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ON A GENERAL APPROACH TO OPTIMALITY IN GAME THEORY

Eduardas Vilkas

January 20, 1976

ON A GENERAL APPROACH TO OPTIMALITY IN GAME THEORY*

by

Eduardas Vilkas**

Introduction

This paper follows the author's work in recent years on the optimality problem in game theory [11, 12]. The general approach to the problem is supplemented by its new applications to most of the well known solution concepts and by some preliminary definitions of a solution for a coalitional game in an extensive form.

The purpose is to make more game theorists familiar with the findings made when the approach appeared in English as well as with the possibilities of its application to such economic phenomena as coalitional dynamics, costly information, uncertainty, etc. These questions were brought to the author's attention by Professor M. Shubik, to whom the author is indebted for many helpful discussions and for his kind invitation to visit Yale.

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1. Definition of a Game

To discuss a general problem of optimality we need an appropriately general definition of a game. The following definition proved to be quite convenient for the purpose:

Definition 1. A game is the collection

$$\langle K, \{X_C\}_{C \in K}, \{S_C(x_C)\}_{C \in K, x_C \in X_C}, \{\succ_C\}_{C \in K} \rangle$$

where K is an arbitrary set of coalitions $C \subset N$, N being a set of players, X_C is a set of strategies of coalition $C \in K$, $S_C(x_C)$ is a set of outcomes of the game when strategy x_C is used, and \succ_C is a preference of coalition C over all outcomes of the game,

$$S = \bigcup_{C, x_C} S_C(x_C) .$$

This definition describes any known class of games in a natural way of specifying the components of game. However, some further remarks seem to be useful.

It is supposed that the game is over when collection of strategies $x(P) = (x_C)_{C \in P}$ is chosen, where P is any coalitional structure, i.e. any maximal collection of nonintersecting coalitions $C \in K$. At the same time there is no need to limit ourselves to this strategic form of the game. Nevertheless, even when realization of the game is of no importance, it is useful to keep the sets of strategies X_C in the definition for explicit indication of the arguments of players and coalitions taken into account, i.e. for specifying with extent of what arguments, actions, etc. the outcome of the game is optimal or stable.

The set of strategies X_C of coalition C need not be independent of the information, coalition C possesses at the moment or, in more narrow sense, of the situation appearing in the game. In a similar way, a set of possible outcomes $S_C(x_C)$ may depend on the previous choices by other coalitions and by itself. In this case, of course, appropriate specification of sets X_C and S_C is needed to keep the model logical.

An economic or social situation in any description can be treated as an outcome of the game. But usually we do not need any description other than the situation $(x_C)_{C \in P}$ when the coalitional structure is fixed, or the coalitional structure P when any coalition has only one strategy, or $(P, (x_C))$ in the most general case. Anyway, the choice of the form of outcome depends on the problem under consideration.

More discussions on Definition 1 one can find in the author's work [11], but we will, in fact not need any of them here.

2. A General Approach to Optimality

Optimality is the notion to say what strategies and preferences are given for. Strategy, of course, is a tool to achieve better outcomes or to expel those undesirable. Therefore, informally, an outcome is said to be optimal if no coalition objects to it (by certain strategy), or, if someone does, then there is a coalition which neutralizes the objection making it impossible or defends the outcome under consideration in one or another way. Unfortunately, because of multiplicity of goals, the general meaning of optimality does not work properly without further specification, which obviously is not unique.

Strategic concept of solution. Let us consider the strategic side of the solution's concept first.

Definition 2. A pair (C, x_C) , $C \in K$, $x_C \in X_C$ is said to be an objection against the outcome $s \in S$, if $s' \succ_C s$ for all $s' \in S(x_C)$.

Definition 3. A pair (Q, x_Q) , $Q \in K$, $x_Q \in X_Q$ is said to be a counter-objection to an objection (C, x_C) against s if

- (i) $s'' \succ_Q s'$ for all $s'' \in S(x_Q)$ and some $s' \in S(x_C)$
- (ii) $Q \cap C \neq \emptyset$,

i.e. counter-objection is an objection of intersecting coalition against an outcome from $S(x_C)$.

Definition 4. An outcome s is said to be optimal if to every objection against it there is a counter-objection. Denote the set of all optimal outcomes by φ .

It was shown in [11], that for all games (Definition 1) under purely mathematical restrictions of compactness and continuity, with transitive preferences \succ_C , the solution set φ is non-empty. Even stronger result is valid: one can put an additional restriction $Q \not\subset C$ on counter-objection [12].

We will return to the counter-objection and the solution φ after some remarks on non-strategic properties of an optimal solution.

Non-strategic properties of solution. The solution given in Definitions 2-4, is perhaps, the weakest possible. As we shall see in the following discussion, even in zero-sum two-person games we need a certain restriction of solution φ to get the usual optimal strategies, which is, in fact, non-strategic.

The most general mathematical description of needed properties of a solution of the game can be made by means of some functional equations and of boundary conditions to them. Let $f(\Gamma)$ be a solution of

the game $\Gamma \in G$, and let $\sigma : G \rightarrow G$, $\tau : H \rightarrow H$, where H is a set of values (maybe, subsets of S) of f on G . Then the equation,

$$f(\sigma\Gamma) = \tau f(\Gamma) \quad (1)$$

will describe the change of the solution imposed by the change of a game. Examples of such equations are well known: L. Shapley's additivity axiom, axioms due to A. Sobolev [7], compositions for von Neuman-Morgenstern solution (L. Shapley [5]), and for the kernel and nucleolus (M. Megiddo [1, 2, 3], Ć. Šimelis [8, 9, 10]). Despite mathematical attractiveness and easiness to interpret the equation (1) the approach did not become basic in game theory. It seems that not only mathematical difficulties caused this, but the tradition of strategic investigation of economic problems did as well.

One can notice that using continuity, monotonicity, symmetry, etc. known functional conditions for a solution can be regarded as a part of the described approach.

3. Solution for Non-Cooperative Game

In a non-cooperative game we have $K = \{i\}_{i \in N}$, $N = \{1, \dots, n\}$, $S = \{(x_1, \dots, x_n) : x_i \in X_i, i \in N\}$, and $x >_i y$, $x, y \in S$ iff $f_i(x) > f_i(y)$, where f_i is a payoff of player i .

Because there are no intersecting coalitions, no counter-objections can arise by Definition 3. It is easy to see that for a two-person zero-sum game with a payoff of the 1st player f a

$$\omega = \{(x_1, x_2) : \max_{x_1} \min_{x_2} f(x_1, x_2) \leq f(x_1, x_2) \leq \min_{x_2} \max_{x_1} f(x_1, x_2)\}$$

i.e. a general solution coincides with the classical one in the sense of the outcome value. However, they differ quite a lot in the sense of optimal behavior of players, because our notion simply does not say anything about it. For a more complete comparison we need some definition of optimal strategies. Naturally, we can call a strategy optimal, if it gives not less than the optimal payoff whatever the behavior of the partners would be. The definition leads to the classical optimality notion, but still is insufficient to get an equilibrium. This is not surprising, because the equilibrium is a special kind of behavior. The outcome $(x_1^*, \dots, x_n^*) \in \varphi$ is the equilibrium if it is still the optimal outcome when the set of strategies X_i is reduced to only one strategy x_i^* for all players $i \neq j$ and if it is true for all $j \in N$ [11].

When players are not going to play a game and the realization of strategies is transferred to the "third" party, it does not matter, of course, which situation from φ is chosen. A general solution concerns namely this case and the maxim or equilibrium strategies are possible, but not unique, specifications of a general solution.

Kernel. Many of the known solution concepts even for cooperative games may be rewritten in a strategic form. Generally, we can put into the sets of strategies the primary definition, or part of it, or some things, used by the authors to argue applicability of their solution concept. Doing so we make the definition of solution more precise. We will define the kernel as general solution to a non-cooperative game over the primary game, of which characteristic function is v .

Let $P = (P_1, \dots, P_m)$ be a coalitional structure and

$$S = \{x : \sum_{i \in P_j} x_i = v(P_j), j = 1, \dots, m; x_i \geq v(i), i \in N\} .$$

Let $e(C, x)$ be the excess,

$$e(C, x) = v(C) - x(C), \quad x(C) = \sum_{i \in C} x_i$$

and $s_{ij}(x)$ be the demand

$$s_{ij}(x) = \max_{C \in T_{ij}} e(C, x),$$

$$T_{ij} = \{C : C \subset N, i \in C, j \notin C\}.$$

Denote $x \parallel y_i, y_j$ the imputation, in which i^{th} component x_i is changed by y_i and j^{th} by y_j for $i, j \in P_k \in P$. Define the strategy set of player i as a family of sets $X_i(x)$, $x \in S$,

$$X_i(x) = \{y = x \parallel y_i, y_j : y_i \leq x_i + (s_{ij}(x) - s_{ji}(x)), \\ y_j = x_j - (y_i - x_i) > v(j), i, j \in P_k\}.$$

With

$$K = \{\{i\}\}, \quad i \in N$$

$$x >_i y \iff x_i > y_i$$

the game in the sense of Definition 1 is given.

Obviously, it is a non-cooperative game. Therefore the solution ϕ consists of the outcomes x , against which no player has objections.

If there is an objection to player i against x , then for some $j \in P_k \ni i$

$$x_i < y_i \leq x_i + (s_{ij}(x) - s_{ji}(x))$$

$$y_j = x_j - (y_i - x_i) > v(j)$$

which means

$$s_{ij}(x) - s_{ji}(x) > 0, \quad x_j > v(j) \quad (2)$$

and x is not in the kernel. And on the contrary if (2), then there is a $y \in X_1(x)$ with $y_1 > x_1$, which means objection of player i against x . So solution φ for this game is the kernel for the usual cooperative game v .

4. Counter-Objection: Discussion and Examples

Informally a counter-objection of coalition Q is an objection which makes the primary objection of coalition C against $s \in S$ doubtful, because it is useful for players from $C \cap Q \neq \emptyset$ to join coalition Q instead of coalition C . This is a reason to keep s on the list for further consideration.

This notion of counter-objection is as simple as possible, but the trouble is that solution φ is generally very large (sometimes all S , which is not necessarily the wrong thing) and therefore not very informative. In the previous section we gave the example (kernel) in which the objections were specified. Now we are going to make some remarks about possible concretization of counter-objections by adding new requirements for them.

The general purpose of the additional requirements is to point out that a counter-objection is stronger, more probable than an objection. As we mentioned above, objection and counter-objection are simply competitive and each of them is possible, but not both.

Counter-objection to any issue. Our further consideration is based on classical cooperative game, therefore we need it written in terms of Definition 1.

Model 1.

$$K = 2^N \setminus \{\emptyset\}$$

$$X_C = \{x_C = (x_i)_{i \in C} : x(C) \leq v(C), x_i \geq v(i), i \in C\},$$

$$S_C(x_C) = \{y : y_C = x_C, y(N) \leq v(N)\},$$

$$x >_C y \iff x_i > y_i \text{ for all } i \in C.$$

Model 2.

K and $>_C$ are the same.

$$X_C(x) = \{x : x(C) \leq v(C), x_i \geq v(i), i \in N, x(N) = v(N)\}$$

$$S_C(x) = \{x\}$$

The existence of a solution for model 1 follows from the general existence theorem [11]. For model 2 the existence is rather obvious under super-additivity condition. Indeed, if (C, y) is an objection without a counter-objection, then

$$y(C) \leq v(C)$$

(3)

$$y(S) \geq v(S), S \cap C \neq \emptyset.$$

Let \bar{y} maximize $y(C)$ under condition (3). Then coalition C has no objection without counter-objection against \bar{y} , and inequalities (3) mean that there are no objections against \bar{y} from coalitions intersecting C . But it follows from super-additivity, that any T , $T \cap C = \emptyset$ has no objection, too:

$$\bar{y}(T) + \bar{y}(C) \geq v(T \cup C) \geq v(T) + v(C), \bar{y}(T) \geq v(T).$$

Hence $\bar{y} \in \emptyset$.

We can compare the two solutions by changing the notion of counter-objection for model 1 in the following way.

Definition 5. Let us say that there is a counter-objection to objection (C, x_C) if there is an objection (Q, y_Q) , $Q \cap C \neq \emptyset$, to any $x \in S(x_C)$ instead of a certain.

It is easy to see that the solution for model 1 with counter-objection by Definition 5 will coincide with the solution for model 2 whenever Definition 4 or 5 is applied.

The von Neumann-Morgenstern solution. Model 2 is quite adequate to a classical cooperative game, nevertheless a solution \emptyset for it differs from the von Neumann-Morgenstern solution. To get the last by our general approach we need some further restriction on counter-objection. The restriction is of the kind mentioned above: a counter-objection must be stronger than an objection. Let V be the set of "strong" imputations.

Definition 6. We will say that there is a V -counter-objection to objection (C, x) if there exists an objection (Q, y) against x , $Q \cap C = \emptyset$ or not, but $y \in V$.

Any von Neumann-Morgenstern solution V equals the solution \emptyset with counter-objection by Definition 6. To prove it, it is sufficient to link the two solutions' definitions. By definition of the von Neumann-Morgenstern solution V no $x \in V$ is dominated by other $y \in V$ (there is no objection (C, y) , $y \in V$ against $x \in V$), and every $z \notin V$ is dominated by some $x \in V$ (to any objection (C, z) , $z \notin V$, there is a V -counter-objection in the sense of Definition 6). Therefore, to any objection against $x \in V$ there exists a counter-objection, and $x \in \emptyset$.

Against any $z \notin V$ there is an objection (C,x) , $x \in V$, which, therefore has no counter-objection.

A similar result, as well as others, for model 1 one can find in the author's paper [13] in English.

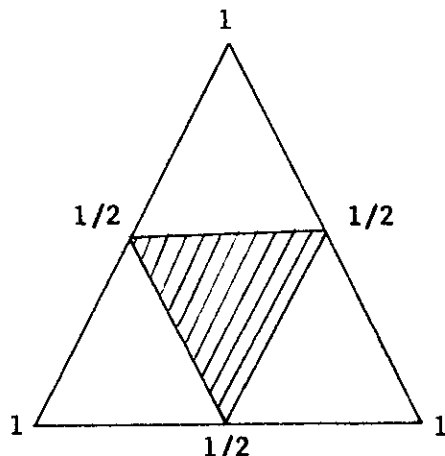
Other restrictions on counter-objection. All real games are hardly separable from player's psychology and they grow strong temptations to invent new notions of counter-objection on this ground. Two feelings--fear and resistance--are quite common and have been used in game theory literature.

Definition 7. A pair (Q,z) is said to be a counter-objection to objection (C,y) against x if (Q,z) , $Q \cap C \neq \emptyset$ is an objection against y and $z <_i x$ for some $i \in C$.

The explanation of the counter-objection is that it is dangerous to player i to join coalition C because it may lead to some loss of his initial payoff. This notion can be generalized to a sequence $(Q_1, z_1), \dots, (Q_r, z_r)$ of subsequent objections with the loss of player $i \in C$ in z_r and the existence theorem can be proved at least in this special case [11].

Definition 8. A pair (Q,z) is said to be a counter-objection to objection (C,y) against x if (Q,z) , $Q \cap C \neq \emptyset$, is an objection against y and $z \succ_Q x$.

It means that coalition Q can defend the initial imputation x and therefore one cannot remove it from φ . We do not know any general existence theorem for this case. However, it is easy to show that the solution related to Definition 8 for three-person simple game with all coalitions winning, but $\{i\}$, $i = 1, 2, 3$, is as shadowed in the



picture. It is obvious that one can think of the core of the game as such a solution in which counter-objections are made impossible.

Nucleolus. It is rather surprising that the nucleolus could be described in strategic form of solution φ , more, it could be done simply and with new interesting social interpretation of nucleolus.

For a classical cooperative game let strategy sets be

$$X_C = \{X_C(x)\}_{x \in X}$$

$$X_C(x) = \{y : e(Q, y) \leq e(Q, x) \text{ for all } Q \in T(C, x) \\ e(Q, y) \leq e(C, y) \text{ for all } Q \notin T(C, x)\} .$$

the $e(Q, x)$ being the excess of coalition Q at x , and

$$T(C, x) = \{Q : e(Q, x) > e(C, x)\} .$$

One can interpret all this by saying a coalition is permitted to improve its position if and only if this does not make it worse for the "neediest" coalitions T and does not change its status in the "social hierarchy."

It is proved by the author [13] that solution φ equals the nucleolus of game v , and hence does core and von Neumann-Morgenstern's solution

5. Elements of Coalitional Dynamics

One of the first attempts to consider the dynamics of the coalitional game is due to R. D. Luce [14]. Luce's ψ -solution is defined for (P, x) as outcomes of the game, where P is a coalitional structure and x is imputation. Given rule $\psi(P)$ of possible changes of coalitions, the following two conditions for $(P, x) \in \psi$ are introduced:

- (i) $x(C) \geq v(C)$ for every $C \in \psi(P)$
- (ii) $x_i = v(i) \implies \{i\} \in P$.

The first condition says that no coalition possible at the moment, has any objection against x . The second says that every player must gain something from joining a coalition.

Now we can define a general game, the core of which is ψ -solution. Let the set of possible coalitions be $K = 2^N \setminus \{\emptyset\}$ and the set of outcomes

$$S = \{(P, x) : x_i > v(i) \text{ if } \{i\} \notin P, x(N) = v(N), x_i \geq v(i)\}.$$

Let \tilde{P}/C be any coalitional structure with $C \in \tilde{P}/C$ and

$$X_C(P) = \begin{cases} \{x : x(C) \leq v(C), (\tilde{P}/C, x) \in S\}, & C \in \psi(P) \\ \emptyset, & C \notin \psi(P). \end{cases}$$

We will finish the definition of game, letting

$$S_C(x) = \{(\tilde{P}/C, x)\}$$

$$(P, x) \succ_C (P', y) \iff x_i > y_i \text{ for every } i \in C.$$

Of course, there are many other definitions of game in terms of Definition 1, for which the set of outcomes without objections turns out

to be the ψ -solution. We just wanted to give an example of how a general approach works in coalitional dynamics.

To complete the definition of a general coalitional game in extensive form we need not so much as to add to the previous model. Let $S = T \cup W$, $T \cap W = \emptyset$, where W is the set of final positions and T is of the non-final. Every coalition has preference among final positions W and a set of strategies (alternatives) $X_C(t)$ at any non-final position $t \in T$, which may be empty. The last will simply indicate that the coalition C at t cannot be formed. In this formulation there is no necessity to keep set K in the definition of the game; it may be changed by the set of players N . What the strategies are for is limitation of the set of possible states and outcomes of the game. That is, what sets $S_C(x_C^t) \subset T$ or W do for every $x_C^t \in X_C(t)$.

Thus, a coalitional game in extensive form can be defined as the collection,

$$\langle N, T, \{X_C(t)\}_{C \subset N, t \in T}, W, \{S_C(x_C^t)\}_{C \subset N, x_C^t \in X_C(t)}, \{>_C\}_{C \subset N} \rangle \quad (4)$$

Obviously, the very meaning of the components of game imposes certain restrictions on these sets. E.g. the highly realistic "something must happen after every sequence of feasible strategies" leads to the constraint of kind $S_C(x_C^{t_1}) \cap S_{C'}(x_{C'}^{t_1}) \neq \emptyset$. We will not discuss this question here and limit ourselves with the final remark, that the model is capable of taking into account, at least formally, informational, institutional, probabilistic, etc. urgent realities of economy (see for argumentation, M. Shubik [6]).

Following the general approach we shall define an objection of a

coalition C against an outcome w for this game. Denote first $W(x_C^t)$ to be a set of final states of the game which follow the choice of strategy x_C^t .

Definition 9. A pair (C, x_C^t) is said to be an objection against $w \in W$ if $x_C^t \in X_C(t) \neq \emptyset$ and $w' \succ_C w$ for every $w' \in W(x_C^t)$.

Definition 10. A pair (Q, x_Q^t) is said to be a counter-objection to objection (C, x_C^t) if it is an objection against some $w' \in W(x_C^t)$ and $Q \cap C \neq \emptyset$.

Now everything which was applied to Definitions 2 and 3 can be used for Definitions 9 and 10 with more variety, of course.

The game (4) has never been investigated in the game theory literature, except in one very special case (R. Selten [4]). Selten discussed a cooperative solution (valuation) for usual non-cooperative game in extensive form. Nevertheless some axioms used by Selten, especially these about information, sound well in model (4).

Besides concretizations of solution φ referred to it would be very interesting to find out a solution concept defined by an axiom similar to Bellman's optimality principle. Of course, this is not the only problem: this sketch on optimality proves there are many more open problems than solved.

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