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AN ANALYSIS OF THE STIRRING AND MIXING PROCESSES IN INCOMPRESSIBLE FLUIDS

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INTRODUCTION

In the laboratory, the phenomena accompanying the mixture of two fluids are so commonplace that they do not seem to merit investigation. In the ocean and atmosphere, the phenomena are presumably very similar, but the lack of a good theory on which to base calculations complicates the problem much more than in the laboratory.

It is useful to consider a trivial experiment by way of introduction: the mixing of coffee and cream. Three more or less distinct stages can be observed:

1. *The initial stage*, in which rather large volumes of cream and coffee are distinctly visible; there are sharp gradients at the interfaces between the volumes, but elsewhere the gradient is practically zero. Averaged over the entire volume, the gradient is small. If motion of the liquids is avoided, this state persists for a considerable time.

2. *The intermediate stage*, after motion has been induced by stirring the liquids; the masses of cream and coffee are distorted, with a rapid increase in the extent of the interfacial regions having high concentration gradients. The average value of the gradient is correspondingly increased.

3. *The final stage*, in which the gradients disappear, apparently quite suddenly and spontaneously, with the liquid becoming homogeneous.

It is a reasonable working hypothesis to assume that these three stages (or at least the second and third) also occur in the ocean and atmosphere when concentration or temperature differences arise.

The potentially long duration of the first stage may be ascribed to the slowness of the diffusion or heat conduction process which results from the small average value of the gradients. During the second stage, the average value of the gradient is increased by the relative

motion of different parts of the liquid as a result of stirring. The disappearance of the gradients during the third stage is presumably caused by molecular diffusion. This is more rapid than during the first stage, because stirring has increased the gradients and thus has also increased the diffusion currents. For convenience, this last stage may be called mixing, to distinguish it from stirring.

TWO EXAMPLES OF STIRRING

As a first example of stirring, one may consider the formation of a steep thermocline by lateral advection; Fig. 1 represents a vertical
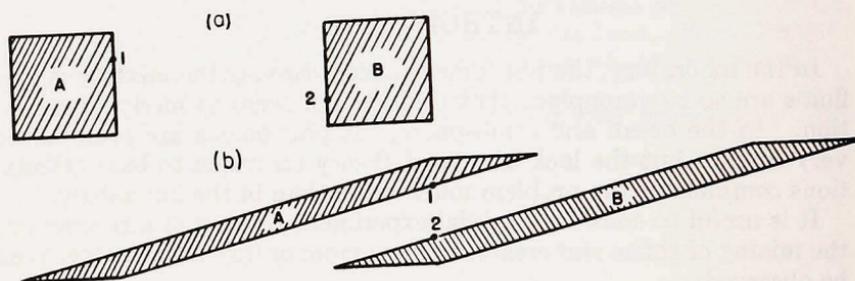

Figure 1.　Stirring action of laminar flow.

section of the fluid. Suppose that initially there is a small horizontal gradient, but that the regions A and B (Fig. 1a) are at appreciably different temperatures because of the relatively large distance separating them. Suppose further that the fluid everywhere moves horizontally, but that there is a vertical velocity gradient; the motion is laminar, with shear across the horizontal planes. Then, since the upper parts of the regions A and B move with a higher velocity than the lower parts, these regions will be distorted until eventually the configuration of Fig. 1b is produced. The particles 1 and 2 will have moved much closer together, although the temperature difference between them is unaltered. Thus the original small horizontal gradient will have been converted into a much larger vertical gradient.

This example suggests that the large thermal gradients shown on the bathythermogram of Fig. 2 may be indicative of regions of relatively large velocity gradients, and that the whole trace is indicative of horizontal shear in the layers between the depth of 120 and 240 feet. Possibly this suggestion could be tested experimentally.

As a second example, suppose the initial temperature distribution to be as before, but that an eddy is somehow set up with its axis horizontal and perpendicularly bisecting the line joining the centers of A and B

(Fig. 3a).   In this eddy, let the orbits of the fluid particles be circular and be described with an angular velocity ω that is a function of their distance from the axis of the eddy.   Then, after the lapse of successive intervals of time, the regions A and B of Fig. 3a will be deformed and
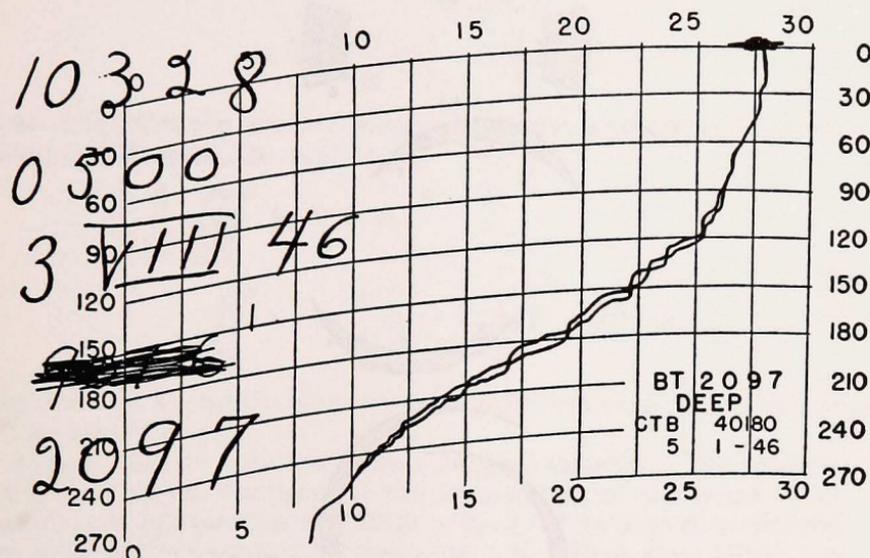

Figure 2.   Bathythermogram taken on August 3, 1946, from U. S. S. INGRAHAM at Lat. 11° 48′ N., Long. 164° 46′ E.   Temperature in °F., depth in feet.

assume the configuration shown in Figs. 3b and 3c.   It is clear that such an eddying motion will be very effective, not merely in increasing the mean gradient but also in increasing the interfacial area of the regions A and B and in decreasing the interfacial distance between them.   All three effects are such as to increase the flow of heat by conduction.   During the stage of stirring these three processes dominate; ultimately they may so increase the conduction of heat that the latter dominates and causes the mean gradient to diminish toward zero.

It is not difficult to show that, because of the stirring action, the mean gradient at a distance r from the center of the eddy is

$$\tfrac{1}{2} \mid ar \, (d\omega/dr) \, t \mid ,$$

where a is the initial horizontal gradient, and t is the elapsed time since the initial instant.   Thus, if the eddy persists long enough, the gradient, and consequently the heat flow, may become very large indeed.

Having considered these two examples of stirring, the next section

will examine the laws according to which heat conduction or diffusion operates to reduce the mean gradient. In the succeeding section, the manner in which these laws are modified by stirring will be treated in some generality.
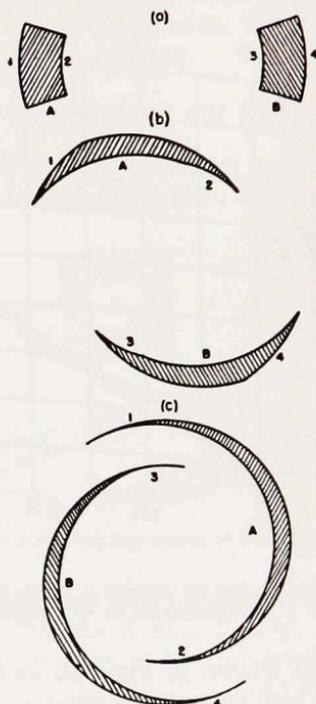

Figure 3.   Stirring action of circular eddy.

THE EQUALIZATION OF GRADIENTS BY CONDUCTION OR DIFFUSION

In order to avoid, initially, the complication of the stirring process, consider conduction or diffusion in a solid substance. The differential equation governing this is

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta , \tag{1}$$

where κ is the conductivity or diffusion constant, and θ the temperature or concentration. On differentiating this, one obtains

$$\frac{\partial}{\partial t} \nabla \theta = \kappa \nabla^2 \nabla \theta ; \tag{2}$$

multiplying scalarly by $\nabla\theta$, the result is

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla\theta)^2 = \kappa\,\nabla\theta\,\cdot\,\{\nabla^2\,\nabla\theta\}$$

$$= \kappa\,\nabla\cdot\{\nabla\theta\,\nabla^2\theta\} - \kappa\,\{\nabla^2\theta\}^2$$

$$= \nabla\,\cdot\,\left\{\frac{\partial\theta}{\partial t}\nabla\theta\right\} - \kappa\left\{\nabla^2\theta\right\}^2. \tag{3}$$

If one integrates the terms of this equation over a volume and uses the divergence theorem, the result is

$$\frac{1}{2}\frac{d}{dt}\iiint\limits_{vol}(\nabla\theta)^2\,d\tau = \tag{4}$$

$$\iint\limits_{bndry}\nu\,\cdot\,\nabla\theta\,\frac{\partial\theta}{\partial t}\,d\sigma - \kappa\iiint\limits_{vol}\{\nabla^2\theta\}^2\,d\tau\,,$$

where $d\tau$ and $d\sigma$ are the elements of volume and area, and ν the unit vector normal to $d\sigma$.

At first, assume that the surface integral vanishes. This will certainly be the case if either: (a) the temperature of each point of the boundary is constant, so that $\partial\theta/\partial t = 0$, or (b) no heat flows through any point of the boundary, so that $\nu\cdot\nabla\theta = 0$. Then, since $\{\nabla^2\theta\}^2 \geqslant 0$, the right side of (4) is either negative or at most zero. Hence the integral on the left must decrease, or at most remain constant.

Define the rms gradient, G, by the equation

$$G^2 = \frac{1}{V}\iiint\limits_{vol}(\nabla\theta)^2\,d\tau\,, \tag{5}$$

where V is the volume of the region under consideration. The quantity I, defined by

$$I^2 = \frac{1}{V}\iiint\limits_{vol}(\nabla^2\theta)^2\,d\tau\,, \tag{6}$$

is a convenient measure of the inhomogeneity of the material inside V. Using these definitions,

$$\frac{1}{2}\frac{dG^2}{dt} = -\,\kappa I^2\,. \tag{7}$$

For a given inhomogeneity, therefore, the mean square gradient decreases at a rate *directly* proportional to the constant κ.

The activity of the material inside V is measured by the quantity A, defined by

$$A^2 = \frac{1}{V} \iiint\limits_{vol} \left(\frac{\partial\theta}{\partial t}\right)^2 d\tau .$$

(8)

Because of (1), (7) may also be written

$$\frac{1}{2}\frac{dG^2}{dt} = -\frac{1}{\kappa}A^2 .$$

(7.1)

For a given activity, the mean square gradient thus decreases at a rate *inversely* proportional to the constant κ.

However, the most important conclusion is that whenever the surface integral of (4) vanishes the rms value of the gradient never increases. In the case of a small solid object in the laboratory, the conditions for the surface integral to vanish are easily fulfilled. However, if an arbitrary volume of the ocean or the atmosphere is considered, this is no longer true. The conduction of heat through the boundary, and its consequent change in temperature, may become very important. Moreover, the process of stirring must be considered.

MODIFIED EQUALIZATION THEOREMS WHEN THE SUBSTANCE IS AN INCOMPRESSIBLE FLUID

When the substance under consideration is an incompressible fluid, rather than a solid, (1) becomes

$$\frac{D\theta}{Dt} = \kappa\nabla^2\theta ,$$

(10)

or, using the indicial notation,

$$\frac{D\theta}{Dt} = \kappa\frac{\partial^2\theta}{\partial x_i^2} ,$$

(11)

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i} .$$

The differentiation of this equation (as above) encounters a complication, because

$$\frac{\partial}{\partial x_j}\frac{D}{Dt} = \frac{D}{Dt}\frac{\partial}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial}{\partial x_i} .$$

(12)

The analogue of (2) is therefore

$$\frac{D}{Dt}\frac{\partial\theta}{\partial x_i} = \kappa\frac{\partial}{\partial x_j}\frac{\partial^2\theta}{\partial x_i^2} - \frac{\partial u_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}, \tag{13}$$

and involves the velocity gradient $\partial u_i/\partial x_j$.

Multiplying by $\partial\theta/\partial x_i$, and summing,

$$\frac{1}{2}\frac{D}{Dt}\left(\frac{\partial\theta}{\partial x_i}\right)^2 = \kappa\frac{\partial}{\partial x_j}\left(\frac{\partial\theta}{\partial x_i}\frac{\partial^2\theta}{\partial x_i^2}\right)$$

$$- \kappa\left\{\frac{\partial^2\theta}{\partial x_i^2}\right\}^2 - \frac{\partial u_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}. \tag{14}$$

In order to obtain simple integrated results, it is now necessary to choose a volume V whose boundary is not stationary but which moves so that it always encloses the same fluid. Integrating over such a moving volume, the result is

$$\frac{1}{2}\frac{dG^2}{dt} = \iint_{bndry} \nu\cdot\nabla\theta\frac{D\theta}{Dt}d\sigma - \kappa I^2 - S, \tag{15}$$

where the additional term is

$$S = \iiint_{vol}\frac{\partial u_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}d\tau \tag{16}$$

and is a measure of the rate at which stirring decreases the mean square gradient.

The other terms of (15) have been adequately discussed above, so that it remains only to discuss the integrand of (16). The velocity gradient is usually resolved into two components

$$\frac{\partial u_i}{\partial x_j} = u_{[ji]} + u_{(ji)}, \tag{17.1}$$

where

$$u_{[ji]} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right\} \tag{17.2}$$

is recognized as being (except for a factor $-\frac{1}{2}$) the curl or vorticity of the motion, while the tensor

$$u_{[ji]} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\} \tag{17.3}$$

is called the deformation tensor and will be recognized from the familiar Stokes law of molecular viscosity.

It may be somewhat surprising to note that the integrand of (16) is independent of the curl, $u_{[ij]}$, but it can be shown that

$$\frac{\partial u_i}{\partial x_j}\frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_j} = u_{(11)}\left(\frac{\partial \theta}{\partial x_1}\right)^2 + 2u_{(12)}\frac{\partial \theta}{\partial x_1}\frac{\partial \theta}{\partial x_2} + 2u_{(13)}\frac{\partial \theta}{\partial x_1}\frac{\partial \theta}{\partial x_3}$$

$$+ u_{(22)}\left(\frac{\partial \theta}{\partial x_2}\right)^2 + 2u_{(23)}\frac{\partial \theta}{\partial x_2}\frac{\partial \theta}{\partial x_3}$$

$$+ u_{(33)}\left(\frac{\partial \theta}{\partial x_3}\right)^2.$$

This is an important result, because it is known that curl or vorticity is very difficult to generate in a large mass of fluid whose coefficient of viscosity is small. If it should have appeared that the opposite were true, that S depended only on the curl, then the hypothesis under investigation would have to be rejected. Such a result would have made the process of mixing so strongly dependent on viscosity as to be untenable. The astonishment at the result arises from the popular confusion of curl or vorticity with eddy motion. The second example above shows that eddy motion is very effective in mixing, possibly more so than laminar motion. On the other hand, it is a fact that in laminar motion the curl cannot be zero, whereas it is easy to give an example of eddy motion with circular orbits for which the curl vanishes; e. g., $\omega(r) = 1/r$. Thus, there is no inconsistency between (16) and the two examples discussed earlier.

In the examples, and in the more general discussion of stirring, it was indicated that the effect of the velocity gradients is usually to increase the mean temperature or concentration gradients. Therefore, it might be expected that the quantity S is always negative. This cannot be proven; in the case of an incompressible fluid, the velocity must satisfy the equation

$$\nabla \cdot u = 0,$$

or

$$u_{(11)} + u_{(22)} + u_{(33)} = 0.$$

Hence, usually at least one of the three terms of this equation (say $u_{(11)}$) will be positive and at least one (say $u_{(22)}$) will be negative. Under these conditions, a gradient having the x_1 direction would make $S > 0$, while a gradient having the x_2 direction would make $S < 0$. Thus S may have either sign, and stirring may sometimes have the effect of decreasing the mean gradient. This situation will be clarified by a re-examination of the first example.

MORE DETAILED EXAMINATION OF
MIXING BY LAMINAR FLOW

The example of stirring by laminar motion is simple enough so that fairly detailed calculations can be made. If the temperature (or concentration) at time $t = 0$ is

$$\theta = ax + by + c , \tag{18}$$

it is required to find a solution of

$$\frac{\partial \theta}{\partial t} + u(y) \frac{\partial \theta}{\partial x} = \kappa \nabla^2 \theta , \tag{19}$$

which also reduces to (18) when t vanishes.

When $\kappa = 0$, this solution is unique, being

$$\theta \equiv \theta_o = a[x - u(y)t] + by + c . \tag{20}$$

This represents the effect of advection or stirring. The component of the gradient in the y-direction is

$$\frac{\partial \theta}{\partial y} = b - at \frac{du}{dy} .$$

For small values of t, its magnitude may either increase or decrease, depending on the relation between a, b and du/dy. For sufficiently large values of t, however, the second term must dominate, and hence the magnitude of the gradient will ultimately increase.

This illustrates two general propositions. It has already been noted above that advection may either increase or decrease the gradients. The new proposition is that, ultimately, advection usually increases the magnitude of the gradient. This is not without exception. In the textbook example of small surface waves, the motion is everywhere periodic and everywhere has the same period. Consequently, the changes caused by advection must also be periodic and they therefore constitute an exception to the rule. Actual surface waves, however, are not periodic, and their advective effects presumably conform to the rule. Because of these exceptional cases, a general proof of the proposition will be difficult and will be analogous to the proof of the ergodic theorem.[2]

Turning next to the case $\kappa \neq 0$, one may set

$$\theta = \theta_o + \vartheta , \tag{21}$$

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[2] G. D. Birkhoff. Proof of a recurrence theorem for strongly transitive systems. Proc. nat. Acad. Sci., *17*: 650–655 (1931).

where ϑ is determined by the equation

$$\frac{\partial \vartheta}{\partial t} + u(y)\, \frac{\partial \vartheta}{\partial x} - \kappa\, \Delta^2 \vartheta \;=\; -\, a\kappa t\, \frac{d^2 u}{dy^2}\,. \qquad (22)$$

At $t = 0$, one must require $\vartheta = 0$; it is characteristic that this requirement does not uniquely determine ϑ. Physically this means that conditions at the boundary of the medium are important: the sun may rise at some time after $t = 0$ and cause warming, etc.

However, special solutions of (22) can easily be obtained. One, which is independent of x, is the series

$$\vartheta \;=\; -\, a\sum_{m=2}^{\infty} \frac{\kappa^{m-1} t^m}{m!} \left(\frac{d}{dy}\right)^{2m-2} u(y)\,, \qquad (23)$$

which can be considered as the solution that is applicable in the absence of such external effects as were just mentioned. It therefore determines the mixing process that is induced by laminar flow.

Since the series for ϑ begins with a term in t^2, it follows that, for sufficiently small values of t, the effect of mixing will be negligible compared to the advective effect which is measured by the term $-\, aut$ in (20). The time, t_c, at which the first term of the series for ϑ becomes equal to the advective term, is given by

$$t_c \;=\; 2u \Big/ \left(\kappa\, \frac{d^2 u}{dy^2}\right)$$

and is a function of y.

Since $u/(d^2 u/dy^2)$ has the dimensions of an area, it may be set equal to l^2, where l is a length that measures the homogeneity of the velocity distribution. Hence

$$t_c \;=\; 2l^2/\kappa. \qquad (24)$$

For water, the thermal diffusion constant is $\kappa = 1.4 \times 10^{-3}$ cm²/sec., while for the diffusion of dissolved salts, $\kappa = 2 \times 10^{-5}$ cm²/sec. This leads to the following values for t_c:

Mixing Times

l	thermal	salt
1 m.	168 d.	32 yr.
1 cm.	24 min.	28 hr.
0.1 mm.	0.15 sec.	10 sec.

These values suggest that the saline microstructure of the sea may be much more pronounced than the thermal microstructure, since diffusion can scarcely counteract the effect of mixing.

The solution for ϑ in the case of a circular eddy is more complicated but should be derivable without excessive difficulty.

THE EFFECT OF VISCOSITY

The two examples considered above are both steady states of flow, and it is unlikely that steady motion will persist over the relatively long time interval t_c.   In particular, it can be shown that viscosity will usually modify the motion considerably in such a length of time, unless it is balanced by a suitable pressure distribution.

The viscous effects are governed by equations that are very analogous to (10), except that κ is replaced by the kinematic viscosity n. For water n/κ (Prandtl's number) has the value 7.5.   Consequently, the viscous effects will become appreciable in a time that is less than that required for thermal mixing.   They will, in general, tend to increase l and hence slow down the mixing process.   However, this tendency may be counteracted by pressure gradients or wind stresses.

CONCLUSIONS

1. In all except certain very special cases, advection alone will ultimately increase the mean value of any initial gradient.

2. This effect of advection is appropriately called stirring.

3. Stirring is independent of the vorticity of the motion and can occur even if the motion is not turbulent.   It may be the explanation of the high gradients shown by some bathythermograms.

4. The effect of conduction or diffusion is to decrease the mean value of the gradient.

5. This is appropriately called mixing.

6. Ordinarily, the early stages of a process in which both stirring and mixing occur will be dominated by the advective processes.

7. These may so increase the mean gradient that the mixing process will ultimately dominate over the stirring process.

8. Viscosity, if not counteracted by other factors, tends to stop the stirring process before an appreciable amount of mixing can occur.