2-1-1977

An Extension of the Brown-Robinson Equivalence Theorem

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COWLES FOUNDATION DISCUSSION PAPER NO. 444

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February 1, 1977

Revised June 1, 1977

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I. Introduction

In this note, we show in Theorem 1 that if the initial allocation is integrable, in the sense that "numerically negligible" coalitions are "economically negligible," then every core allocation of a nonstandard exchange economy is integrable. This result allows us to extend the Equivalence Theorem of Brown-Robinson.

Brown-Robinson (see Theorem 1 in [1]), assumed that all allocations--including the initial allocation--were standardly bounded in showing that an allocation is in the core of a nonstandard exchange economy iff it is a competitive allocation. We show in Theorem 2 that their proof is essentially valid for a much larger class of nonstandard exchange economies, where only initial allocations are required to be integrable.

As a consequence of their Equivalence Theorem, Brown-Robinson in [2] proved a limit theorem on the existence of "approximately" competitive prices for core allocations in a sequence of economies. Of course, the conditions on the economies in the sequence are such as to guarantee--among other things --that the initial allocations and the core allocations in the nonstandard

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*The research described was undertaken by grants from the Ford Foundation and the National Science Foundation.

limit economy are standardly bounded. Consequently, Theorem 2 in this paper suggests a generalization of the Brown-Robinson Limit Theorem to a wider class of sequence economies, where no assumptions are required on the core allocations and whose associated nonstandard limit economies need only have integrable initial endowments. This generalization is given in Theorem 3.

II. The Results

In this note we conform to the notation and concepts as used by Brown-Robinson [1]. However, wherever they use the term allocation or final allocation, we shall take it to mean instead any assignment $Y(t)$ from the set of traders, $\{T = 1, 2, \ldots, \omega\}$, into ${}^*\Omega^n$ such that

$$(1/\omega) \sum_{t=1}^{\omega} Y(t) \simeq (1/\omega) \sum_{t=1}^{\omega} I(t) .$$

We shall also need the following definition: An allocation $Y(t)$ is said to be integrable if

$$(1/\omega) \sum_{t=1}^{\omega} Y(t) \text{ is finite and } |S|/\omega \simeq 0 \Longrightarrow 1/\omega \sum_{t \in S} Y(t) \simeq 0 .$$

In [1], Brown-Robinson assumed that $I(t)$, the initial allocation, was standardly bounded. This is assumption (ii) on page 44 in their paper. We shall replace (ii) by the weaker assumption: (ii)' $I(t)$ is integrable.

Theorem 1. If $\mathcal{E}$ is a nonstandard exchange economy satisfying assumptions (i) to (iv) (see [1], page 44), with (ii)' substituted for (ii), then every allocation X in the core of $\mathcal{E}$ is integrable.

This Theorem reduces to the following two lemmas:

<u>Lemma 1</u>. $(\forall \epsilon > 0)(\exists \bar{n} \in N)\{(|S^{\bar{n}}|/\omega) \geq 1-\epsilon\}$ where $S^n = \{t \in T | X(t) \leq ne\}$.

<u>Proof</u>. $(|S^{\nu}|/\omega) \simeq 1$ $(\forall \nu \in {}^*N-N)$. If not, we contradict the fact that X is an allocation by virtue of (ii)'. Let $\mathcal{A} = \{n \in {}^*N | (|S^n|/\omega) < 1-\epsilon\}$. If $\mathcal{A}$ is empty, the proof is finished. If not, being an internal, star-finite set, $\mathcal{A}$ has a greatest element, say ρ , and $\rho \notin {}^*N-N$. Let $\bar{n} = \rho+1$, which completes the proof.

<u>Lemma 2</u>. If X is any allocation in the core of $\mathcal{E}_\omega$, then for any commodity r and any internal, negligible set V , $(1/\omega) \sum_{t \in V} X_r(t) \simeq 0$.

<u>Proof</u>. Suppose not, i.e. there exists a set C of commodities, say 1, 2, ..., k , $k \leq n$; and internal, negligible sets V_i such that $(1/\omega) \sum_{t \in V_i} X_i(t) \not\simeq 0$. $(\forall i \in C)$. Let $V = \bigcup_{i \in C} V_i$ and $(1/\omega) \sum_{t \in V} X_i(t) = m_i$ $(\forall i \in C)$. Certainly $|V|/\omega \simeq 0$ and $m_i \not\simeq 0$ for all $i \in C$.

Let $H^i_n = \{t \in T - V | X_i(t) \geq 1/n\}$ for all $i \notin C$. We can assert that there exists $r_i \in N$ such that $|H^i_{r_i}|/\omega \not\simeq 0$. Suppose not. Then by Robinson's Theorem ([4], page 65), there exists $\nu \in {}^*N-N$ such that $|H^i_\nu|/\omega \simeq 0$. Certainly $(1/\omega) \sum_{t \in T-H^i_\nu} X_i(t) \simeq 0$. If $(1/\omega) \sum_{t \in H^i_\nu} X_i(t) \not\simeq 0$, $i \notin C$, then we contradict the definition of C . Thus $(1/\omega) \sum_{t \in T} X_i(t) \simeq 0$, and we have a contradiction, by virtue of (iii), to the fact that X is an allocation. If $r = \operatorname{Max}_{i \notin C} r_i$, and $\tilde{H}_r = \bigcup_{i \notin C} H^i_r$, then $|\tilde{H}_r|/\omega \not\simeq 0$. By Loeb's Theorem, see Appendix of [3], we can pick $H_r \subset \tilde{H}_r$ such that H_r is internal and $0 \not\simeq |H_r|/\omega \not\simeq 1$. Let $|H_r|/\omega = \epsilon$.

Lemma 1 guarantees the existence of $v \in N$ such that $|S^v|/\omega \geq 1 - (\varepsilon/2)$. Let $W = H_r \cap S^v$, then $|W|/\omega \not\simeq 0$. If not, $|H_r \cup S^v|/\omega = |H_r|/\omega + |S^v|/\omega - |H_r \cap S^v|/\omega \simeq 1 + \varepsilon/2$, which contradicts that $H_r \subset T$ and $S^v \subset T$.

For any $t \in W$ let $Y(t) = X(t) + (\omega/2|W|)\ (m_1, \ldots, m_k, 0, \ldots, 0)$. From (iv)$\gamma$ $\mu(Y(t)) >_t X(t)$. Define $\Delta = \{\delta(t)|\bar{S}(Y(t), \delta(t)) >_t X(t)\}$ where $\bar{S}(x, \nu)$ is a closed ball of radius ν centered at x and $\bar{S}(x,\nu) >_t X(t)$ means all points in the closed ball are preferred to $X(t)$. Δ is Q-closed and given the irreflexivity of $>_t$, bounded from above. Thus $\underset{\delta(t)\in\Delta}{\text{Max}}\ \delta(t)$ exists. Denote it to be $\bar{\delta}(t)$. $\bar{\delta}(t) \not\simeq 0$; otherwise we contradict that $\mu(Y(t)) >_t X(t)$ if and only if $S(Y(t), \delta) >_t X(t)$ for some $\delta \not\simeq 0$, a fact proved by Khan in [3], page 561.

Let $\bar{\delta} = \underset{t\in W}{\text{Min}}\ \bar{\delta}(t)$. $\bar{\delta}$ is well defined since W is an internal star-finite set and $\bar{\delta}(t)$ has been chosen in an internal manner. Let $\eta = \text{Min}(1/r, \bar{\delta})$, $B = T-V$ and considering the following

$$Z(t) = Y(t) - \{0, \eta/2\} \qquad (\forall t \in W)$$

$$= X(t) + (\omega/2|B-W|)\ (m_1, \ldots, m_k, \eta(|W|/2\omega), \ldots, \eta(|W|/2\omega)) \qquad (\forall t \in B-W).$$

$\{0, \eta/2\}$ is a vector with 0 in the first k coordinates and $\eta/2$ elsewhere. By construction, $\mu(Z(t)) >_t X(t)$ $(\forall t \in W)$. By (iv)β, $\mu(Z(t)) >_t X(t)$ $(\forall t \in B-W)$.

Now, for any $i \in C$,

$$(1/\omega) \sum_{t \in B} (Z_i(t) - I_i(t)) = (1/\omega) \sum_{t \in B} (X_i(t) - I_i(t)) + m_i/2$$

$$\lneq (1/\omega) \sum_{t \in B} (X_i(t) - I_i(t)) + m_i = (1/\omega) \sum_{t \in B} (X_i(t) - I_i(t)) + (1/\omega) \sum_{t \in V} X_i(t)$$

$$\simeq (1/\omega) \sum_{t \in T} (X_i(t) - I_i(t)) \quad \text{since} \quad 1/\omega \sum_{t \in V} I_i(t) \simeq 0 .$$

For any $i \notin C$,

$$(1/\omega) \sum_{t \in B} (Z_i(t) - I_i(t)) = (1/\omega) \sum_{t \in B} (X_i(t) - I_i(t))$$

$$- (1/\omega) \sum_{t \in W} \eta/2 + (|W|/|B-W|) \sum_{t \in B-W} \eta/4\omega$$

$$\lesssim (1/\omega) \sum_{t \in B} (X_i(t)) - (1/\omega) \sum_{t \in T} I_i(t) - (1/\omega) \sum_{t \in W} \eta/2$$

$$\lneq 0 \quad \text{since} \quad (1/\omega) \sum_{t \in W} \eta/2 \gneq 0 .$$

Thus we have an allocation Z that blocks X via the non-negligible coalition B, a contradiction to the fact that X is in the core.

Q.E.D.

<u>Proof of Theorem 1</u>. $(1/\omega) \sum_{t=1}^{\omega} X(t)$ is finite because of (ii)'. Thus suppose that there exists a set of commodities C and an internal negligible set V such that $(1/\omega) \sum_{t \in V} X_r(t) \not\simeq 0$ for all $r \in C$. We then have a contradiction to Lemma 2.

<u>Theorem 2</u>. If $\mathcal{E}$ is a nonstandard exchange economy satisfying assumptions (i) to (iv), with (ii)' substituted for (ii), then an allocation X is in the core of $\mathcal{E}$ if and only if there exists a price vector p such that (p, X) is a competitive equilibrium of $\mathcal{E}$.

Proof of Theorem 2. In view of Theorem 1, we know that under (ii)', all core allocations are integrable. Thus all we need to show is that the proof of Theorem 1 of [1] carries over to the case where X and I are integrable, instead of being standardly bounded.

The first change in the proof is on page 47 where $F_n(t)$ is now defined as $\{\bar{x} \in R_n | (\forall \bar{w} \in S_{1/n}(\bar{x}))\bar{w} >_t X(t), \bar{x} - I(t) \text{ finite}\}$ $G_n(t)$ is defined as before, i.e. $G_n(t) = F_n(t) - I(t)$, and each vector in $G_n(t)$ is finite, although $I(t)$ might be infinite. Letting $G(t) = \bigcup_{t \in N} G_n(t)$ and $\Delta(U)$ = the S-convex hull of $G(t)$, we see that $\Delta(U)$ is near standard. Consequently we can apply the separating hyperplane theorem if we prove the Principal Lemma. For this make no change until the fourth paragraph of page 48 where now the arbitrary t_0 in U is chosen such that $X(t_0)$ is finite. The remainder of the proof of the Principal Lemma goes through.

The first half of Theorem 1 of [1] needs no changing, since the only property of standardly bounded allocations they use is $|S|/\omega \simeq 0 \implies \frac{1}{\omega}\sum_{t \in S} X(t) \simeq 0$, i.e. the defining property of integrable allocations.

Starting in the second paragraph on page 50, let X be in the core of $\mathcal{E}_\omega$. and U be a full set of traders in the Principal Lemma. Then by the S-separation lemma there exists a standard $\bar{p} \neq \bar{0}$ s.t. $x \in S - \text{Int}(\Delta U)$, $\bar{p}\cdot\bar{x} \geq 0$. Hence $\bar{y} \in S - \text{Int}(G(t)) = G(t)\bar{p}\cdot\bar{y} \gtrsim 0$. This is equivalent to saying that $\bar{p}\cdot\bar{x} \gtrsim \bar{p}\cdot I(t)$ for all $x \in S - \text{Int}(F(t)) = F(t)$.

Let $\mathcal{F} = \{t \in T | X(t)$ and $I(t)$ are finite$\}$ $\mathcal{F}$ is external but this won't be a problem. Clearly $\mathcal{F} \neq \emptyset$. For all $t \in \mathcal{F}$ and all standard $\bar{z} \geq \bar{0}$ we can show that $\bar{p}\cdot(\bar{z} + X(t)) \gtrsim \bar{p}\cdot I(t)$, same reasoning as appears on page $\mathcal{F}$. But this implies that $\bar{p} \geq \bar{0}$ and $\bar{p}\cdot X(t) \gtrsim \bar{p}\cdot I(t)$ for all $t \in \mathcal{F}$.

Let $A_m = \{t \in T | \bar{p}\cdot X(t) + 1/m \leq \bar{p}\cdot I(t)\}$ for each $m \in N$.

Suppose for some $m \in N$, that $|A_m|/\omega = \delta \not\simeq 0$. Since for $\epsilon = \delta/4$, there exists $\ell \in N$ such that $|\{t \in T | X(t)| \geq \ell\}|/\omega < \epsilon$ and $|\{t \in T | |I(t)| \geq \ell\}|/\omega < \epsilon$, we see that $|\{t \in A_m | |X(t)| < \ell, |I(t)| < \ell\}|/\omega \geq \delta/2$. But this contradicts $\bar{p}\cdot X(t) \gtrsim \bar{p}\cdot I(t)$ for all $t \in \mathcal{J}$. Hence A_m negligible for all $m \in N$ and therefore except for a negligible set of t, $\bar{p}\cdot X(t) \gtrsim \bar{p}\cdot I(t)$.

We can now show that except for a negligible set of t, $\bar{p}\cdot X(t) \simeq \bar{p}\cdot I(t)$. If for some non-negligible internal set, S, we have $\bar{p}\cdot X(t) \gtrapprox \bar{p}\cdot I(t)$, then $\frac{1}{\omega}\sum_{t \in S} \bar{p}\cdot X(t) \gtrapprox \frac{1}{\omega}\sum_{t \in S} \bar{p}\cdot I(t)$, which contradicts the assumption that X is in the core, i.e. $\frac{1}{\omega}\sum_{t \in T} X(t) \simeq \frac{1}{\omega}\sum_{t \in T} I(t)$.

To complete the proof we must show that $X(t)$ is maximal in t's budget set. We will first show that $\bar{p} \gg \bar{0}$. Suppose not: let $p^1 \simeq 0$, say. Since $\bar{p}$ is standard some coordinate of $\bar{p}$ is not infinitesimal, say $p^2 \gtrapprox 0$. But $\frac{1}{\omega}\sum_{t \in T} I^2(t) \gtrapprox 0$. Since X is an allocation it follows that $\frac{1}{\omega}\sum_{t \in T} X^2(t) \gtrapprox 0$, so there must be a non-negligible internal set of traders S, for whom $X^2(t) \gtrapprox 0$. Let $|S|/\omega = \delta \gtrapprox 0$. Pick $\epsilon = \delta/4$, then there exists $\ell \in N$ such that $|\{t \in T| |X(t)| \geq \ell\}|/\omega < \epsilon$ and $|\{t \in T| |I(t)| \geq \ell\}|/\omega < \epsilon$. Therefore $|\{t \in S| |X(t)| < \ell, |I(t)| < \ell\}|/\omega \geq \delta/2$. Let $\{t \in S| |X(t)| < \ell, |I(t)| < \ell\}$. Now for any trader t, it follows from desirability that $X(t) + (1, 0, \ldots, 0) >_t X(t)$. Choosing $t \in E$ we see by continuity that for some sufficiently small $\epsilon \gtrapprox 0$, $X(t) + (1, -\epsilon, 0, \ldots, 0) >_t X(t)$. Hence $X(t) + (1, -\epsilon, 0, \ldots, 0) \in F(t)$. Therefore $\bar{p}\cdot I(t) \lesssim \bar{p}\cdot[X(t) + (1, -\epsilon, 0, \ldots, 0)] = \bar{p}\cdot X(t) + p^1 - \epsilon p^2 \lnapprox \bar{p}\cdot X(t)$. Therefore $\bar{p}\cdot I(t) \lnapprox \bar{p}\cdot X(t)$ for all $t \in E$, but $|E|/\omega \gtrapprox 0$ which contradicts the fact proved above that except for a negligible set of t, $\bar{p}\cdot X(t) \simeq \bar{p}\cdot I(t)$. Consequently $\bar{p} \gg \bar{0}$.

Now for all $t \in \mathcal{J}$ the proof that $X(t)$ is maximal goes thru in the

the same manner as it appears in the last paragraph of page 50. Suppose for some $m \in N$, that $B_m = \{t \in T | \exists \bar{y} \in B_{\bar{p}}(t), \bar{y} \succ_t X(t), |\bar{y} - X(t)| \geq 1/m\}$ is non-negligible, say $|B_m|/\omega = \delta \gneq 0$. As before we can show that at least half of the traders in B_m are in $\mathcal{J}$, which is a contradiction. Hence B_m negligible for all $m \in N$ and therefore $X(t)$ is maximal in all but a negligible set of traders' budget sets. This completes the proof.

We now introduce the terminology necessary to state our limit theorem, Theorem 3.

We will assume that all agents in the economy have the same consumption set Ω_n, the positive orthant of R_n. The tastes of an agent are represented by a preference relation $\succ$ on Ω_n. The set of all preference relations is denoted by $\mathcal{P}$. $\not\succ$ will denote the complement of $\succ$ in $\Omega_n \times \Omega_n$. If $\mathcal{P}$ is given the topology of closed convergence, then $\mathcal{P}$ is a compact separable metric space. Henceforth we assume that $\mathcal{P}$ has the topology of closed convergence.

If $\bar{x} \in R_n$ and ϵ a real positive number then
$B_\epsilon(\bar{x}) = \{z \in R_n | |\bar{z}-\bar{x}| < \epsilon\}$.

If C and D are subsets of R_n, then $C \not\succ D$ means that $\exists c \in C$, $\exists d \in D$ such that $c \not\succ d$.

A preference $\succ \in \mathcal{P}$ is said to be monotone if $\bar{x} > \bar{y}$, i.e. $x_i \geq y_i$ for all i and for some j, $x_j > y_j$, then $\bar{x} \succ \bar{y}$.

A standard exchange economy $\mathcal{E}$ is a triple $\langle T, \rho, I \rangle$, where T is a finite initial segment of the natural numbers, i.e., $|T| = n \in N$; T is the set of traders; ρ is a function from T into $\mathcal{P}$. $\rho(t)$ is the preference relation of trader t and will be denoted $\succ_t$; I is a function from T into Ω_n. $I(t)$ is the initial endowment of trader t;

An allocation X is a function from T into Ω_n such that $\frac{1}{|T|} \sum_{t \in l} X(t) \leq \frac{1}{|T|} \sum_{t \in T} I(t)$.

An allocation X blocks an allocation Y via a coalition E if $\frac{1}{|T|} \sum_{t \in E} X(t) \leq \frac{1}{T} \sum_{t \in E} I(t)$ and $(\forall t \in E) X(t) >_t Y(t)$.

The core of $\mathcal{E}$, $\mathcal{C}(\mathcal{E})$, is the set of unblocked allocations.

A sequence of standard exchange economies $\mathcal{E}_n = \langle T_n, \rho_n, I_n \rangle$ for $n = 1, 2, \ldots$, is said to be purely competitive if

(1) $|T_n| \to \infty$ as $n \to \infty$;

(2) $\lim_{n \to \infty} \frac{1}{|T_n|} \sum_{t \in T_n} I_n(t)$ exists;

(3) $E_n \subseteq T_n$ and $\lim_{n \to \infty} \frac{|E_n|}{|T_n|} = 0 \Longrightarrow \lim_{n \to \infty} \frac{1}{|T_n|} \sum_{t \in E_n} I_n(t) = \overline{0}$.

<u>Theorem 3</u>. Let $\mathcal{E}_n = \langle T_n, \rho_n, I_n \rangle$ be a purely competitive sequence of standard exchange economies, where

(1) $(\exists \overline{\beta} \gg \overline{0})(\forall n \in N)(\forall t \in T_n) I_n(t) \geq \overline{\beta}$;

(2) For all $\epsilon > 0$, there exists a compact set of monotone preferences $\mathcal{K}$ and there exists $n_0 \in N$ such that for all $n \in N$, if $n \geq n_0$, then

$$\frac{|\{t \in T_n | t \in \mathcal{K}\}|}{|T_n|} \geq 1-\epsilon .$$

Then $(\forall \delta > 0)(\exists n_0 \in N)(\forall n \in N)(\forall X \in \mathcal{C}(\mathcal{E}_n))(\exists \overline{p})$

$$\left[n \geq n_0 \Longrightarrow \frac{|\{t \in T_n | (\exists \bar{y} \in \Omega_n) \bar{p} \cdot \bar{y} \leq \bar{p} \cdot I(t) \text{ and } B_\delta(\bar{y}) \succ_t B_\delta(X(t))\}|}{|T_n|} < \delta \right.$$

$$\left. \text{and } \frac{|\{t \in T_n | \bar{p} \cdot X(t) \geq \bar{p} \cdot I(t) + \delta\}|}{|T_n|} < \delta \right] .$$

Proof. The proof of Theorem 3 is essentially the same as the proof of the Brown-Robinson Limit Theorem in [2], hence we will not repeat it here.

ACKNOWLEDGMENTS

The idea for the proof of Theorem 2 is due to Abraham Robinson. We are also grateful for helpful discussions with Mr. Salim Rashid.

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