soliton perturbed by $|1,1\rangle$ (darker/dashed) and $|1,2\rangle$ (lighter/dotted).

However, the eigenstate expansion does not explicitly conserve the linear momentum of the system. Structurally, the eigenstate expansion is not translationally invariant and therefore does not have linear momentum as a conserved quantity.\(^\text{**}\)

We can also see this by considering the time evolution of the perturbed wavefunc-

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\(^1\)See the Madelung representation of this problem as discussed in eg. Hui et al.\(^14\) and Hui\(^68\).

\(^\text{**}\)By comparison, the eigenstates and perturbation theory are rotationally invariant, and so angular momentum is explicitly conserved.
tion considered above,

$$\psi(t) = e^{-iE_0 t} \left( |00\rangle + c e^{-i(E_n - E_0) t} |n, \text{odd} \ell\rangle \right). \quad (4.19)$$

Even if the imaginary part of c is zero at $t = 0$, the perturbation develops a nonzero relative phase at a later time, and the system does develop a nonzero momentum (although with a zero time average value). Interestingly, in our simulations, the relative phase of the $|10\rangle$ term with the ground state remains constant at approximately zero, consistent with a vanishing momentum.

While the above suggests an underlying structural problem with any odd $\ell$ mode, Figs. 4.7 and 4.8 show that significant discrepancies are only evident for the lowest energy $\ell = 1$ state. We attribute this to the fact that this mode generates the largest coherent momentum of the system. Higher energy modes have multiple nodes resulting in reversals of the velocity direction and higher $\ell$ modes result in a less coherent motion, and therefore a smaller net linear momentum. Furthermore, while the perturbative theory generically permits coupling across $\ell$ modes, this is not allowed at the lowest order as discussed above. Therefore, even $\ell$ modes do not excite the $|01\rangle$ mode, maintaining good agreement with the perturbative results.

4.4 ULDM HALO

We now investigate the eigenstate decomposition and evolution of a ULDM halo. This system can be treated as a solitonic core with an NFW skirt

$$\rho(r) = \begin{cases} 
\rho_{\text{sol}}(r), & 0 \leq r \leq r_a \\
\rho_{\text{NFW}}(r), & r_a \leq r \leq r_{\text{vir}}.
\end{cases} \quad (4.20)$$

The border between the skirt and the core falls in the range $3r_c \leq r_a \leq 4r_c$, 

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where \( r_c \) is the FWHM of the solitonic core and the exact value of \( r_\alpha \) is determined by setting the mass of the halo \( M_h \) and requiring the profile be continuous. To generate a halo profile that could be described by Eq. 4.20, we use CHPLULTRA to collide 8 randomly placed equal mass solitons\(^{141}\). We then average the resultant late-time profile over 0.9 code time units. See Fig. 4.9 for an illustration of our averaged profile compared with instantaneous profiles at different times, and Fig. 4.10 for the corresponding potentials.

![Graph showing density profile and fluctuations](image)

**Figure 4.9:** The radially and time-averaged (from \( t = 0.1 \) to \( t = 1.0 \) code units) density profile of our ULDM halo is shown in blue. We use the potential corresponding to this profile to calculate our eigenstates. Snapshots of instantaneous density profiles at \( T = 0.2, 0.4, 0.6, 0.8 \) and 1.0 are shown in grayscale (light to dark, respectively). The size of their fluctuations relative to the averaged profile are given in the lower panel. Experimenting with differently time-averaged potentials yielded only small fluctuations in the mass normalization of the resulting eigenstates. All data is shown in code units.

We construct the eigenstates for the potential seeded by the spherically- and time-averaged ULDM density profile. Next, we analyze the 3D simulation of the 8-soliton collision that led to our profile by decomposing it into its constituent
Figure 4.10: The gravitational potentials corresponding to the density profiles in Fig. 4.9. Note that the \( \sim 40\% \) fluctuations in density correspond to \( \sim 10\% \) fluctuations in potentials.

\[ |C_\ell(t)|^2 \] indices.\(^{††}\) The results are shown in Fig. 4.11. At each timestep, the \( |00\rangle \) state accounts for the solitonic core at the center of the halo profile, while a superposition of higher modes results in the NFW skirt. We find that the \( \ell = 0 \) mode dominates, accounting for just over 35% of the simulated mass, with almost the entirety being in the soliton itself (\( \ell = 0, n = 0 \)). Higher \( \ell \)-modes account for the halo’s NFW skirt, with the \( \ell = 1 \) making up about 10% of the wavefunction, albeit with large fluctuations. Modes with \( \ell = 2 \) contributions account for a little more than 8%, while \( \ell = 3 \) and \( \ell = 4 \) terms account for around 6% each. The modes presented in Fig. 4.11 account for \( \sim 67\% \) of the halo’s mass, with the rest being in higher modes.

\(^{††}\)In this section we also sum over \( m \)-modes, as our halo is not axisymmetric and \( m \neq 0 \) modes contribute significantly.
Figure 4.11: The time evolution of $|C_\ell|^2$ in a 3D box with an 8-soliton merger ending in a ULDM profile. The colored lines represent the $\ell$ modes as indicated in the legend and the gray background tracks the evolution of the soliton ($n = 0$, $\ell = 0$, $m = 0$). The vertical line denotes the approximate point of halo formation at time $t = 0.1$. Note that the $\ell = 0$ line dominates throughout the simulation and is almost entirely composed of the ground state soliton, oscillating around 30%. The $\ell = 1$ modes make up around 10% of the halo; the $\ell = 2$ modes make up around 8%; and higher $\ell$-modes account for around 6% of the halo wavefunction each.

As in Fig. 4.6, the mean amplitude of each $|C_\ell|^2$ line is roughly constant—albeit with relatively large excursions—suggesting that mass is primarily exchanged between modes with the same $\ell$ number. Mapping to astrophysically reasonable units, the evolution of the system is shown for approximately 23 Gyrs, the halo mass is $M_h \sim 15 \times 10^8 M_\odot$, and its radius is $r_h \sim 20 \text{kpc}$ (see Table 2.1). We find no signs of the eigenstate decomposition tending towards a perfectly relaxed state over this time period, even though the density profile of the halo appears to be more stable (as shown in Fig. 4.9). It is also possible that this is a result of the artificial construction of this halo, and that the asymmetry in the initial conditions somehow still persists. We plan to explore decompositions for a larger variety of halos in future work.

The relatively large amplitude of non-solitonic modes making up $\sim 70\%$ of this halo suggest that our perturbative approximations cannot be applied as simply as in the case of mildly perturbed solitons. In principle, we could attempt to use
Eq. 4.18 and significantly increase the L, N cutoff values (i.e., keep track of many more modes) to attempt to find an approximate perturbative match to the full solution. Furthermore, since the differential equations for the time dependent perturbation theory are exact, one could imagine exactly evolving the full system (including a complete calculation of the potential) for a truncated basis. This might provide some advantages over the full Schrödinger-Poisson solvers.

4.5 Discussion

In this paper we solved for the eigenstates and eigenenergies of the Schrödinger-Poisson system. We assume that the potential is constant in time, consistent with Li et al.\textsuperscript{149}. Once we obtain the eigenstates of the system, we see phenomena familiar from simulations of ULDM halos. Perturbing the ground state soliton with an $\ell = 0$ component, we recovered the familiar “breathing mode” exhibited by ULDM solitonic cores; $\ell = 1$ perturbations cause the center of the soliton to move in ways reminiscent of the random walk of the core found in simulations\textsuperscript{149,163}, $\ell = 2$ perturbations resulted in a “cross” oscillation pattern characteristic of the quadrupole moment. We examined the dependence of our eigenstates on the size of our outer boundary condition $r_{\text{max}}$ and found that higher excited states can be strongly impacted by this choice, but not by our choice of potential.

We tested the accuracy and utility of our perturbative approximation by comparing it with the evolution of the full non-linear Schrödinger-Poisson system. We began by comparing the evolution of a radially symmetric system, where the ground state was perturbed by the $|100\rangle$ state, which we found to be an excellent match when tracking $N \geq 10$ states in our perturbation theory calculation. Additionally, this remains true when the ground state is perturbed with different higher $n$ modes. Finally, we characterized the sensitivity of this approach to the perturbation amplitudes, finding that amplitudes in $\psi$ of order $10\%$ quickly begin to di-
verge from the full solution but amplitudes of 5% or less match.

Extending our perturbation theory calculation to include non-radially symmetric components, we likewise found that full simulation results match the perturbative prediction. Both of these numerical experiments show that by accounting for the perturbations in the potential, $\Delta \Phi_{jk}$, we were able to achieve a better match between predicted and simulated mode evolution than by simple superposition of modes and their appropriate $e^{-iE_n t}$ evolution used in Li et al.\textsuperscript{149} and Dalal et al.\textsuperscript{164}. The largest divergence between our simulated and perturbative calculations arises because linear momentum is not conserved in our perturbative eigenstate expansion. This effects only odd $\ell$ modes due to the antisymmetric nature of odd spherical harmonics; furthermore, it is negligible for all except the lowest $\ell = 1$ state, which generates the largest coherent momentum.

We created a ULDM halo in \textsc{chplultra} by colliding eight randomly placed solitons. We decomposed each snapshot of this simulation into $|n \ell\rangle$ eigenstates and tracked the evolution of $|C_{\ell}|^2$ modes. We found:

- the soliton accounts for around 30\% of the halo’s mass;
- higher $\ell = 0$ modes account for very little ($\sim 5\%$) of the halo mass relative to the soliton;
- the $\ell = 1$ modes account for $\sim 10\%$, while $\ell = 2, 3, \text{and} 4$ account for around 8\% or less each;
- the halo does not appear to relax even when evolved over timescales longer than the current age of the Universe.

The relatively large amplitudes of excited modes show that while the perturbative expansion provides insight into the dynamics, fully reproducing its behaviour would require a significant number of terms and accounting for mode-mode interactions.
There are a number of opportunities created by this work. First, as highlighted by Li et al.\textsuperscript{149}, this eigenstate expansion provides a useful language for describing the evolution of ULDM systems and a computationally cheap way of synthesizing realistic ULDM halos. Conversely, this approach has the ability to create benchmark numerical solutions to validate codes that solve the SP system and provides a framework with which to understand the impact that different boundary conditions could have on results. The machinery developed here promises to be useful in analyzing ULDM systems with significant symmetry, such as binary soliton mergers; we will develop this possibility in future work. Moreover, although we restricted our discussion to small perturbations of solitons, our approach could form the basis of a simulation tool built around the time evolution of a sum of (appropriately designed) eigenstates, as opposed to a spatially discretized wavefunction. Finally, we speculate that these techniques could provide complementary tools to better understand questions like the mechanisms by which ULDM systems gravitationally relax and hope to explore these questions in the future.

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Wisdom comes from experience. Experience is often a result of lack of wisdom.

Terry Pratchett

5

Core-Halo Mass Relation:
an Eigenstate Perspective

5.1 Introduction

The development of accessible and accurate cosmological simulations has revolutionized the way we visualize the imprint of dark matter in observational data\textsuperscript{165}. While simulations of the cold dark matter (CDM) model have been shown to accurately reproduce large-scale cosmological structures, there exist a number of apparent discrepancies on kiloparsec (kpc) scales\textsuperscript{17}. One such small-scale dis-
crepancy, referred to in the literature as the “core-cusp problem”, has arisen due to differences between computationally predicted dark matter density profiles and observed profiles from rotation curves of dwarf galaxies\textsuperscript{18–20}. Whereas CDM-only simulations predict a ‘cuspy’ internal profile\textsuperscript{21}, observations tend to favour a flatter central ‘core’\textsuperscript{24,25}. This suggests that state of the art CDM-only simulations do not paint the full picture. Additional small-scale problems include observations of fewer satellite dwarf galaxies than simulations would suggest (“the missing satellites problem”)\textsuperscript{27,28}, an overabundance of isolated dwarfs\textsuperscript{166}, and CDM predictions of massive subhalos which are too dense to host any of their bright satellites (“the too big to fail problem”)\textsuperscript{31}.

While it has been shown that the core-cusp problem can be somewhat ameliorated with the addition of baryons to CDM simulations, other compelling solutions exist\textsuperscript{34,35,167}. One such alternate solution to the above problems is a dark matter candidate consisting of ultra-light, axion-like particles with a mass of order $10^{-22}$ eV. Various names exist for such a candidate, which we refer to here as Ultra-light Dark Matter (ULDM), emphasising the constituent particle’s extremely small mass.\footnote{This candidate is also sometimes known as fuzzy dark matter, scalar-field dark matter, $\Psi$ dark matter, or BECDM. See Ferreira\textsuperscript{44} for a recent review of nomenclature subtleties.} An important consequence of this is that, by virtue of the Heisenberg uncertainty principle, quantum effects become apparent on scales of a few kpc, corresponding to the de Broglie wavelength of the ULDM particle. In particular, a quantum pressure mechanism prevents the formation of a cuspy central halo profile during collapse, and instead supports the formation of a Bose-Einstein condensate core with a solitonic density profile\textsuperscript{136}. Hence, ULDM presents as a natural candidate to avoid the ‘core-cusp’ problem of CDM, without the need to invoke baryonic feedback. Meanwhile, simulations suggest that the outer region of a ULDM halo resembles the NFW profile of CDM\textsuperscript{21}, such that above scales of about 1 kpc the two models are mutually consistent\textsuperscript{168,169}. An general review of ULDM physics can be
found in Ref.\textsuperscript{14}, while more recent reviews of theoretical frameworks and observational signatures can be found in Grin et al.\textsuperscript{170} and Ferreira\textsuperscript{44}.

With the solitonic core being the most obvious prediction of ULDM haloes, much attention has been devoted to predicting the relative mass of the core with respect to the entire halo. Dubbed the core-halo mass relation, this dependence is key to motivating observational searches for ULDM and placing bounds on the constituent particle’s mass. Nevertheless, disagreement around the exact scaling remains\textsuperscript{42,141,171,172}. In this paper, we seek to reconcile these disparate results and add a different perspective to the literature by decomposing our merger products into eigenstates of the system via the formalism we developed in Zagorac et al.\textsuperscript{3}. Our halos are formed through collisions of ULDM solitons; we arrange the solitons in highly symmetric configurations and use our pseudo-spectral code, CHPLULTRA\textsuperscript{154}, to conduct the simulations. In particular, we are interested in the relationship between the relative mass of the core and halo parameters, as well as the dependence of this relationship on merger history. Both of these points are of keen interest to establishing hierarchical halo formation in a Universe with ULDM as the dominant component of dark matter.\textsuperscript{173–175}.

This paper is organized as follows: In Section 5.2, we present the formation of halo cores in the collisions and the dependence on initial conditions. In Section 5.3, we present the scaling we find between halo parameters and relative core mass, and discuss the effect of merger histories. Finally, we discuss our results and future work in Section 5.4. We present our code and its units in Sec. 2.4.3.

5.2 Formation of Halos

5.2.1 The Schrödinger-Poisson Eigensystem

In the non-relativistic regime, ULDM can be described by a macroscopic wavefunction evolving under the influence of its own self-gravity. The dynamics are then
governed by the Schrödinger-Poisson (SP) system of equations:

\[ \frac{i \hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\Phi \psi, \tag{5.1} \]

\[ \nabla^2 \Phi = 4\pi G m |\psi|^2, \tag{5.2} \]

where \( \psi \) is the wavefunction of the dark matter, \( \Phi \) is the gravitational potential, and \( m \) is the mass of the constituent particle. As the ULDM field configuration may be described by a single wavefunction \( \psi \), the dark matter density traces the quantum mechanical probability distribution and is defined as \( \rho = m|\psi|^2 \). The ground state of the SP system is called a soliton. The SP system can also be studied in the fluid representation by application of the Madelung transformation:\(^{69,137}\):

\[ \psi = \sqrt{\rho} e^{iS/\hbar}, \quad v = \nabla S/m \tag{5.3} \]

which provides a useful definition of velocity as the gradient of the phase of \( \psi \).

We can define three pertinent energies in the system: the classical kinetic energy \( K \), the quantum energy \( Q \), and the gravitational potential energy \( V \). The three components are set as follows:\(^{14}\):

\[ K = \frac{1}{2} \int dr \rho v^2, \tag{5.4} \]

\[ Q = \frac{1}{2} \int dr \nabla \sqrt{\rho}^2 \tag{5.5} \]

\[ V = -\int dr \rho \Phi \tag{5.6} \]

In the case of solitons interacting without significant overlap in their wavefunctions, the kinetic term evaluates to the familiar \( K = \Sigma \frac{1}{2} M_i v_i^2 \). The quantum energy \( Q \) corresponds to the quantum pressure term which keep the soliton stable without an internal velocity dispersion. Finally, the gravitational potential energy \( V \)
includes both self-gravitation and multi-body interactions, but is dominated by the former. Thus changing the distance between solitons only negligibly impacts the potential energy, but changing the mass (and therefore $\Phi$) has a larger impact. For this reason, we will only vary the mass and number of colliding solitons in our halos rather than their relative distances.

ULDM haloes have a characteristic profile, consisting of a soliton core and an NFW skirt. The radial density of such a halo can be parametrized as

$$\rho(r) = \begin{cases} 
\rho_{\text{sol}}(r), & 0 \leq r \leq r_\alpha \\
\rho_{\text{NFW}}(r), & r_\alpha \leq r \leq r_{\text{vir}}
\end{cases}$$

(5.7)

where $r_\alpha$ is the transition radius between the soliton and the skirt and is equal to approximately three full-width half maxima of the soliton, $r_\alpha \sim 3r_c$. The virial radius of the NFW skirt is $r_{\text{vir}}$. Note that, while the soliton is an eigenstate solution to the SP system, a ULDM halo is not. In the literature, three ways of numerically forming a ULDM halo stand out: through collision of multiple solitons, through spherical or elliptical collapse, or through a cosmological simulations where halos that have formed and decoupled from Hubble expansion are then further evolved in an SP-solver. In this paper, we will consider the products of solitonic collisions, though we will compare our findings with the results of cosmological simulations in Schive et al.

In order to effectively track the evolution of the solitonic core in a halo, we first calculate the eigenstates of the SP system. We begin by verifying that the potential $\Phi$ of the halo is approximately constant in time after soliton mergers. This sug-

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1In addition to the newer method by Li et al. discussed in the previous chapter.
gests the following approximation for the Schrödinger equation:

\[-i \frac{\partial}{\partial t} \psi = \left[ -\frac{1}{2} \nabla^2 + \langle \Phi \rangle \right] \psi \]  

(5.8)

where \( \langle \Phi \rangle \) is a time-averaged potential and we have set \( m_a = h = G = c = 1 \) for simplicity. This approximation allows us to solve for the eigenstates and eigenenergies of the system for a given \( \langle \Phi \rangle \); our choices for the time-averaged potentials are discussed below. For a more detailed explanation of the procedure, see Zagorac et al.\(^3\).

5.2.2 Emergence of Halo Cores

![Figure 5.1: Left: The evolution of the ground state of each halo is shown as a function of time. The subsample of our runs shown here are indexed by the number of simultaneously merging solitons. Note that the merger is evident from the sudden creation of a core (for instance, at \( t = 0.15 \) for the \( N = 2 \) case). After the merger each halo core oscillates around a constant number; we take the average between \( 0.2 < t < 0.3 \) as the relative halo mass, \( M_c / M_{\text{halo}} \). This region is marked with a dashed rectangle and a zoomed-in version is shown on the right, where we have also marked the average value of each line with dashed lines.](image)

We initialize our first set of runs with equal-mass solitons arranged in symmetric configurations, ranging from 2 to 15 solitons. The case of two solitons is a simple binary mergers, with higher even numbers corresponding to the vertices of Platonic solids: the tetrahedron (4), the octahedron (6), the cube (8), the icosahedron (12), and a superposition of the octahedron and cube (14). Odd numbers of initial solitons are achieved by adding a soliton to the center of these configura-
tions. Note that we don’t perform a merger of 10 or 11 solitons.

The solitons start at rest, and have formed a halo with a core and an NFW-like skirt halfway through the simulation or earlier. For each case, we calculate the time-averaged density profile over the last third of the simulation, corresponding to 0.1 code units of time. We solve for the potential corresponding to this profile using the Poisson equation, and use that time invariant potential to calculate the eigenstates of the system. We verify that the ground state corresponds to the shape of the core of the halo, equivalent to a soliton of appropriate mass. Finally, we isolate the contribution of the ground state to the halo’s wavefunction by projecting our $\psi$-grid $|\psi\rangle$ into the wavefunction of the ground state soliton $|0\rangle$:

$$c_0 \equiv \langle 0 | \psi \rangle.$$  \hspace{1cm} (5.9)

Recall that the density of the dark matter is given by $\rho = |\psi|^2$; we therefore choose to normalize our eigenstates by the total halo mass such that $\Sigma_{n\ell m} |c_{n\ell m}|^2 = 1$. Consequently, the relative mass of the soliton core to the halo is given by $|c_0|^2 = M_c/M_{\text{halo}}$.

In each case, the formation of the soliton core is clearly seen in the sudden increase in the amplitude of $|c_0|^2$, as illustrated in Fig. 5.1. After the merger, the contribution of the ground state core oscillates around an approximately constant number. We therefore define the relative mass of the core, $M_c/M_{\text{halo}}$, as the average of $|c_0|^2$ between times $0.2 < t < 0.3$.$^3$

Isolating the core using the eigenstates of the system is a new approach. In previous works, the mass of the core was determined by fitting the central core of the halo to a mass profile. In Ref.$^{168}$, the profiles are fit around density maxima using

$^3$We use this time interval because we know halo cores have been created in all of our simulations by that point; varying this time interval does not significantly impact our results.
the soliton profile approximation.

\[ \rho_c(r) \simeq \rho_0 \left[ 1 + 0.091 \cdot \left( \frac{r}{r_c} \right)^2 \right]^{-8} \quad (5.10) \]

where \( \rho_0 \) is the central density and \( r_c \) is the FWHM of the soliton. We perform the same analysis on our data using averaged halo profiles over the interval \( 0.2 < t < 0.3 \), in parallel with our eigenstate analysis. We will compare the results of the two approaches and their effect on the core-halo mass relation in the following Section.

5.3 Core-Halo Mass Relationship

5.3.1 Review of Previous Work

There is previous work concerning the scaling of the relative core mass with different parameters in the system. We focus on results derived from simulations where halos were produced through soliton mergers rather than through spherical or elliptical collapse. In particular, Schwabe et al.\textsuperscript{168} and Mocz et al.\textsuperscript{172} have found that the dimensionless parameter

\[ \Xi \equiv \frac{|E|}{M^3/(Gm/\hbar)^2} \quad (5.11) \]

is instrumental in determining how much of the total mass will end up in the final halo core. Here, \( E = K + Q + V \) is the total energy, and \( M \) is the total mass of the system. The proposed relationship is a power law

\[ M_c/M \propto \Xi^\alpha, \]

where the power index \( \alpha \) is subject to debate. Schive et al.\textsuperscript{42} originally found \( \alpha = 1/2 \), which Chavanis\textsuperscript{178} was later able to reproduce using an effective thermody-
namic model. Since then, Schwabe et al.\textsuperscript{168} found it to be in the range $\alpha = 1/6 - 1/4$ for equal-mass soliton binary mergers and $\alpha = 1/2$ for unequal-mass soliton binary mergers. Mocz et al.\textsuperscript{172} found $\alpha = 1/3$ by merging many solitons of different masses and initial conditions. Using a slightly different approach, Du et al.\textsuperscript{171} considered the merger history of ULDM halos with stochastic merger trees, and found that the core-halo mass relation depends only on the mass fraction loss of cores during binary mergers, $\beta$, such that $\alpha = 2\beta - 1$. Their data from cosmological simulations were fit well with $\beta = 0.7$, leading to an exponent of $\alpha = 0.4$.

A thorough summary of much of the work discussed above can be found in Nori and Baldi\textsuperscript{176}, in which the authors also ran their own tests using halos formed through spherical collapse in cosmological simulations; their results were consistent with $\alpha = 1/3$, but not with the original relation found in Schive et al.\textsuperscript{42}. This (mis)match is somewhat curious, given that Schive et al.\textsuperscript{42} also performed cosmological simulations with expanding backgrounds, while Schwabe et al.\textsuperscript{168} and Mocz et al.\textsuperscript{172} perform idealized soliton mergers to construct halos. In this work we will exclusively analyze halos formed through soliton mergers, and will primarily compare our results with references considering the same formation mechanism.

5.3.2 SIMULTANEOUS SOLITON MERGERS

We calculate the parameter $\Xi$ using the initial energy and mass in our system and plot it against our derived relative core masses, as shown in Fig. 5.2. We repeat simulations for three different initial soliton masses, $M = 40, 50$ and 60 in code units (see Appendix 2.4.3 for definitions of code units). Fitting a power law to our data, we find $\alpha = 0.215$ (illustrated in the black dashed line). This is closest to the $\alpha = 1/4$ scaling derived in Schwabe et al.\textsuperscript{168} (illustrated in the gray line). Note that there is more visible scatter present as $\Xi \rightarrow 10^{-3}$, corresponding to more massive halos formed through collisions of higher numbers of solitons. We have verified
Figure 5.2: We present the log-scaled relationship between the relative core mass and $\Xi$ in our data for three different masses of merging solitons. The power law fit to our data yields an exponent of $0.215$, closest to the $1/4$ scaling from Schwabe et al.\textsuperscript{168}.

that the scatter is not due to resolution issues and seems to be intrinsic to the data. We attribute the scatter to numerical effects arising from the large number of interacting solitons.

We also illustrate the results from fitting cores and compare them with results from calculating soliton eigenstates in Figure 5.3. Note that fitting cores produces a different value for $\alpha$. The fitted data also shows a large scatter at high values of $\Xi$ and almost universally overestimates the mass of the soliton when compared to the eigenstate data. The relative overestimation leads to a shallower core-halo mass relation, approximately proportional to $\Xi^{1/8}$.

This is further illustrated by comparing the results of our equal mass binary mergers with the results of Schwabe et al.\textsuperscript{168}: our method reveals a core of about 57% of the halo mass for binary mergers, whereas Schwabe et al. find closer to 70%. Note, however, that our fitting reveals binary-produced cores of closer to
FIGURE 5.3: Shown is the core-halo mass relation for cores derived from eigenstates (black circles) and from fitting functions (purple open diamonds). Both sets of data include mergers of $M = 40$, $M = 50$, and $M = 60$ solitons. Gray lines with exponents of $1/4$ and $1/8$ are shown as a reference.

60%, suggesting that the different results can be attributed to more than just the different core extraction methods. Furthermore, the result from Schwabe et al. is consistent with the findings of Du et al. $^{171}$, which found that $\beta = 0.7$ fits the results of their cosmological simulation at different redshifts. This discrepancy requires additional attention, and we plan to investigate it further.

5.3.3 SEQUENTIAL SOLITON MERGERS

Interestingly, this relationships we found in Sec. 5.3.2 cease to hold when halos are formed through non-simultaneous mergers of solitons. We again initialize numerical boxes populated with solitons, this time arranged so that two stages of mergers happen: two simultaneous initial mergers of $2 \geq N_{\text{sol}} \geq 7$ solitons, and a subsequent merger of the two products. Therefore a sequential merger of e.g. 8 solitons will involve two 4-soliton simultaneous mergers and the subsequent bi-
nary merger of the two products. The $\Xi$ parameter of such a sequential system is almost identical to the $\Xi$ parameter of a simultaneous merger of 8 solitons (under the assumption all the solitons have equal masses). Therefore, if the only parameter controlling the core-halo mass relation is $\Xi$, we expect the final halos in both these cases to have identical core-to-halo mass ratios.

Nevertheless, Fig. 5.4 reveals a different trend for sequential mergers, which are best fit with a power law index $\alpha = 0.277$. Whereas the data presented in Sec. 5.3.2 supported the findings of Schwabe et al.\textsuperscript{168} and a slope of $1/4$, this slope approaches the $1/3$ index found in Mocz et al.\textsuperscript{172}. We attribute this difference to the differences in how simulations were performed in the two references. Schwabe et al. simulated only binary soliton collisions (with most of their range in $\Xi$ provided by varying the mass ratio $\mu$), whereas Mocz et al. looked at scenarios involving a group of soliton cores of different masses that merge to form a final virialized halo. The latter setup all but guarantees a sequence of multiple mergers, whereas solitons in binaries by definition merge simultaneously. The correspondence of our results to those of Schwabe and Mocz based only on changes in merger order suggests that merger history plays a significant role in determining the core-halo mass relation of individual dark matter halos.

5.3.4 Unequal Mass Soliton Mergers

Lastly, we relax our assumption of equal mass mergers. In this section, we systematically investigate the effects of changing mass ratio $\mu \equiv M_1/M_2$ in binary soliton mergers. In performing binary mergers with $\mu \geq 1$, Schwabe et al.\textsuperscript{168} found that for $\mu \sim 7/3$ the smaller progenitor soliton is completely disrupted and forms a diffuse halo around the more massive progenitor, which forms the halo’s core. We therefore restrict ourselves to investigating $1 \leq \mu \leq 3$, as we expect the same qualitative behavior of larger and larger mass ratios. We observe this effect in the $\mu = 3$
Figure 5.4: Shown is the core-halo mass relation for halos formed through simultaneous mergers (black circles) and through sequential mergers (purple diamonds). Both sets of data include mergers of $M = 40$, $M = 50$, and $M = 60$ solitons. The power-law exponents of the two data sets vary significantly, with halos formed through sequential mergers exhibiting an exponent of closer to $1/3$ as in Mocz et al.\textsuperscript{172} and Nori and Baldi\textsuperscript{176}.

Specifically, we consider binary soliton mergers with $\mu \in \{1.05, 1.10, 1.25\}$, expecting the results to fall close to the those of the appropriate equal-mass binary mergers on our scatter blot.
Figure 5.5: We show the core-halo mass relation of halos formed through binary mergers of unequal mass (pink triangles) relative to previous data. For comparison, two gray lines are shown illustrating $\alpha = 1/3$ and $\alpha = 1/4$ exponents, as in previous figures, with an additional $\alpha = 1/2$ gray line. Note that this figure is shown on a slightly different scale than previous similar figures.

Bizarrely, we found that the evolution of their ground states is much more perturbed than examples shown in Fig. 5.1, leading to poorly-motivated estimates of $M_c/M_{\text{halo}}$ when averaging over the cores’ mass contribution to the halo. At the moment, we are ascribing this to a numerical issue affecting these simulations in particular and planning to investigate the issue further. As such, future results may affect the scaling relation in Fig. 5.5 more significantly than others.

5.4 DISCUSSION

In this work, we presented a novel way of isolating soliton cores inside ultralight dark matter halos by calculating the ground state of the halo. We create halos through mergers of different numbers and masses of solitons. The ground state (and, indeed, the higher states) are seeded by the isolated halo’s approximately
stationary potential, as described in Zagorac et al.\(^3\). We can then perform a dot product of the calculated ground eigenstate and the simulated halo wavefunction at each timestep, revealing the time evolution of the halo’s core. After a quick process of formation, we find the contribution of the ground state stays approximately constant, allowing us to define a relative core-halo mass, \(M_c/M_{\text{halo}}\).

This new approach to calculating the relative mass of the core and the halo also opens a door towards commenting on the core-halo mass relation of ULDM. Previous work (starting with Schive et al.\(^42\) and Schive et al.\(^43\)) has established the core-halo mass relation as the scaling of the relative mass \(M_c/M_{\text{halo}}\) with a parameter \(\Xi \equiv |E|/M^3\). Generally, this relationship is taken to be some power law \(M_c/M_{\text{halo}} \propto \Xi^\alpha\), with the exponential \(\alpha\) varying from \(1/6\) to \(1/2\) in the literature.

Those results are products of forming halos through soliton collisions and fitting for their central cores. We compared fits to central cores with our eigenstate approach and found that fitting consistently overestimates the relative core mass with respect to our method, yielding a shallower power law such that \(\alpha_{\text{fit}} < \alpha_{\text{eig}}\). This discrepancy may be due to the fact that fitting is done on the density of the halo rather than its wavefunction: as the former is the square of the latter, \(\rho = |\psi|^2\), fitting the density may include cross-terms which are not present when dealing with the wavefunction directly. However, while the wavefunction is easier to simulate, the density itself is the observable of the system. Thus, understanding the subtleties of these different approaches will be crucial to looking for cores in observational data; we intend to investigate this relationship thoroughly in the future.

Using our eigenstate approach, we simulated soliton collisions in three regimes:

1. simultaneous mergers—where some \(N\) solitons of identical masses merge simultaneously at the center of our box;

2. sequential mergers—where two groups of \(N\) same-mass solitons merge at
the same time, after which the two initial merger products collide at the center of the box, forming the final halo; and

3. unequal mass mergers—where we merge two solitons with a given mass ratio, $\mu = M_2/M_1$, forming a halo at the center of the box.

All of our data are summarized in Fig. 5.6, with an overall best fit of $\alpha = 1/4$. In each of these regimes, we find a different $\alpha$-value: $\alpha \sim 0.25$ for simultaneous mergers, $\alpha \sim 0.33$ for sequential mergers, and $\alpha \sim 0.50$ for unequal mass mergers. These values correspond to the three most often cited values in the literature and suggest the dependence of the core-halo mass relation on a given halo’s formation and merger history (consistent with the findings of Du et al.\textsuperscript{171}). Furthermore, our assumptions in each of the three regimes mirror the ones made in Schwabe \textit{et al.} (Schwabe et al.\textsuperscript{168}) and Mocz \textit{et al.} (Mocz et al.\textsuperscript{172}). Schwabe \textit{et al.} merge binaries of solitons of equal mass and unequal mass, finding a scaling of 0.25 for the former
and 0.5 for the latter. Mocz et al. (along with Nori and Baldi\textsuperscript{76} after them), find a value of $\alpha = 1/3$. They merge many solitons with stochastic initial conditions; in this regime, most mergers will be binary collisions between halo-like products of previous mergers, corresponding qualitatively to our sequential mergers.\textsuperscript{8}

Thus, we are able not only to recreate the most commonly cited core-halo mass relation values, but also to link them to halos’ merger histories. While our results may help to tie together previous findings, they also suggest a troubling dependence of the core-halo mass relation on individual numerical setups used to investigate it. One way to alleviate this possibility would be to use our method for calculating eigenstates on halos produced by different numerical ULDM solvers. In this way, we would be able to compare halos (and their cores!) quantitatively, eigenmode by eigenmode, and tease apart sources of numerical uncertainty in code size and the effects of merger history more precisely.

5.5 Acknowledgments

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\textsuperscript{8}Note, however, that there is no requirement that the mergers in the two references be of equal-mass halos, such as in our setup!
A good bookshop is just a genteel Black Hole that knows how to read.

Terry Pratchett, Guards! Guards!

A Library of Soliton Binaries

Having dedicated some thought to what halos formed through multi-soliton mergers reveal about the core-halo mass relation, let us now revert to simple binary collisions. After all, halos in the Universe are built up hierarchically, where each merger in a halo’s history is most likely a binary. Thus, instead of constructing synthetic scenarios where we maximally vary $\Xi$, we turn to varying other parameters in a binary soliton merger and commenting on the resultant halo.
6.1 Halo initial conditions

6.1.1 Colliding Solitons

As discussed in the preceding chapter, previous work has investigated ULDM halo formation through soliton mergers (including binaries) as a function of initial box parameters, total mass $M$, total energy $E$, and total angular momentum $L$. In particular, Schwabe et al.\textsuperscript{168} and Mocz et al.\textsuperscript{172} have found that the $\lambda$-invariant unitless parameter

$$\Xi \equiv \frac{|E|/M^3}{(G\ m/h)^2} \quad (6.1)$$

is instrumental in determining how much of the mass of the system will end up in the final halo core, such that

$$\frac{M_c}{M_h} \propto \Xi^\alpha \quad (6.2)$$

Regardless of the correct value for the scaling parameter $\alpha$, we have found that $\Xi$ is not very descriptive for binary collisions due to its small variation (Fig. 6.1). As can be seen from the definitions of the three energies in Eqs. 5.5-5.6, the quantum and potential self-energies of each soliton scale as $M^3$, thus contributing only a constant value to $\Xi$. The kinetic energy (Eq. 5.4) and two-body interaction term of the potential energy (Eq. 5.6) are sub-dominant in the total energy term and scale as $M$ and $M^2$ respectively.

This lack of variability in $\Xi$ for a two-body colliding system leads us to conclude that in a bound binary collision, the percentage of the total mass $M$ that ends up forming the soliton core is roughly constant. This means we expect all the resul-
tant soliton profiles from our runs to be self similar as:

\[ \rho(r) = \frac{\rho_0}{\left(1 + 0.091 \left(\frac{r}{r_c}\right)^2\right)^8} \]  \hspace{1cm} (6.3)

where \( \rho_0 \propto M^4 \) and \( r_c \propto 1/M \). For an illustration, see Figure 6.3.

![Figure 6.1: Example of \( \Xi \) parameter scaling as initial velocities and mass ratio of solitons is changed. The white dashed line in the upper left denotes where the total energy of the system \( E = 0 \), such that parameters to the left of the line will produce an unbound system. Varying mass ratio produces a larger scatter in the \( \Xi \) parameter than varying mass but keeping \( \mu = 1 \); nevertheless, there is still less than an order of magnitude of variation in \( \Xi \).]

6.1.2 Initial Conditions

Each run is initialized in an \( L^3 \) box with \( N^3 \) grid units. Each of our colliding solitons is given a mass, position, and velocity value in code units such that the total linear momentum in the box is zero and the collision happens in the center of the box. This is equivalent to a translation in the center of mass frame, and does not
physically restrict our system while allowing us to maximally resolve the collision.

We investigate the effects of varying the initial mass of the colliding solitons $M_1 = M_2 = M$, the mass ratio of the solitons $M_1/M_2 = \mu$, their initial velocities $v_1 = v_2 = v$, and their impact parameter $b$. See Fig. 6.2 for an illustration. In the equal mass case, we vary the soliton mass $25 < M < 200$, with the limits set by mandating the solitons be small enough to fit in the box while remaining puffy enough to resolve their full-width at half-max (FWHM) with our grid. In the unequal mass case, we fix our smaller soliton mass $M$ to be $M = 50$ code units and vary the mass ratio $1 \leq \mu \leq 3$ providing an order of magnitude of variability in the resulting potential and quantum energies, $Q$ and $V$. We vary the initial velocity such that $0 \leq v \leq 15$ code units, thereby changing $K$. Since the total potential energy only weakly depends on the initial positions of the two solitons, we fix the distance along the x-axis to $2a = 0.6$ such that the $M = 50$ soliton doesn’t signif-
icantly overlap with either the edge of the box or the other soliton. We vary the impact parameter along the perpendicular y-axis, $0.0 < b < 0.6$, thus considering collisions with initial angular momentum.

6.2 AN EXAMPLE MERGER

In comparing simulations of bound binary soliton collisions with different initial parameters, a relatively invariant qualitative picture of the final state core emerges. One example is shown in Fig. 6.3, which illustrates a halo core produced through the collision of two solitons with equal masses $M = 50$, impact parameter $b = 0.0$, and initial velocities $v = 0$. We present this example to highlight some aspects of the final state; we will discuss the dependence of this state on initial conditions in the following Section.

In our example of two merger the two solitons start at rest and merge approximately at $T = 0.15$ in code units (see the top panel of Fig. 6.3); we evolve the simulation to $T = 3.0$. At $T = 2.25$, the grid shape is still visible in the interference pattern of the ULDM density; by $T = 3.0$, this effect has been replaced by the characteristic “lumpiness” of the ULDM profile skirt. Nevertheless, the symmetry of the initial conditions is still evident in the system, which remains symmetric around the two Cartesian axes of the plane. Throughout our simulation the total energy in our numerical box stays constant to better than 1% and mass in conserved to better than 2%. As shown in the lower right panel of Fig. 6.3, potential energy is the dominant form of energy, with quantum kinetic energy being secondary. Classical kinetic energy is initially zero as expected, and it peaks at the time of the solitons’ merger at $T = 0.15$.

The product or the merger quickly starts to resemble a ULDM profile with a solitonic core and skirt falling off as $\sim r^{-3}$. See the left panel of Fig. 6.3, where we

\footnote{For a choice of $\lambda = 1.8$, $T = 3.0$ will correspond to about 7 Gyrs and the solitons will have masses on the order of a few times $10^7 M_\odot$. See Table 2.1 for other unit conversion options.}
Figure 6.3: An illustration of halo formation via a binary soliton merger.  
**Top:** Density planes of the simulation at different times: $T = 0.001$, $T = 0.150$, $T = 0.225$, and $T = 0.300$.  
**Left:** Mass density profiles from $T = 0.2$ to $T = 0.3$ (in color) and the average of those profiles (black).  
**Right:** Potential Energy (purple), quantum energy (pink), and kinetic energy (orange) as a function of time. Total energy is shown with the gray dashed line.

show instantaneous profiles for the last 100 saves ($0.2 < T < 0.3$) in color and their average $\langle \rho \rangle$ in black. Note that averaging over the last 0.15 time units (corresponding to immediately after the merger) instead provides a very similar average profile. Given a time-averaged radial profile $\langle \rho \rangle$, we are able to calculate the corresponding time-independent potential $\langle \Phi \rangle$, which we in turn use to derive the eigenstates of the merger product. These eigenstates are qualitatively indistinguishable from those we derive from the profile of a single soliton in Zagorac et al.\textsuperscript{3}, though their masses are appropriately scaled for the problem at hand.

This calculation further allows us to decompose our simulation box into individual eigenstates; the results are shown in Fig. 6.4. Odd-$\ell$ modes are not excited above the threshold of box noise, nor are $m > 0$ nodes, so we do not show them. The soliton density ($\langle 000|000 \rangle$) is excited fairly immediately at the time of collision and plateaus at around 0.57 with small oscillations. Subsequent $\langle n00|n00 \rangle$ modes
Figure 6.4: The amplitudes of $|c_{n\ell}|$ modes are shown on a log scale. The top panel shows $\ell = 0$ modes; the second, $\ell = 2$ modes; and the third, $\ell = 4$ modes. We only show $n < 5$ modes in each panel. The merger happens around $T = 0.15$ as indicated by the vertical line; pre-merger data is indicated by dashed lines. The lowermost panel shows the contribution for all $\ell = 0$ modes (black), $\ell = 2$ modes (pink), and $\ell = 4$ modes (yellow), as well as their total (dot-dashed, black).
are excited below an amplitude of 0.16, with higher states having progressively lower amplitudes. The \( |n20\rangle \langle n20| \) states are excited to amplitudes around 0.1 or less, while \( |n40\rangle \langle n40| \) modes do not exceed amplitudes of 0.01.

In the bottom panel of Fig. 6.4, we show the total amplitude per \( \ell \) mode coefficients defined as \( |C_\ell|^2 \equiv \sum_{n,m} |c_{n\ell m}|^2 \), where \( |m| \leq \ell \). The \( \ell = 0 \) modes oscillate around an amplitude of about 0.8; the \( \ell = 2 \) modes oscillate around about 0.2; and the \( \ell = 4 \) modes oscillate around about 0.05. The sum of these modes (up to \( \ell = 4 \) and \( n = 25 \)) make up almost 100% of the system immediately after the solitons merge, but fall of slightly and plateau around 85% at \( T = 0.3 \). This suggests that modes higher than \( \ell = 4 \) or \( n = 25 \) do not get excited immediately upon halo formation, but shortly thereafter. Furthermore, the number of additional excited modes seems to say approximately level (making up 15% of the halo’s mass). Such behavior is more reminiscent of the perturbed solitons presented in Zagorac et al. than the halo in the same paper. While the percentage of the mass in excited modes is too high to expect perturbation theory to suffice in modelling this system, the amplitude of the soliton’s oscillations is significantly smaller. Thus, we expect ULDM halos formed from highly symmetric initial conditions to exhibit a more stable core.

The presence of certain modes is also evident in halo dynamics illustrated by the top panel of Fig. 6.3. The third and forth column capture the halo core oscillating in a “cross” pattern from an ellipsoid along one axis (a prolate ellipsoid), to a sphere, to an ellipsoid along the perpendicular axis (an oblate ellipsoid), and back to a sphere. This is a tell-tale sign of the quadrupole moment of the halo core being excited through the binary soliton collision, as confirmed by the significant contribution of \( \ell = 2 \) modes (see Fig. 6.4). Another visible effect is the so-called soliton ”breathing mode,” which refers to the contracting and expanding of the
soliton caused by excited spherically symmetric modes \((n > 0, \ell = 0)\). The behavior is particularly evident in the two rightmost panels of the top panel of Fig. 6.3. Since there was no initial angular momentum in the box, the final core is not rotating, though this does happen in runs where \(b > 0\).

### 6.3 Mergers with Different Initial Conditions

Having shown what a simple merger product looks like in terms of its eigenstates, we now vary each of the parameters in the simulation: velocity (and therefore kinetic energy), the mass of the solitons, and finally the mass ratio of the solitons. In each case, we derive the eigenstates as described above and verify that the ground state corresponds to a soliton the size of the halo’s core.

#### 6.3.1 Mass

Firstly, we perform equal-mass mergers of solitons of different masses \(M\). We restrict ourselves to initial masses in the range \(M \in \{25, 50, 100, 150, 200\}\) in code units. The lower limit is set so each soliton isn’t too puffy for the size of the box, while the upper limit is set so that we still have sufficient resolution to analyze the resultant halo’s core.

The results are presented in Fig. 6.5. In the top panel, we show the formation of the soliton eigenstate, \(|c_{000}|\). In each of the five cases, the resultant core is approximately \((75\%)^2 \approx 57\%\) on the halo’s mass. Higher-\(M\) initial conditions lead to a more sudden soliton formation, almost approximating a step function, while the \(M = 25\) case grows more smoothly. The total contribution of \(\ell = 0\) modes, \(|C_0|^2\), is slightly higher than the soliton contribution, closer to 70%; the \(\ell = 2\) modes contribute between 10 and 25%, depending on the run; and \(\ell = 4\) modes contribute less than 5% in each case.

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†In nuclear physics, “breathing mode of solitons” refers to multipole modes above the spherically symmetric mode, \(\ell = m = 0\).
Figure 6.5: The dependence of a halo’s eigenstate makeup on different initial soliton masses, $M$. We highlight the soliton eigenstate (top), the total contributions of the $\ell = 0, 2, \text{ and } 4 \text{ modes (middle), and their sum (bottom). The approximate soliton contribution (75\%) is shown with the gray dashed line in the top panel. Each merger happens at a different time, and pre-merger data is indicated by dashed color lines.}
Totaling the contribution of $\ell \leq 4$ modes, the $M = 25$ and $M = 50$ cases account for around 85% of the total halo, while $M = 100$ and higher runs only encapsulate around 75%. The figure suggests mergers of lower M-mass excite more higher modes, but not to the detriment of the mass of the final core. Therefore, we might expect halos formed from different M binary mergers to have the same relative size of core but a different skirt. This could furthermore impact the oscillation timescales of the core’s breathing mode, which arises due to the presence of $\ell = 0$ excited modes perturbing the soliton ground state. We will investigate this further in the next section.

6.3.2 Mass Ratio

Next we vary the mass ratio of the two solitons $\mu \equiv M_1/M_2$. We set the smaller soliton mass to $M_2 = 50$ and vary $M_1$ so that $\mu \in \{1.0, 1.5, 2.0, 2.5, 3.0\}$. Thus, the smallest total initial mass in our box is 100 code units and the largest is 200, roughly spanning the same total mass as in the previous section. We limit ourselves to $\mu < 3.0$ in accordance with Schwabe et al.\textsuperscript{168}, where the authors found that binary mergers with $\mu \gtrsim 7/3$ cause the total disruption of the less massive soliton and virtually no change in the more massive soliton. This transition is also evidenced in our data: the $\mu = 3.0$ case looks qualitatively different than the other runs immediately after the merger, though it ultimately settles to approximately the same values as the others.

Similarly to varying the initial mass, we present our data in tiers in Fig. 6.6. We again note the quick and evident formation of the soliton cores in the top panel, with the caveat that $\mu = 3.0$ looks different for the reasons stated above. In the middle tier, we show the total contribution of $\ell = 0$, 1, and 2 modes now. Unlike all other mergers presented in this paper, $M_1 \neq M_2$ mergers excite odd-$\ell$ because they

\textsuperscript{1}This is potentially due to the fact that more massive solitons are also more compact, whereas lighter solitons are puffier and more easily disrupted.
Figure 6.6: The dependence of a halo’s eigenstate makeup on different initial soliton mass ratios, \( \mu \). We highlight the soliton eigenstate (top), the total contributions of the \( \ell = 0, 1, \) and 2 modes (middle), and their sum (bottom). The approximate soliton contribution (75\%) is shown with the gray dashed line in the top panel. Each merger happens at a different time, and pre-merger data is indicated by dashed color lines.
break symmetry\(^8\). We thus chose to show \(\ell = 1\) and \(\ell = 2\) modes, which make up \(10 - 25\%\) of the halos’ masses. We have verified that higher \(\ell\)-modes contribute significantly less, resembling \(|C_4|^2\) as illustrated in Fig. 6.5. The lowest tier indicates the sum of all modes with \(0 \leq \ell \leq 4\). Note that, unlike with varying mass, higher \(\mu\) values sum to closer to 100\%, suggesting that lower mass ratios excite more modes higher than \(\ell = 4\). The soliton’s oscillation frequency is once again visibly different when varying \(\mu\).

6.3.3 Kinetic Energy

As illustrated in Fig. 6.1, changing the initial velocities of the solitons has a small impact on the \(\Xi\) parameter when the mass ratio \(\mu = 1\), and even less of an effect for \(\mu > 1\). Nevertheless, we varied the initial velocity \(v\) between \(v = 0\) and \(v = 15\) code units; above that, the system is unbound and the merger never happens.

The data are shown in Fig. 6.7 in three tiers, as before. Naturally, each merger will happen at a different time, with \(v = 15\) happening first. However, in order to compare the resultant modes more easily, we scaled our results so that all mergers happen at \(T = 0.15\), corresponding to the \(v = 0\) case. This has revealed that the \(0 \leq v \leq 4\) cases are virtually indistinguishable. The \(v = 10\) case is qualitatively different but similar, producing a core of similar relative mass, while the \(v = 15\) is both qualitatively and quantitatively different, producing a soliton core closer to 40\% of the halo’s mass.

Higher modes follow a similar pattern, with \(v \leq 4\) cases dominating both in \(|C_0|^2\) and \(|C_2|^2\). In the case of \(|C_4|^2\) modes, each run contributes less than 20\% to the halo’s mass. Finally, the total contribution of \(\ell < 4\) modes in the case of the low-velocity runs is around 85\%, while higher-velocity runs contribute closer to 75\%. This suggests that high-velocity runs either excite more higher-\(\ell\) modes, or

\(^8\)Note the zero contribution of the \(\mu = 1.0\) run to \(|C_1|^2\)
Figure 6.7: The dependence of a halo’s eigenstate makeup on different initial velocities, $v$. We highlight the soliton eigenstate (top), the total contributions of the $\ell = 0, 2, \text{ and } 4$ modes (middle), and their sum (bottom). The approximate soliton contribution (75%) is shown with the gray dashed line in the top panel. Each merger happens at a different time, but we translate the data in time so that they happen at $T = 0.15$, as indicated by the vertical line. Pre-merger data is indicated by dashed color lines.
excite the same number of those modes to higher amplitudes, compared to the low-velocity runs. Despite these differences, the solitons’ oscillations maintain similar frequencies in all cases.

6.3.4 Angular Momentum

Finally, we vary the initial angular momentum of our binary mergers by changing the impact parameter $b \in \{0.0, 0.2, 0.4, 0.6\}$. The results are shown in Fig. 6.8. Halos formed through these collisions rotate in the plane of the collision—an obvious difference from previous runs when examining cross-sections of the box as a function of time. Despite this difference, the halos’ eigenstate makeup changes remarkably little with increasing $b$, remaining similar to other cases without angular momentum.

As with other $\mu = 1$ runs, the collision excited neither odd-$\ell$ nor $m > 0$ modes. Similarly to changing the velocities, the soliton evolution remains remarkably similar—almost indistinguishable. The sum of spherically symmetric $\ell = 0$ modes is also approximately constant, just under 80%. The contribution of $\ell = 4$ nodes is maximal for $b = 0.0$, around 20%. The total contributions of the $\ell \leq 4$ modes fall with from $\sim 80\%$ to $\sim 60\%$ with rising angular momentum, suggesting that mergers with angular momentum produce more higher-order modes, though again not to the detriment of the soliton.

6.4 Soliton Timescales

In each of the halos produced in Sections 6.3.1-6.3.4, the final soliton core shows oscillations around $|c_{000}|^2 \approx 57\%$ of the halo mass. Even when evolving some of the runs beyond $T = 0.3$ code units this behavior remains robust and the soliton does not either shrink or grow on average over long timescales. While the average relative mass of the soliton isn’t significantly augmented by varying any of our four
Figure 6.8: The dependence of a halo’s eigenstate makeup on angular momentum, indexed by different initial impact parameter $b$. We highlight the soliton eigenstate (top), the total contributions of the $\ell = 0, 2,$ and $4$ modes (middle), and their sum (bottom). The approximate soliton contribution (75%) is shown with the gray dashed line in the top panel. Each merger happens at a different time, and pre-merger data is indicated by dashed color lines.
parameters, the frequency of oscillations is visibly different in some cases, but not others. For example, varying initial kinetic energy (Fig. 6.7) and initial angular momentum (Fig. 6.8) both produce solitons which are remarkably similar, suggesting that neither kinetic energy nor angular momentum set the oscillation frequency. On the other hand, varying $M$ and $\mu$ produces very clearly different oscillatory behavior and timescales.

When varying total mass in the box, we expect to find a $\nu \propto M^2$ scaling relation due to eigenenergies. As discussed in Zagorac et al.\textsuperscript{3}, a ULDM wavefunction can be written in the interaction picture as

$$\psi(t) = \sum_{n=1}^{N} c_{n \ell m}(t) \exp(-i E_{n \ell m} t) |n \ell m\rangle.$$  \hspace{1cm} (6.4)

In the simplest example, we can imagine a soliton $|000\rangle$ perturbed only by a single excited mode, $|n \ell m\rangle$. In this case, the frequency of the solitons oscillations are set by the difference of eigenenergies, $\nu \propto (E_{n \ell m} - E_{000})$.

The values of the eigenenergies scale with the mass of the potential from which they are seeded, in our case, the potentials are set by the mass of the halo’s core $M_c$, meaning that the energies scale with core mass $E \propto M_c^2$. Thus, as the oscillation frequencies are related to the difference of two eigenenergies, we expect that $\nu \propto M_c$ as well.

A Fourier decomposition of the oscillations in the $|000\rangle$ ground state of each run reveals the dominant frequency peaks below the Nyquist frequency of the box. These data are illustrated in Fig. 6.9. The frequencies of these oscillations roughly scale as $\nu \propto M_{\text{tot}}^2$, where $M_{\text{tot}}$ is the total initial mass in the box. We note that the $\mu \geq 2.0$ frequencies depart from the best fit line, as does the $M = 100$ run. In each case the resultant frequency is close to zero, despite the mean of the data being subtracted before Fourier decomposing the data using a Lomb-Scargle peri-
Figure 6.9: We show the quadratic dependence of frequency of core oscillations on total initial mass. 

**Frequency:** Peak frequencies from Fourier transforming our soliton oscillation data are shown in filled circles: black for $\mu = 1$ runs from Sec. 6.3.1, pink for $\mu > 1$ runs from Sec. 6.3.2. The gray dashed line illustrates a best fit of $\nu \propto M^2$ to the data. 

**Energy:** Eigenenergy differences $\Delta E = E_{100} - E_{000}$ are shown in open diamonds with corresponding colors. This scaling is shown to motivate the expected scaling of the frequencies.

It is possible that this is due to grid noise in the spatial resolution of the resulting cores or aliasing due to under-sampling the high-frequency oscillations in the time domain.

Additionally, we have scaled frequencies with initial mass $M_{\text{tot}}$ rather than final core mass $M_c$ for simplicity. However, in the previous chapter, we found that $\mu = 1$ soliton mergers produce a different core-halo mass relation than $\mu > 1$ mergers; thus, scaling with $M_c$ might be more predictive of oscillation frequencies (though the value of $M_c$ is less likely to be known than the total mass of a given halo). We will investigate both of these options more in the future, as the period of the soliton’s oscillations could be key in searching for ULDM observables in galaxies. Note, however, that any relation will depend both on the final mass of the core
and the halo’s formation and merger history. Thus, a tighter understanding of the core-halo mass relation presented in the previous chapter will be a necessary prerequisite for reliable predictions of solitons’ dynamical timescales.

6.5 Discussion

In summary, binary soliton mergers don’t show enough variation in the $\Xi$ parameter to probe the core-halo mass relation thoroughly and easily; however, they are excellent test beds for qualitatively describing halo products based on the initial conditions of the merger. This is an especially interesting case for hierarchical halo formation, as mergers featuring more than two solitons would be exceptionally rare in the Universe due to the fine-tuning of initial conditions necessary to achieve such a merger.

We varied the initial mass, mass ratio, kinetic energy, and angular momentum of our binary mergers. In each case we decomposed the final product into its constituent eigenstates. Regardless of initial conditions, the final soliton on average accounted for 57% of the final halo’s mass, as predicted by the small variation in $\Xi$. Increasing initial soliton velocity (and, consequently, kinetic energy) results in halos virtually indistinguishable from each other for $v < 10$; above that, the system is close to unbound, and the contribution of $\ell = 0, 2, 4$ modes is suppressed. This suggests high-velocity mergers excite more $\ell > 4$ modes than low-velocity mergers. Increasing angular momentum similarly produces virtually indistinguishable solitons, though increasing the impact parameter suppresses $\ell = 2$ modes, again suggesting that $\ell > 2$ modes make up a higher percentage of the halo’s mass.

Varying mass shows more evident differences in the final halos. Increasing the mass of the solitons (but keeping their mass ratio at one) suppresses the total contribution of $\ell = 0, 2, 4$, presumably exciting higher modes to larger amplitudes. Runs with mass ratios $\mu > 1$ paint a more complicated picture: not only do
they cause more turbulence in the core’s evolution, runs with \( \mu \geq 2.0 \) excite fewer modes with \( \ell > 2 \). Evident differences between \( \mu > 1 \) and \( \mu = 1 \) simulations were also noted in the previous chapter, opening two possibilities: either these cases tell us something important about what ULDM halo phenomenology would look like, or these runs suffer from more adverse numerical effects. We will certainly run significantly more tests to quantify the latter option before attempting to publish any work involving these runs in the coming months.

We additionally consider oscillation frequencies of halo cores produced through mergers. While changing initial kinetic energy and angular momentum evidently doesn’t impact frequency, tweaking initial mass and mass ratios evidently do. This is expected due to the relationship between frequency and eigenenergy, \( \nu \propto (E_{000} - E_{n\ell m}) \). Our eigenenergies scale as \( M^2 \), and we find roughly the same scaling in frequency when Fourier transforming the evolution of our halos’ cores. Once again, the \( \mu > 1 \) runs seem to be outliers, further prompting additional investigation of these simulations.
Ponder Stibbons was one of those unfortunate people cursed with the belief that if only he found out enough things about the universe it would all, somehow, make sense.

Terry Pratchett, The Last Continent

7

Conclusion and Future Work

7.1 Conclusion

We live in an exciting time of dark matter exploration. After decades of hearing hoof beats and thinking “horses”*—but not seeing them!—the field has begun to move towards considering not just zebras†, but also centaurs, unicorns, and hippogriffs. One such unicorn solution is a tiny particle with a exciting phenomenology called UltraLight Dark Matter. In this dissertation, I explored the phenomenol-

*In this metaphor, the horses are rather…wimpy.
†Though zebras can be quite MONDane, depending on where one lives and works.
ogy of ULDM through HPC simulations. The small mass of the ULDM particle gives rise to wave effects on halo-scales, allowing us to model the candidate’s dynamics through the halo’s eigenstates. This is a novel approach, developed and tested in papers and work that form the bulk of this thesis, and which will continue to drive my postdoctoral research.

In summary, the main results of this dissertation are:

1. ULDM halos with approximately stationary potentials can be decomposed into an appropriate set of eigenstates, with the ground state accounting for the halo core and higher states making up its skirt. We were able to validate the evolution of perturbed solitons in our numerical solver CHPLULTRA with perturbation theory. At present, ULDM halo’s are not good candidates for using a perturbative approach. However, analyzing the evolution of their eigenstates is still an illuminating exercise, which showed that 1) mass is mostly not exchanged between modes with different \( \ell \)-numbers and 2) our halos do not show an obvious relaxation timescale.

2. Halos formed through more symmetric initial conditions are easier targets for eigenstate decomposition; in particular, they are useful for probing the relative mass of the core and the halo known as the core-halo mass relation \( M_c/M_{halo} \propto \Xi^\alpha \). We found that the best-fit scaling parameter \( \alpha \) depends on the formation history of halos. Halo’s formed through simultaneous mergers of \( 2 < N < 15 \) equal mass solitons revealed a scaling \( \alpha \approx 1/4 \), while sequential mergers of similar solitons favored \( \alpha \approx 1/3 \); unequal mass binary mergers followed \( \alpha = 1/2 \). Each of these results has appeared in the literature before under similar conditions, suggesting that the discrepancy in \( \alpha \) could be due to halo’s histories rather than simply the initial conditions captured in \( \Xi = |E|/M^3 \).
3. Halos formed through binary soliton collisions are of particular interest given that halos are known to grow via hierarchical formation. Investigating the effects of different masses, mass ratio, kinetic energy, and angular momentum in the binary has been useful for providing a qualitative picture of what such halos might look like. Somewhat surprisingly, neither the addition of angular momentum nor kinetic energy significantly changes the eigenstate makeup of the final halo product. However, changing the mass or mass ratio of the initial solitons shows qualitative changes in the behavior of the final core; most notably, these mergers result in different oscillation frequencies from the soliton mode which scale as $\nu \propto M^2$. These oscillations may present key ULDM observables, thus calling for a tighter understanding of their frequency and scale.

One additional chapter dealt with primordial black holes in the very early Universe. The results of my black hole analysis suggest:

4. It is possible that a population of GUT-scale PBH existed and merger immediately after the end of inflation, ushering a temporary matter-dominated phase; however, the gravitational signature of merging black holes with a monochromatic mass function would not be observable by aLIGO or LISA.

7.2 Future Directions

The eigenstate analysis developed in this thesis can be applied to elucidate a number of ULDM scenarios, as well as other wave phenomena. One such scenario involves testing observable signatures of ULDM in our galactic neighborhood via stellar streams in the Milky Way. Stellar streams are thin, elongated structures which form from the tidal disruption of globular clusters. They are sensitive to dynamical heating via turbulence—a signature of the quantum nature of ULDM
—which creates caustics in the stream\textsuperscript{164}. These caustics erase the imprint of the de Broglie wavelength of ULDM in density power spectra of streams, but not in quantities such as angular momenta. Identifying individual modes in simulations of stellar streams around a Milky Way-sized ULDM halo will test consequences of ULDM which might remain imperceptible when considering just the holistic halo, as well as map the angular momentum transfer from the halo to the streams. Furthermore, an eigenstate analysis could serve to break degeneracy between observational predictions of dark matter and tease apart non-ULDM sources of dynamical heating, subsequently reducing noise in predicted traces of the particle’s de Broglie wavelength in Gaia data.

Another possible target for my eigenstate analysis are ULDM halos hosting supermassive black holes (SMBH). ULDM halos containing SMBH will develop different oscillatory behavior (the “breathing mode”) and may even eject the black hole depending on initial conditions\textsuperscript{180}. Calculating eigenstates that arise from different initial conditions will lead to predictions of oscillation frequencies set by their eigenenergies, which will in turn lead to predictions of the behavior of the black hole. Considering a range of halo and black hole parameters, it is possible to calculate the expected gravitational wave (GW) background from such ULDM-SMBH interactions (of interest to LISA), which can be subsequently extended to include mergers of ULDM halos hosing SMBH (of interest to aLIGO). The latter will also be useful for building on my core-halo mass relation work by considering hierarchical formation of halos hosting SMBH.

The same mechanism can be applied to studying interactions of the inflaton condensate (governed by the SP system) and primordial black holes (PBH) in the very early Universe. This would be natural extension of my PBH work, which considered only a population of black holes with a monochromatic mass function. Using Chplultra to model the behavior of the inflaton background and its collapsed
structures—solitons, or “inflaton stars”—could reveal a better choice of mass function for such a calculation. While the stochastic GW background from the formation and merger of inflaton stars has been investigated, the subsequent production of and interaction with PBH remains unexplored. A detailed exploration of inflaton-PBH dynamics using CHPLULTRA may reveal a variety of observable consequences, from another source of a primordial power spectrum (of interest to aLIGO and LISA) to impacts on the observable perturbation spectrum (imprinted in early cosmological structures to be imaged by JWST).

Finally, it could be possible to expand the applicability of perturbation theory with the use of machine learning (ML) to find optimal bases in which to calculate the evolution of ULDM systems. As shown Chapter 4, ULDM halos are somewhat tumultuous and do not naturally tend to a relaxed late-time state in an isolated system; this limits the predictive power of the eigenstate picture for halos formed through highly non-symmetric initial conditions. However, it is possible that there exists a basis in which the system can be described by a smaller number of modes, or simply shows fewer fluctuations in time. A better-suited basis for expanding turbulent halos could drastically extend applicability of PT, thus saving on computational resources, maximizing resolution, and extending our work on the core-halo mass relation. Finding such a basis would require alternate solutions of the differential equations governing ULDM: a task very well suited to neural network ODE solvers. Even when a closed-form solution to an ODE does not exist, these solvers are adept at providing approximate solutions.
Calculating the gravitational potential

We require the gravitational potential $\Phi$ from densities of the form $\rho_{lm}(r)Y_{lm}(\theta, \varphi)$

$$\nabla^2 \Phi = 4\pi \rho_{lm}(r)Y_{lm}(\theta, \varphi), \quad (A.1)$$

where we assume that the potential vanishes at infinity. Recalling that the spherical harmonics are eigenfunctions of the angular Laplacian, the solution must have the form $\Phi = \Phi^{\text{rad}}(r)Y_{lm}$. Making the change of variables $y = r\Phi^{\text{rad}}$, the radial part
of Poisson’s equation becomes

\[ \frac{\partial^2 y}{\partial r^2} - \frac{\ell(\ell + 1)}{r^2} = 4\pi r \rho_{lm}(r) \quad (A.2) \]

with boundary conditions

\[ y(r = 0) = 0 \quad (A.3) \]
\[ y(r_{\text{max}}) = -\frac{4\pi}{2\ell + 1} \frac{1}{r_{\text{max}}^\ell} \int_0^{r_{\text{max}}} dr' r'^2 \rho_{lm}, \quad (A.4) \]

where the upper boundary condition follows directly from the Laplace expansion of the Green’s function for a $1/r$ potential, assuming that the density has vanished by $r_{\text{max}}$. Note that for $\ell = 0$, the upper boundary condition is simply $y = -M$ where $M$ is the total mass, as expected for a spherically symmetric problem. We solve this by rewriting the differential equation as a linear algebra problem, similar to our treatment of the Schrödinger equation. Note that we could have just as easily just used the Green’s function, but we find the linear algebra approach more convenient computationally.

Given the potential, we are able to calculate its expectation value with any two other states as follows:

\[ \langle j | \Delta \Phi_{0p} | k \rangle = \int dr d\Omega (f_j^* Y_{jm}^*)(\Delta \Phi_{0p}^{\text{rad}} Y_{0m}^* Y_p)(f_k Y_{km}) \]
\[ = (4\pi)^{-1/2} \int dr f_j^* \Delta \Phi_{0p}^{\text{rad}} f_k \int d\Omega Y_{jm}^* Y_p Y_{km}, \]

where we used the shorthand $j = n_1 \ell_1$ and $k = n_2 \ell_2$ when comparing to Eq. 4.18. Here, we are using $\Delta \Phi_{0p}^{\text{rad}}$ to refer to the radially-dependent piece of the potential arising from the product of state $p$ with the ground state, while its spherical behavior is captured by the two spherical harmonics. Thus, we can split the integration into the radial piece (which is the same as the spherically symmetric case in
Sec. 4.3.2) and a new aspherical piece. Performing the replacement $Y_0 = (4\pi)^{-1/2}$
our angular piece becomes an integral over three spherical harmonics, equivalent
to a Wigner 3j symbol\textsuperscript{181}. 


