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by

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1. INTRODUCTION

The view has been expressed by many that a meaningful capital theory can and should be developed without ever defining an aggregate capital index. A fine prototype of this approach is Malinvaud's now classical paper of 1953. The same banner has been unfurled, though not with full identity of views, in Cambridge, England, and in Cambridge, Massachusetts.

With princely unconcern econometricians have continued to fit aggregate production functions approximating an aggregate output index, for an economy or a sector, by a function $F(L,K)$ of aggregate labor (L) and capital (K) input indices. When the matter of the logical foundations for such a construct is raised, words such as "parable" or "metaphor" are pressed into service.

Coexistence of logically unconnected or even incompatible approaches makes for a rich science. Part of this richness lies in the challenge to find points of view that may tie together what appear to be competing

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approaches. This paper does not attempt to arrive at a definite stand on the issue of capital aggregation. Its more modest purpose is to select a few pieces of work in the literature that have a bearing on the problem, to describe their principal ideas in a summarizing way, and to comment on such insights as they may give in the problem of aggregating production relations. The selection is avowedly subjective, and leaves out some important contributions already extensively discussed in the literature.

There are two other selfimposed constraints. One is the acceptance of that shadowy notion of perfect allocation that is subject to two seemingly opposite interpretations: that of perfect markets guided by complete information and perfect foresight, and that of perfect planning possibly guided by appropriate shadow prices. This constraint is adopted on the hunch that aggregation is simpler within it than without it, while what is learned in this way may still be a worthwhile starting point for the study of more complicated situations. The constraint is applied to that part of the economy whose aggregation is under discussion, and not necessarily to the rest of the economy. It may also be applied to the future under conditions showing that it could not have held in the past.

The second constraint arises from a preference for the notion of elementary processes as the building blocks from which production relations are constructed. In the simplest case each process is defined merely by the ratios of inputs to outputs in any utilization of the process. Use of this simple linear case implies an assumption of constant returns to scale within any one process, possibly subject to an upper bound set by a capacity limit. The assumption of a finite number of processes has the advantage that the micro-data that describe technology in detail often are of this nature. Also, algorithms for marshalling such data to answer broader questions are available. Finally, cases of joint production can readily be included in this way. Generally speaking, however, the discrete representation of processes is more suited for the industrial sector than for agriculture. In the latter case, the differentiable production function normally employed for aggregate relations may well be the appropriate form to represent a family of elementary processes that allows continuous substitution of one factor of production for another. Production relations in which the two forms are combined may, of course, be most appropriate in some cases.

In Section 2 we reason from a given size and composition of the capital stock available to each productive unit and held constant during the single period considered. The object of the analysis is to derive the production function in the space of current inputs and outputs implicit in efficient utilization of its "own" capital stock by each unit. The characteristics of these stocks find expression in the shape of the production locus, but do not explicitly appear as variables. In this context, therefore, the term aggregation refers only to the fact that one production relation for the whole is derived from a number of simpler relations for the parts. There is no attempt yet to reduce the number of variables by the formation of suitable index numbers. Rather, the number of relations is reduced to one, using the assumption of internally efficient utilization (or in some cases non-use) of the individually controlled parts of the capital stock.

In Section 3 the size and composition of the total capital stock do explicitly appear in the model, and can change over time. However, the attention concentrates on the search for a capital stock that if initially given does not change (is "invariant") as a result of optimization of the production path for capital goods and consumption goods over an infinite horizon. Implied in this optimization is that not only the use of individually controlled parts of the capital stock, but indeed the size, composition, allocation and use of the total stock are optimally chosen in a sense to be defined in Section 3. The ideas of Section 3 are presented with the help of a simple example.

Section 4 discusses how such an invariant capital stock may depend on the discount factor for future utility flows that enters into the optimality criterion for paths over time. While normally a larger discount factor (a smaller discount rate) is associated with a larger invariant capital stock, a simple example of the reverse relationship is given.

An Appendix has been added after the S'Agaro conference, to elaborate on statements made in the discussion in response to comments and questions by a number of participants. In particular the Appendix indicates that, in a counterintuitive example of an invariant capital stock that is larger when the discount factor is smaller, that invariant stock is not unique, is not stable under small changes in the initial capital stock, and is bracketed by two other invariant stocks each of which is stable.

2. ONE-PERIOD PRODUCTION RELATIONS

Consider a period short enough so that the size and composition of the stock of fixed capital can be regarded as given and constant within the period. The discussion concerns itself with an aggregate or whole that may be interpreted as a branch of industry, a sector of the production economy, or the entire production system of an economy. We are looking for a procedure that derives the "short-run" production function for the whole from production possibility data associated with the parts (pieces of equipment, departments, plants, firms, or branches of industry) that together make up the whole. In keeping with the short-run point of view, we allow (but do not insist on) an interpretation in which the possibility of transfers of capital goods between parts during the period is ruled out.

The locus classicus is a beautiful brief paper by Houthakker (1955). Variants of his procedure were developed by Levhari (1968) and K. Sato (1969). A fuller and more systematic treatment is given by Johansen (1972) in an important book in which the various production function and supply function concepts are defined, are related to each other, and to empirical data.

Houthakker, Johansen and Levhari represent the production possibilities inherent in the capital stock of any given part by a process vector in the space of input and output commodity flows. Besides indicating the ratios of inputs and outputs by the ratios of its components, the process vector is given a length expressing the absolute inputs and outputs corresponding to full-capacity use of the capital stock for that part. Then, as long as the capital stock is held constant, the collection of process vectors, one for each part, is all one requires for the derivation of the short-run production function for the whole. Information about the physical composition of the capital stock available to each part and the processes involved in its production are needed only at a later stage of analysis where changes in the capital stock are introduced.

We have argued above that, in regard to industrial production, it seems more suitable to treat the number of parts represented by process vectors as finite, but often large. Houthakker, Levhari and, in his Chapters 3, 4, 5, Johansen approximate this discrete collection of vectors by an infinite number of process vectors arranged in a smooth frequency distribution over the entire space of inputs and outputs, or over some subset of this space that may be of lower dimensionality. This has the advantage that each individual process may be thought of as operating only at a level one or zero.

Here I shall use a finite number of process vectors, as is done by Johansen in his first exposition of the short-run production function (Sec. 2.4) and in his applications to the Swedish wood pulp industry (Sec. 8.7) and the Norwegian tanker fleet (Ch. 9). As already explained, let in this case the capacitated process vector $a^j \equiv (a_1^j, a_2^j, \ldots, a_n^j)$ represent inputs and outputs under full utilization of the capital stock of that part. A scalar utilization factor x_j can then be applied to the process vector a^j to represent the input and output flows at feasible activity levels by the scalar product

$$x_j a^j \equiv (x_j a_1^j, x_j a_2^j, \ldots, x_j a_n^j) , \qquad 0 \leq x_j \leq 1 .$$

Figure 2.1* illustrates the construction of the short-run produc-

*Figures 2.1 and 2.2 are found on pp. A1, A2 at the end of this paper.

tion function in the simplest case of one input and one output. This case could serve as a first approximation for a collection of base-load power plants burning clean coal (if we include labor with the capital stock). Process vectors a^1, a^2, a^3, a^4 might represent plants of increasing age with decreasing "efficiency" of conversion of fuel into electric energy. In this simple case the short-run production function is represented by the broken straight line connecting the successive partial sums $0, a^1, a^1 + a^2, \ldots,$ of the given process vectors taken in order of decreasing "efficiency" of conversion. Under a regime of maximization of net revenue from current operations of the whole, a gradual increase of the ratio of the output price to the input price will trace out the production function. Intervals on the relative price axis will correspond to points $a_1, a_1 + a_2, \ldots,$ where the production function has a kink, and relative prices characteristic of successively less efficient pieces of capital will correspond to segments on which these pieces are taken into use in the same order.

If there are $m > 2$ inputs and outputs and n processes, there is no natural linear order of the subsets of processes successively taken into use, and there are price vectors which permit more than one process to be efficiently taken into partial use simultaneously. The construction then is as follows.

Let $\mathcal{J}$ (for "technology") denote the set $\{a^j \mid j = 0, 1, \ldots, n\}$ of all capacitated process vectors a^j, where $a^0 \equiv 0$ (the origin). For any subset $\mathcal{J}'$ of $\mathcal{J}$ (including $\mathcal{J}$ itself but excluding the empty set) let $a(\mathcal{J}')$ be the sum

$$a(\mathcal{J}') \equiv \sum_{a^j \in \mathcal{J}'} a^j$$

of all process vectors of $\mathcal{J}'$. Then the feasible set in the space of commodity flows is the convex hull $\mathcal{H}$ of the set of vectors $a(\mathcal{J}')$ for all $\mathcal{J}'$, that is,

$$\mathcal{H} \equiv \{ \sum_{\mathcal{J}' \subset \mathcal{J}} x(\mathcal{J}')a(\mathcal{J}') \mid \sum_{\mathcal{J}' \subset \mathcal{J}} x(\mathcal{J}') = 1,\ x(\mathcal{J}') \geqq 0 \text{ for all } \mathcal{J}' \subset \mathcal{J} \},$$

the set of all convex-linear combinations of all the partial sums $a(\mathcal{J}')$. Finally, the graph $\mathcal{F}$ of the production function is the efficient boundary of $\mathcal{H}$, that is, the set of those points h of $\mathcal{H}$ such that the only point $h+\epsilon$ of $\mathcal{H}$ with $\epsilon \geqq 0$ is the point h with $\epsilon = 0$.

In this case, as output prices increase and/or input prices decrease, the order in which additional processes are started up under current net revenue maximization depends on the path in the price space followed.

A diagram illustrating this construction in the case of three processes and three goods (two outputs, one input) is given in Figure 2.2. A similar diagram for the case of two inputs and one output is given by Johansen (1972, p. 17) in projection on the input space, with isoquants drawn in to indicate increasing output levels.

3. A CAPITAL STOCK INVARIANT UNDER OPTIMIZATION OVER TIME

In Section 2 simplicity was bought by the assumption of a fixed capital stock and a fixed technology for its utilization. In the present section, in which we consider an intertemporal model of an entire economy, we shall continue to assume a fixed technology, not only in the utilization of capital goods in production, but also in the production of the capital goods themselves. However, we shall treat changes over time in the capital stock as entirely feasible. We also assume absence of institutional barriers to the transfer of capital goods from one control to another. The process notion therefore no longer implies allocative control over a fixed capital stock associated with each part. The process thus becomes a more purely technological concept, in which capital goods are now represented by coefficients for capital inputs required by the process.

We shall also strengthen the assumption of efficient use of resources to one of intertemporal optimality, defined by specifying some suitable social objective function over time. As explained already, this construct can be regarded as a simulation that yields a first approximation either to a centrally planned and managed economy, or to the course over time of a market economy that manages to sustain reasonably full employment. In the latter case the interpretation of the assumed intertemporal preferences is an implicit rather than an explicit one. In both cases, the simulation takes a rosy view of the working of the simulated economy.

We shall utilize an objective function of the form

$$U \equiv \sum_{t=1}^{\infty} (\alpha)^{t-1} u(y^t) \ , \qquad 0 < \alpha < 1 \ .$$

Here y^t denotes a vector of final consumption flows of the various consumables in period t, $u(y)$ the utility flow associated with a consumption flow of y per unit period,* and α a discount factor** per unit of time, applied to future utilities and assumed given and constant. Consumption goods as well as capital goods are produced using processes selected from a finite collection of processes. The inputs to these processes include the utilization of capital goods and the consumptive use of resource flows such as labor, minerals, clean air. Total available flows of resources in each period are again assumed fixed over time (for simplification though not realism).

The grand and difficult problem posed by this model is to associate with any (historically) given initial capital stock an optimal path (if one exists), that is, a path that maximizes the objective function U among all feasible paths over an infinite time period.

However, some further provisional simplification can again be bought by first asking only the following question: <u>Does there exist a capital stock which, if put in the place of the given initial stock, will be reproduced precisely at the end of each period as a result of the optimization</u>? If so, such a capital stock can be regarded as being in equilibrium with the technology, the resource constraints, and the preferences, both intertemporal as given by α, and within each period as given by

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* $u(y)$ is assumed differentiable and concave.

** $(\alpha)^t$ denotes α raised to the power t, in contrast with the use of superscripts t as time labels in y^t, and in x^t, z^t below.

$u(\cdot)$. We shall refer to such a stock as an invariant optimal capital stock.

For the basic case of a single good in the double role of capital good and consumption good, our question has been fully answered by Ramsey (1928) and his followers.* If the output flow of that good produced by a fixed labor force is a strictly concave function of the available capital stock, then there is a unique invariant optimal capital stock. It has been called the golden rule stock modified by discounting. Its dependence on the discount factor α is also well known: As α increases (hence the discount rate $\rho = (1-\alpha)/\alpha$ decreases), the invariant stock increases and approaches the golden rule stock proper as $\alpha \to 1$, hence $\rho \to 0$. It is readily computed from the value of α and the form of the production function, by requiring the marginal productivity of the good as capital in terms of the good as an output flow to be equal to ρ . Hence it does not depend on the shape of the utility function. Finally, for any α , $0 < \alpha < 1$, and any positive initial capital stock, a unique optimal capital path exists which approaches the modified golden rule stock as time proceeds.

Matters are more complicated for an arbitrary number of commodities. An analysis of the general case involving any finite numbers of processes and of the three types of goods (i.e., capital, consumption, resources) is given by Hansen and Koopmans (1972) from both the theoretical and computational points of view, with references to earlier work. Here we consider in some detail an example with one capital good, one resource and two consumption goods. It is hoped that such an exploration will bring out some of the economic content and implications of the

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*For a more recent exposition see Koopmans (1967) in which other literature is also cited.

concept of an invariant optimal capital stock more vividly than can the theorems and algorithms regarding the general case. While the presentation is self-contained and uses only elementary calculus, some unproved statements are supported by the reference cited.

We shall assume that the single-period utility function $u(y_1, y_2)$ is defined for all $y_1 \geqq 0$, $y_2 \geqq 0$, increases strictly with each of the two consumption flows, y_1, y_2 (nonsaturation), is strictly concave and continuously differentiable. As to the constraints, Table 1* gives the input and output coefficients for the four goods for each of three processes. The symbols a_j, b_j, x_j, y_i, z^t represent non-negative scalars. The technical coefficients a_j, b_j are independent of time by assumption. The coefficient vectors are normalized so as to specify a unit input of the single resource (labor, say) for the unit activity level of each process. Also, the units of the two consumption goods are chosen so that one unit of labor is required to produce one unit of either good. As to timing, labor and consumption can be regarded as flows during the period. In those parts of the reasoning in which we consider only one period at a time, no time label will be attached to the x_j, y_i. Capital input is required to be available at the beginning of each period for use during that period. Capital output becomes available at the end of each period. Since capital input and output for a given period may differ, a time superscript t is attached to the symbol z whenever needed.

Capital "output" represents the sum of (already used) capital released for possible use in the next period and new capital goods produced during the period. In principle one should consider two capital goods

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*See p. 3.4a.

Table 1: Technology Matrix for an Intertemporal Model of Production

Notations for Coefficient Vectors				Activity Levels x_i and Technical Coefficients x_1	x_2	x_3	Availabilities and Total "Outputs" $\geqq$	
f_1	f_2	f_3	=	$-a_1$	$-a_2$	$-a_3$	$-z^1$	capital input
				b_1	b_2	b_3	z^2	capital "output"
-1	-1	-1	=	-1	-1	-1	-1	labor
d_1	d_2	0	=	1	0	0	y_1	cons. good "1"
				0	1	0	y_2	cons. good "2"

constructed from the same blueprint but with different lengths of prior use as different capital goods. For simplicity, for the processes $j = 1, 2$, interpreted as producing consumption goods only, we specify $a_j > b_j$ to simulate loss of effectiveness of the capital good by a constant geometric decline per period in its quantity, regardless of the rate of use. In this case the specification $a_3 < b_3$ is essential for an increase or even constancy over time of the capital stock to be compatible with a positive level of consumption. The symbols f_j, d_j are abbreviated notations for the corresponding column vectors of order two for the coefficients of capital goods and consumption goods, respectively. These symbols are used mostly in the diagrams.

The first line of the table expresses the feasibility constraint

$$-a_1x_1 - a_2x_2 - a_3x_3 \geqq -z^1 ,$$

which says that capital in use during the period cannot exceed the amount available at its beginning. The other constraints can be read off accordingly. The entire set of constraints remains the same at all times, and can be read as applying either to the first period, or to any nameless future period. To apply it to a specific period, say the t^{th}, superscripts t are attached to the x_j and y_i, and $t-1$ is added to the superscripts of z^1, z^2.

Note that the only variable occurring in the constraints for two successive periods, say those with labels t, $t+1$, is the variable z^{t+1}. This, together with the additive form of the objective function, makes it possible to carry out the optimization of the entire future path (starting with any prescribed initial stock z^1) in two stages. In the first stage all values z^t for $t = 1, 2, \ldots,$ are held fixed at arbitrary jointly attainable levels, and the attention is directed toward maximizing the term $\alpha^{t-1}u(y^t)$ within each period by choice of the x_j^t, y_i^t subject only to the constraints for that period. The result is a value

$$\max_{x^t, y^t} \alpha^{t-1}u(y_1^t, y_2^t) \equiv \alpha^{t-1}\psi(-z^t, z^{t+1}), \text{ say,}$$

that depends only on the initial and terminal capital stocks of that period. The second stage then consists in maximizing

$$\bar{U} \equiv \sum_{t=1}^{\infty} \alpha^{t-1}\psi(-z^t, z^{t+1})$$

subject to the given initial stock z^1 and such constraints on the pairs (z^t, z^{t+1}), $t = 1, 2, \ldots$, as are implicit in those of Table 1.

While our focus is on initial stocks z^1 that in the second stage yield constant optimal paths $z^t = z^1$, $t = 2, 3, \ldots$, it will help if we do not yet specify $z^t = z^{t+1}$ in describing the first stage. Reverting to the nameless-period notation of Table 1 we therefore now take both z^1 and z^2 as given and possibly different, and drop the factor α^{t-1}. The first observation then is that optimality requires

$$y_1 = x_1, \quad y_2 = x_2,$$

because any slack in the consumption of either good would unnecessarily diminish the utility $u(y_1, y_2)$.

Our procedure for analyzing stage one will be to compare the maximal utility flow, $\psi(-z^1, z^2)$, for the given z^1, z^2, with that attainable flow, to be denoted $\varphi(-z^1, z^2)$, that results if each of the three other constraints is tentatively required also to hold with strict equality,

$$-a_1x_1 - a_2x_2 - a_3x_3 = -z^1$$

$$b_1x_1 + b_2x_2 + b_3x_3 = z^2$$

$$-x_1 - x_2 - x_3 = -1 \; .$$

We shall call these the no-slack constraints for capital input, capital output and labor, respectively. The domain of definition of $\varphi(-z^1, z^2)$ then is that set of points $(-z^1, z^2)$, to be denoted $\mathcal{Z}$ below, for which a nonnegative solution (x_1, x_2, x_3) of the no-slack constraints exists.

We choose our example such that the 3-by-3 matrix of coefficients of the x_j is nonsingular, solve for x_3 from the third equation and substitute the resulting expression in the other two equations, obtaining

$$-(a_1 - a_3)x_1 - (a_2 - a_3)x_2 - a_3 = -z^1$$

$$(b_1 - b_3)x_1 + (b_2 - b_3)x_2 + b_3 = z^2 \; ,$$

with a nonsingular 2-by-2 matrix. Ignoring nonnegativity constraints, these equations define a one-to-one linear mapping from the points (x_1, x_2) to the points $(-z^1, z^2)$.

The non-negativity of the activity levels x_j, $j = 1, 2, 3$,*

*For $j = 3$ the constraint now takes the form $x_1 + x_2 \leqq 1$.

and the identity of the x_j and y_j allow us to enter the level curves (indifference curves) of $u(y_1, y_2)$ in the closed positive quadrant of the (x_1, x_2)-plane (see Figure* 3.1). We recall for later use some elementary mathematical verities. Since the derivatives

$$u_1(x_1, x_2) \equiv \frac{\partial u}{\partial x_1}, \qquad u_2(x_1, x_2) \equiv \frac{\partial u}{\partial x_2},$$

are positive in all points of the quadrant, there is in each point (x_1, x_2) a tangent to the level curve of negative finite** slope. This tangent partitions the set of all directions out of the point (x_1, x_2) into three subsets. As illustrated by arrows labeled $+$, 0, or $-$ in Figure 3.1, in all directions (δ_1, δ_2) leading from (x_1, x_2) to points "above" the tangent the directional derivative

$$\left(\frac{du(x_1 + \lambda\delta_1,\ x_2 + \lambda\delta_2)}{d\lambda}\right)_{\lambda=0}$$

of u is positive; in the two directions along the tangent that derivative is zero; in all directions to points "below" the tangent the derivative is negative. The origin is "below" the tangent***. Finally, an implication of the strict concavity of u should be noted. Proceeding from (x_1, x_2) along a straight line in any direction with a nonpositive direc-

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*Figures 3.1 to 5.2 are found on pages marked A3 to A15 at the end of this paper.

**Our assumptions about $u(\cdot,\cdot)$ imply that the axes are not tangent to any level curve.

***Whenever $(x_1, x_2) \neq (0,0)$.

tional derivative in (x_1, x_2), the function will monotonically decrease along the entire feasible segment of that line. In particular, the maximum of $u(x_1', x_2')$ among points (x_1', x_2') of the tangent in (x_1, x_2) is reached uniquely in (x_1, x_2).

The no-slack constraints for labor allow us to represent the set of attainable activity vectors (x_1, x_2, x_3) by the closed triangle $\mathcal{X} \equiv \mathcal{X}(0, d_1, d_2)$ with vertices $0, d_1, d_2$ in the space of (x_1, x_2) only, since $x_3 = 1 - x_1 - x_2$. The mapping $(x_1, x_2) \Longleftrightarrow (-z^1, z^2)$ in turn transforms this triangle into the triangle $\mathcal{Z} \equiv \mathcal{Z}(f_1, f_2, f_3)$ in the space of $(-z^1, z^2)$ —see Figure 3.2. The triangle $\mathcal{Z}$ represents the set of all those pairs $(-z^1, z^2)$ that are both attainable and consistent with the added no-slack constraints. Any point in this triangle simultaneously represents the pair $(-z^1, z^2)$ by reference to the rectangular coordinate axes of $-z^1$ and z^2, and the pair (x_1, x_2) by reference to a skew coordinate system defined within the triangle by placing the origin in f_3 and unit points on the two axes in f_1 and f_2. Transferred to the new (x_1, x_2)-coordinates defined on $\mathcal{Z}$, the level curves of $u(x_1, x_2)$ now also serve as level curves for the function $\varphi(-z^1, z^2)$ mentioned above (see Figure 3.3). This function is then defined on $\mathcal{Z}$ by

$$\varphi(-z^1, z^2) \equiv u(x_1, x_2) \qquad \text{whenever} \qquad (x_1, x_2) \Longleftrightarrow (-z^1, z^2) .$$

It represents the utility attained in the given period with initial and terminal capital specifications $(-z^1, z^2)$ if each of the five constraints is forced to hold with equality. Because of the linearity of the mapping

φ inherits the continuous differentiability and the strict concavity of u.

The function $\varphi(-z^1, z^2)$ so defined in $\mathcal{Z}$ is not necessarily the same, even within $\mathcal{Z}$, as the function $\psi(-z^1, z^2)$ defined earlier in the entire set of feasible $(-z^1, z^2)$. The difference within $\mathcal{Z}$ is that $\psi(-z^1, z^2)$ is the maximum attainable utility under constraints permitting slacks, whereas $\varphi(-z^1, z^2)$ is the unique utility level that is attainable—hence optimal—under constraints that rule out all slacks. Since narrowing the constraint set cannot increase the maximum attainable utility we must have

$$\psi(-z^1, z^2) \geqq \varphi(-z^1, z^2)$$

in all points $(-z^1, z^2)$ of $\mathcal{Z}$. On the other hand, we must have

$$\psi(-z^1, z^2) = \varphi(-z^1, z^2)$$

in all those points $(-z^1, z^2)$ of $\mathcal{Z}$ in which the maximum utility attainable under constraints permitting slacks is in fact attained for the unique no-slack activity vector $(x_1, x_2) \Leftrightarrow$ the given $(-z^1, z^2)$. We shall now examine for each of the three constraints under what conditions this is the case.

Let $(-z^1, z^2)$ be a point in the interior of $\mathcal{Z}$. Taking first the two capital constraints, we assume that the no-slack constraint for labor is satisfied. Then a slack of $\delta^1 > 0$ in capital input would mean that out of a stock z^1 made available only $z^1 - \delta^1$ is used in production. Similarly, a slack of $\delta^2 > 0$ in capital output would be to produce $z^2 + \delta^2$ but hand on only z^2 . For either of these and any combination of them to decrease utility, it is sufficient* for the two derivatives

$$\varphi_1(-z^1,\ z^2) \equiv \frac{\partial\varphi}{\partial(-z^1)}\ , \qquad \varphi_2(-z^1,\ z^2) \equiv \frac{\partial\varphi}{\partial z^2}\ ,$$

to be negative. Figure 3.3 illustrates that this implies a finite negative slope

$$s \equiv -\frac{\varphi_1(-z^1,\ z^2)}{\varphi_2(-z^1,\ z^2)} = \left(\frac{dz^2}{d(-z^1)}\right)_{\varphi = \text{const.}}$$

for the tangent to the level curve of φ in the point $(-z^1,\ z^2)$.

We now turn to the no-slack constraint for labor, which we examine assuming the no-slack constraints for capital to be satisfied. Let the amount of labor in use be changed from 1 to

$$x_1 + x_2 + x_3 = 1-\epsilon\ , \qquad 0 < \epsilon \leqq 1$$

allowing a slack of ϵ . We choose a particular small value of ϵ and treat the above equation as defining a new experimental equality constraint on labor. This defines a new mapping between the new activity levels $\tilde{x}_1$, $\tilde{x}_2$ and the capital specifications z^1 , z^2 , according to the equations

*In the configuration of Figure 3.3, where the new origin f_3 for the $(x_1,\ x_2)$-coordinate system is "above" and "to the right of" the point $(-z^1,\ z^2)$, it is necessary and sufficient that these derivatives are nonpositive (not both can be zero), thus allowing a vertical or a horizontal tangent. See the implication of strict concavity of u discussed above.

$$-(a_1 - a_3)\tilde{x}_1 - (a_2 - a_3)\tilde{x}_2 - a_3(1-\epsilon) = -z^1$$

$$(b_1 - b_3)\tilde{x}_1 + (b_2 - b_3)\tilde{x}_2 + b_3(1-\epsilon) = z^2 \ .$$

But then those values $\tilde{z}^1$, $\tilde{z}^2$ of the capital stocks that would have produced the present consumption flows $\tilde{x}_1$, $\tilde{x}_2$ in the absence of any labor or capital disposals are related to the specified flows z^1 , z^2 by

$$-\tilde{z}^1 = -z^1 - \epsilon a_3 \ , \qquad \tilde{z}^2 = z^2 + \epsilon b_3 \ .$$

It is these values that are to be tested in relation to the level curves of the function φ in Figure 3.3. This is done in Figure 3.4, where the dashed line connecting the interior point $(-z^1, z^2)$ of $\mathcal{Y}$ with $(-\tilde{z}^1, \tilde{z}^2)$ is drawn so as to be parallel, in slope and direction, to the line $\overline{Of_3}$ connecting the origin of the $(-z^1, z^2)$-plane with the point $f_3 = (-a_3, b_3)$ representing the capital producing process. Therefore, for any slack in labor use to be nonoptimal, the following condition on the directional derivative in the direction $(-a_3, b_3)$ is both necessary* and sufficient,

$$\left(\frac{d\varphi(-z^1 - \lambda a_3, \ z^2 + \lambda b_3)}{d\lambda}\right)_{\lambda=0} = -a_3\varphi_1(-z^1, \ z^2) + b_3\varphi_2(-z^1, \ z^2) \leqq 0 \ .$$

Since we assume capital slacks to be nonoptimal, φ_1 and φ_2 must be nonpositive, which precludes $\varphi_2 = 0$. Therefore the condition just obtained is equivalent to the slope condition

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*The necessity is achieved by the inclusion of the = sign, on the strength of the strict concavity of φ .

$$s(-z^1, z^2) \equiv - \frac{\varphi_1(-z^1, z^2)}{\varphi_2(-z^1, z^2)} \geqq - \frac{b_3}{a_3} ,$$

as illustrated in Figure 3.4.

We have now derived conditions on the nonoptimality of slacks by testing the effect of the no-slack constraints one or two at a time while assuming the other(s) to be satisfied. Due to the differentiability and strict concavity of φ, the conditions so obtained can be combined to form one condition on the nonoptimality of any combination of non-negative slacks in the three constraints with $\delta^1 + \delta^2 + \epsilon > 0$, that leaves the resulting point $(-z^1 + \delta^1 - \epsilon a_3, z^2 + \delta^2 + \epsilon b_3)$ in $\mathcal{Z}$. In terms of the slope $s \equiv s(-z^1, z^2)$ of the tangent to the level curve of φ in $(-z^1, z^2)$ that comprehensive sufficient (necessary)* condition then is

$$- \frac{b_3}{a_3} \leqq s(-z^1, z^2) < (\leqq) \ 0 .$$

Finally, in all interior points $(-z^1, z^2)$ of $\mathcal{Z}$ in which the comprehensive sufficient condition is satisfied we must have, as explained above,

$$\varphi(-z^1, z^2) = \psi(-z^1, z^2) .$$

Therefore, the level curves of φ and ψ coincide in that part of $\mathcal{Z}$ in which the sufficient slope condition is satisfied. Moreover, in those points, we can look upon the negatives

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*See footnote on page 3.11.

$$q^1 \equiv -\varphi_1(-z^1, z^2) , \quad q^2 \equiv -\varphi_2(-z^1, z^2) , \quad r \equiv a_3\varphi_1(-z^1, z^2) - b_3\varphi_2(-z^1, z^2) ,$$

of the directional derivatives used in testing for the tightness of the capital and labor constraints as non-negative shadow prices associated with the corresponding inputs and outputs. These prices are expressed in units of marginal utility discounted to time $t = 1$.

For all points $(-z^1, z^2)$ of $\mathcal{Z}$ for which the no-slack conditions are met, stage one, the discussion of optimization within single periods for which $(-z^1, z^2)$ is given, has now been completed. It will turn out that the stage one analysis for this subset of $\mathcal{Z}$ is sufficient for our present exploratory purpose.

We are therefore now ready for stage two, the search for invariant optimal capital stocks. We now want to examine points $(-z^1, z^2)$ for which $z^1 = z^2 = z$, say. In the diagrams these points are denoted

$$f = (-z, z) = z \cdot (-1, 1) \equiv z \cdot e ,$$

that is (see Figure 3.5), points of the line $\mathcal{L}$ through the origin and of slope -1 . Again, we first limit our search to points $(-z, z)$ in which the reproduction of $z^2 = z$ from $z^1 = z$ is achieved optimally without slacks. This limits the search first of all to points of the segment $\mathcal{S}$ in which $\mathcal{L}$ intersects $\mathcal{Z}$. Since by previous assumptions about the a_j , b_j the points f_1 , f_2 are "below" $\mathcal{L}$, f_3 "above" $\mathcal{L}$, the segment $\mathcal{S}$ intersects $\mathcal{Z}$ in its interior. It is the segment $\mathcal{S}$ minus its end points in which we shall now search.

Secondly, we shall at first use the slightly more restrictive sufficient slope condition

$$-\frac{b_3}{a_3} < s(-z,\ z) < 0$$

for the within-period optimality of the no-slack activity vector $(x_1,\ x_2) \Longleftrightarrow (-z,\ z)$.

Assuming that such a point $(-z,\ z)$ exists we must now find a test whether the maximization of

$$U = \sum_{t=1}^{\infty} \alpha^{t-1} \psi(-z^t,\ z^{t+1})$$

subject to $z^1 = z$ and the within-period constraints for each period is achieved by $z^t = z$ for all $t > 1$. Trimming our sails once more, we shall first study a weaker test, necessary but perhaps not sufficient, obtained by specifying

$$z^1 = z = z^3 = z^4 = \ldots ,$$

which leaves only z^2 free to vary. We then need to consider only the maximization of

$$V \equiv \psi(-z,\ z^2) + \alpha\psi(-z^2,\ z)$$

with respect to z^2 . Finally, since u and therefore φ have continuous first derivatives, the present slope condition is satisfied also in a neighborhood $\mathcal{N}$ within $\mathcal{Z}$ of the point $(-z,\ z)$. Therefore, restricting z^2 further such that both $(-z,\ z^2)$ and $(-z^2,\ z)$ are in $\mathcal{N}$, we are in fact maximizing

$$W \equiv \varphi(-z,\ z^2) + \alpha\varphi(-z^2,\ z) .$$

The test then is whether the maximum of W within $\mathcal{N}$ is attained for $z^2 = z$. For this to occur, it is necessary that

$$0 = \left(\frac{dW}{dz^2}\right)_{z^2 = z} = \varphi_2(-z,\ z) - \alpha\varphi_1(-z,\ z)\ ,$$

or, equivalently*,

$$s(-z,\ z) = -\frac{\varphi_1(-z,\ z)}{\varphi_2(-z,\ z)} = -\frac{1}{\alpha}\ .$$

It has been proved elsewhere** that this necessary condition for the maximality of U in the constant program $z^t = z$ is also sufficient. Figure 3.5 illustrates the construction. One scans the points of $\mathcal{S}$ to find one or more points where the slope of the level curve of φ has the value $-1/\alpha$ for the given α . Any point in the interior of $\mathcal{Z}$ satisfying this "slope condition" represents an invariant optimal capital stock, to be denoted $\hat{z}$, provided the prescribed slope $-1/\alpha$ itself meets our present "no-slack" slope constraint***

$$-\frac{b_3}{a_3} < -\frac{1}{\alpha} = s(-\hat{z},\ \hat{z})\ .$$

The "slope condition" for an invariant capital stock we have found has a natural interpretation in terms of the shadow prices q^t , r^t associated with the constant program $z^t = \hat{z}$. The condition specifies

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*Since not both, hence neither, of φ_1 , φ_2 can vanish in $(-z,\ z)$.

**Hansen and Koopmans (1972).

***We do not need to reiterate the constraint $s < 0$ because the specification $0 < \alpha < 1$ requires that $s < -1$.

$$q^2 = \alpha q^1 \;, \quad q^3 = \alpha q^2, \; \ldots, \quad \text{so} \quad q^t = (\alpha)^{t-1} q^1 \;, \quad t = 2, 3, \ldots,$$

a geometric decline in marginal utility of the invariant capital stock $z^t = \hat{z}$, in the ratio α per period equal to the discount factor prescribed by the objective function U .

The condition $q^2 = \alpha q^1$ extends to the end points of $\mathcal{S}$ and to points of $\mathcal{L}$ for which an associated optimal activity vector involves slacks. It also generalizes to similar models with any number of capital goods, resources and consumption goods. In these cases, q^1 , q^2 are to be regarded as vectors of shadow prices (dual variables). Where these vectors are not uniquely determined, the condition $q^2 = \alpha q^1$ requires only that one can find values q^1 , q^2 within the permissible joint range of (q^1, q^2) that meet the condition.

4. THE RELATION BETWEEN THE DISCOUNT FACTOR AND AN INVARIANT OPTIMAL CAPITAL STOCK

In the preceeding Section 3, the element of intertemporal preferences was introduced by a discount factor α applicable to future utilities. At the end of the Section it was found that, if an invariant capital stock is indefinitely maintained, that same factor α also applies in the definition of shadow prices of goods. The reason is simple. As long as the capital stock, the consumption vector, and the one-period utility function $u(\cdot,\cdot)$ do not change over time, the same holds for the marginal utilities of the goods in question. Therefore the discount factor for utilities equals that for goods in this case.

It is of interest to study the relation between the discount factor α and the associated value or values of the invariant capital stock $\hat{z}(\alpha)$. This may be defined as the set of compatible pairs $(\alpha, \hat{z}(\alpha))$. This notion is applicable equally to the perfect market interpretation and to the perfect planning model. Its principal weakness in either case is the disregard of technical change. A second weakness is the circumstance that for a historically given initial capital stock, even without further technical change, continued growth toward an attractive invariant capital stock is likely to be, in most if not all existing economies, the first recommendation of the criterion U on which the concept rests. For most policy problems knowledge of the characteristics of the near-future segment of that path is the most urgent requirement.

However, we have to crawl before we can walk, and walk before we can run. It is hoped that an analysis of the relation between α and $\hat{z}(\alpha)$ may add precision to intuitions and ideas with a long history in economic theory. It may also turn out to be a useful preparation for the more difficult problems associated with a path that chases a capital stock which itself is in a moving equilibrium over time with changing technology, changing resource availability, and changing momentary and intertemporal preferences.

Figure 4.1 illustrates that there can easily be more than one invariant capital stock for a given value of α. The diagram exhibits two distinct points of $\mathcal{S}$ with identical slopes $-1/\alpha$, and between them a third point with a different slope $-1/\alpha'$, where $\alpha' > \alpha$. Such a pattern is entirely compatible with strict concavity of the function $\varphi(-z^1, z^2)$.

Figures 4.2 and 4.3 illustrate that this cannot occur if the two consumption goods are <u>normal</u> goods. By this I mean that, for any fixed positive relative prices p_1, p_2, the utility-maximizing consumption pair y_1, y_2 attainable within a given budget b at those prices increases strictly in both components as b increases. In that case the absolute value of the slope

$$|s^*(y_1, y_2)| \equiv \frac{u_1(y_1, y_2)}{u_2(y_1, y_2)}$$

of the level curve of u in the point (y_1, y_2) increases if y_2 increases with y_1 held constant and decreases if y_1 increases with y_2 constant. In that case, if one follows any straight line with negative slope such as $\overline{d_1'd_2'}$ in Figure 4.2, the absolute slope of the level curve increases as y_2 increases (and hence y_1 decreases at the same time). Choosing $\overline{d_1'd_2'}$ in such a way that by the mapping $(y_1, y_2) = (x_1, x_2) \Longleftrightarrow (-z^1, z^2)$ it transforms into the segment $\mathcal{S}$ in Figure 4.3, we find that $\hat{z}(\alpha)$ increases as α increases. The interpretation is that a higher discount factor (a lower real interest rate) is associated both with a higher equilibrium capital stock per worker and with a proportionately higher consumption of the good "2", which is more capital intensive in its production than good "1". In contrast, in Section 5 we shall consider the counterintuitive case of decreasing $\hat{z}(\alpha)$, where a higher discount factor is associated with a smaller invariant capital stock and, indeed, a lower utility level in each period. In that case the more capital-intensive good "2" is superior* to good "1".

For the present case of normal consumption goods, we shall describe without full proof the $(\alpha, \hat{z}(\alpha))$ pairs for α and/or z at the end points of their permitted ranges. If, as in Figure 4.4 on page $s(-z, z)$ reaches its algebraic upper bound -1 in some point $(-\hat{z}(1), \hat{z}(1))$ of $\mathcal{S}$ interior to $\mathcal{Z}$, then

$$\hat{z}(1) \equiv \lim_{\alpha \to 1} \hat{z}(\alpha)$$

is an analogue of the (undiscounted) golden rule capital stock of the one-sector model. Values $z > \hat{z}(1)$ then cannot occur as invariant capital stocks for any permitted value of α . If as in Figure 4.5 the slope $-1/\bar{\alpha}$ of φ at the (boundary) point $(-\bar{z}, \bar{z})$ of $\mathcal{Z}$ with the highest attainable value $\bar{z}$ of z satisfies

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*At least in a neighborhood of the set of consumption vectors $\hat{y}_1(\alpha)$, $\hat{y}_2(\alpha)$ associated with the pairs $(\alpha, \hat{z}(\alpha))$ in question.

$$-\frac{b_3}{a_3} < s(-\bar{z}, \bar{z}) \equiv -\frac{1}{\bar{\alpha}} \leqq -1 ,$$

then $\bar{z}$ is an invariant stock for the following set of values of α,

$$\hat{z}(\alpha) = \bar{z} \qquad \text{for} \quad \bar{\alpha} \leqq \alpha \leqq 1 .$$

Most intriguing is the situation for the lowest value

$$\alpha = \underline{\alpha} \equiv \frac{a_3}{b_3}$$

of α permitting within-period optimality without labor slack in combination with the corresponding invariant stock of $\hat{z}(\underline{\alpha})$, if a stock satisfying the "slope condition" exists for that $\underline{\alpha}$. In Figure 4.5, this $\hat{z}(\underline{\alpha})$ is in the interior of $\mathcal{J}$. It could also be the lower end point $\underline{z}$ of $\mathcal{A}$. In either case, this same value $\alpha = \underline{\alpha}$ can also be associated with any stock z in the range

$$0 \leqq z \leqq \hat{z}(\underline{\alpha})$$

in the role of an invariant stock for that $\underline{\alpha}$, with unemployment increasing as z decreases.

Figure 4.5 shows for any given such z the determination of the <u>unemployment</u> $\epsilon = 1-x$, say, where $x = x_1 + x_2 + x_3$ is the remaining <u>employment</u>, both measured in a total-labor-force unit. It is now preferable to give up the principle underlying Figure 3.2, the representation of (x_1, x_2) and $(-z^1, z^2)$ by a single point referred to two different coordinate systems. If we were to insist on maintaining this principle in the presence of unemployment, we would have to use a coordinate system of (x_1, x_2) that moves, level curves and all, with the axes remaining parallel to themselves but the origin in the point $(-a_3x, b_3x)$ sliding along $\overline{Of_3}$ in step with the employment x. It is simpler to retain the old origin f_3 for the coordinate system of (x_1, x_2). Using z as a parameter, and denoting the corresponding employment and activity levels by

$$\hat{x}(z),\ \hat{x}_1(z),\ \hat{x}_2(z)\ , \qquad 0 \leqq z \leqq \hat{z}(\underline{\alpha})\ ,$$

these quantities are determined with the help of a displaced vector

$$(-\tilde{z}^1,\ \tilde{z}^2) \equiv (-z,\ z) + (1-x)(-a_3,\ b_3)$$

similar to that used in the tightness test for the labor constraint. For given z and variable x, the point so defined moves from $(-z, z)$ along a straight line of slope $-b_3/a_3 = -1/\underline{\alpha}$. To determine the value $x = \hat{x}(z)$ of x corresponding to the given z one extends this line, if possible, until it is tangent to an (undisplaced) level curve of $\varphi(-z^1, z^2)$. The value of x at the point of tangency $\hat{f} \equiv \hat{f}(z) \equiv (-\tilde{z}^1(z), \tilde{z}^2(z))$, say, is the desired $\hat{x}(z)$. The values of $\hat{x}_1(z)$, $\hat{x}_2(z)$ are read off from the mapping relation

$$(\hat{x}_1(z),\ \hat{x}_2(z)) \longleftrightarrow (-\tilde{z}^1(z),\ \tilde{z}^2(z))\ ,$$

taken in the reverse direction. If no tangency point exists,

$(-\tilde{z}^1(z), \tilde{z}^2(z))$ is a boundary point of $\mathcal{Z}$. Which one it is is determined by rules similar to those applicable in the end points of $\mathcal{S}$. As to the latter, if in no point of $\mathcal{S}$ interior to $\mathcal{Z}$ there is a tangent to $\varphi(-z^1, z^2)$ of slope $-1/\underline{\alpha}$, then the boundary point $(-\underline{z}, \underline{z})$ of $\mathcal{Z}$ on $\mathcal{S}$ nearest to the origin will take the place of $(-\hat{z}(\underline{\alpha}), \hat{z}(\underline{\alpha}))$ in the above description, and will also serve as a no-slack invariant capital stock for all α such that

$$-\frac{1}{\underline{\alpha}} \leqq -\frac{1}{\alpha} \leqq s(-\underline{z}, \underline{z}) .$$

How does the ratio a_3/b_3 come to have such an important role as a critical value $\underline{\alpha}$ of the discount factor in the present problem? The answer lies in a connection between the present model and the von Neumann model obtained from Table 1 by discarding all but the first two constraints. Since process "3" has the highest ratio b_j/a_j of capital output to input among the three processes, the requirement of fastest capital growth implicit in the von Neumann model can be met only by shifting all labor from the production of consumption goods to that of capital goods—a feat easier in the so truncated model than in reality. A counterpiece to this observation arises in the present model. If impatience rises, hence the discount factor sinks, below the critical value $\underline{\alpha}$,

$$0 < \alpha < \underline{\alpha} \equiv a_3/b_3 ,$$

then the only invariant capital stock in existence is the null stock,

$$\hat{z}(\alpha) = 0 , \quad \text{with} \quad \hat{x}_j(\alpha) = 0 , \quad j = 1, 2, 3 .$$

At the precise point $\alpha = \underline{\alpha}$, a whole family of invariant capital stocks z , $0 \leqq z \leqq \hat{z}(\alpha)$, and associated employment levels $\hat{x}(z)$ varying continuously from 0 to 1 , maintains the connectedness of the set of all points $(\alpha, \hat{z}(\alpha), \hat{x}_j(\alpha), j = 1, 2, 3)$ in 5-dimensional space.

Figure 4.5 illustrates the dependence of $\hat{z}(\alpha)$ on α and shows one particular possible geometrical form for the family of $\hat{z}(\underline{\alpha})$ associated with $\alpha = \underline{\alpha}$. Figure 4.6 exhibits a corresponding curve for the dependence of the $\hat{x}_j(\alpha)$, $j = 1, 2$, on α in the (x_1, x_2)-plane, again with a one-parameter family of points for the value $\alpha = \underline{\alpha}$.

5. APPENDIX:*

INSTABILITY OF THE INVARIANT CAPITAL STOCK IN THE COUNTER-INTUITIVE CASE

So far we have only asked for a capital stock invariant under optimization. We now raise the question of stability of an invariant capital stock under small perturbations of the initial stock. An invariant optimal capital stock will be called stable (under optimization) if optimal paths starting from initial stocks in some neighborhood of the invariant stock will converge over time to that invariant stock.

Figure 5.1 illustrates a simple heuristic (nonrigorous) test of the stability of an invariant optimal (scalar) capital stock in one important case. It is based on properties of the function

$$W \equiv W(z^1, z^2, z^3; \alpha) \equiv \varphi(-z^1, z^2) + \alpha\varphi(-z^2, z^3)$$

that go beyond those studied above in connection with the test of invariance of an initial stock $z = z^1 = z^3$. Recall that the latter test confirms the invariance of such a z if the maximum of W with respect to z^2 is attained when also $z^2 = z$. Let this be the case.

The heuristic test of stability then applies in the case where, for all z^1 and z^3 in a neighborhood $(z-\epsilon, z+\epsilon)$ of z , the value $\hat{z}^2$ of z^2 that maximizes W is a strictly increasing function both of z^1 and of z^3 . While this, let us say, strong smoothness condition on W may seem arbitrary, it becomes more natural if we think in terms of a class of functions W which, in the limit for smaller and smaller time units, permits a smooth transition to a continuous time variable.

*As indicated at the end of Section 1, this Appendix was added after the Conference.

To apply the stability test to the invariant stock z , let $z < z' < z+\epsilon$, and now take $z^1 = z' = z^3$. write $\hat{z}'$ for the value of z^2 maximizing $W(z', z^2, z')$. The strong smoothness condition then requires that $\hat{z}' > z$. The stability test then says that,

$$\left.\begin{array}{l} z \text{ is stable if } z < \hat{z}' < z' \\ z \text{ is unstable if } z' < \hat{z}' \end{array}\right\} \text{ whenever } z < z' < z+\epsilon .$$

To be conclusive the test would also need to be applied symmetrically to all z' with $z-\epsilon < z' < z$.

Figure 5.1 illustrates by heavy lines a case where the stability test is met by a z' with $z < z' < z+\epsilon$. The thin lines give plausibility to the test by suggesting a limiting process converging to the optimal path from an initial stock $z^1 = z'$ by alternately holding the capital stocks constant in odd-numbered and even-numbered points of time while optimizing at all other points. The convergence follows from the strong smoothness condition.

So far the discussion has been concerned with the stability of a capital stock $z \equiv \hat{z}(\alpha)$ invariant for a given value of the discount factor α . We shall now show that the same test also answers the question whether, for a discount factor α' slightly larger (or smaller) than α , we have the intuitive case where the corresponding invariant stock $z' \equiv \hat{z}(\alpha')$ is also larger (smaller) than z , or the counterintuitive case where $z' < (>)\ z$.

The conditions defining z and z' are

$$W(z^2) \equiv \varphi(-z, z^2) + \alpha\varphi(-z^2, z) \text{ is maximal for } z^2 = z ,$$

$$W'(z^2) \equiv \varphi(-z', z^2) + \alpha'\varphi(-z^2, z') \text{ is maximal for } z^2 = z' .$$

Necessary and sufficient conditions for these to hold are, respectively,

$$\varphi_2(-z,\ z) \quad - \alpha\, \varphi_1(-z,\ z) \quad = 0 \ ,$$

$$\varphi_2(-z',\ z') - \alpha' \varphi_1(-z',\ z') = 0 \ .$$

The stability test involves a third function of a W-type, viz.,

$$W^{(')} \equiv \varphi(-z',\ z^2) + \alpha\varphi(-z^2,\ z') \ ,$$

and, is, on account of the strict concavity of $W^{(')}$, equivalent to

$$z \text{ is } \begin{bmatrix} \text{stable} \\ \text{unstable} \end{bmatrix} \text{ if } \left(\frac{dW^{(')}}{dz^2}\right)_{z^2 = z'} = \varphi_2(-z',\ z') - \alpha\varphi_1(-z',\ z') \begin{bmatrix} < \\ > \end{bmatrix} 0$$

for $|z - z'| < \varepsilon$. But then, since this expression vanishes if α is replaced by α' , and since φ_1 is positive, we find that z is stable if $\alpha < \alpha'$, unstable if $\alpha > \alpha'$. Note that these are precisely the intuitive and the counter-intuitive case, respectively, with regard to the direction of change of the invariant capital stock when the discount factor is changed.

To illustrate the implications of this finding (see Figure 5.2), assume that the counter-intuitive behavior of $\hat{z}(\alpha)$ applies throughout the interval $\mathcal{A}$ of Figure 3.5. Let as before $\underline{z}$, $\bar{z}$ denote the capital stocks corresponding to the lower and upper end points of $\mathcal{A}$, and $\bar{\alpha}$, $\underline{\underline{\alpha}}$, respectively, the corresponding discount factors. Then $\underline{z} < \bar{z}$, $\underline{\underline{\alpha}} < \bar{\alpha}$, and $\underline{z} = \hat{z}(\bar{\alpha})$, $\bar{z} = \hat{z}(\underline{\underline{\alpha}})$ are invariant capital stocks for the discount factors shown, provided $\underline{\underline{\alpha}} \geqq \underline{\alpha} \equiv b_3/a_3$. Now take an α with $\underline{\underline{\alpha}} < \alpha < \bar{\alpha}$, and study the dependence of the optimal path z^t on the prescribed initial value z^1 . Then, if by chance $z^1 = \hat{z}(\alpha)$, the path z^t continues on the constant level of the unstable invariant capital stock $\hat{z}(\alpha)$. If $\underline{z} < z^1 < \hat{z}(\alpha)$, by however little, z^t decreases until the level $\underline{z}$ is reached, whereupon the path continues at that level. Likewise, if $\hat{z}(\alpha) < z^1 < \bar{z}$, the path increases until it becomes constant at the level $\bar{z}$. In fact, $\underline{z}$ is a stable invariant stock $\hat{z}(\alpha')$ for all α' such that $\underline{\underline{\alpha}} \leq \alpha' < \bar{\alpha}$, and $\bar{z}$ is a stable stock for all α' such that $\underline{\underline{\alpha}} < \alpha' \leq 1$. Hence, under the present assumptions, the endpoints $\underline{z}$, $\bar{z}$ are prototypes of empirically meaningful invariant capital stocks, while $\hat{z}(\alpha)$ is a freak, a knife-edge occurrence. Its only conceivable empirical significance is a signal that for z^1 in a neighborhood of $\hat{z}(\alpha)$ the many features of reality not expressed in an otherwise acceptable model will influence the outcome of the toss of a coin.

Iwai [1975] has confirmed the heuristic reasoning of this appendix by a rigorous application of stability analysis that examines the behavior of second derivatives of the function $W(z^1, z^2, z^3; \alpha)$ for the z^t in a neighborhood of an invariant capital stock $\hat{z}(\alpha)$. His analysis also includes the case where strong smoothness of W is not assumed. It can then happen that, for some $z^1, z^3 > \hat{z}(\alpha)$, the value of z^2 maximizing W satisfies $z^2 < \hat{z}(\alpha)$. In such a case optimal paths can oscillate between values above and below $\hat{z}(\alpha)$, with stability not governed by the criterion found above.

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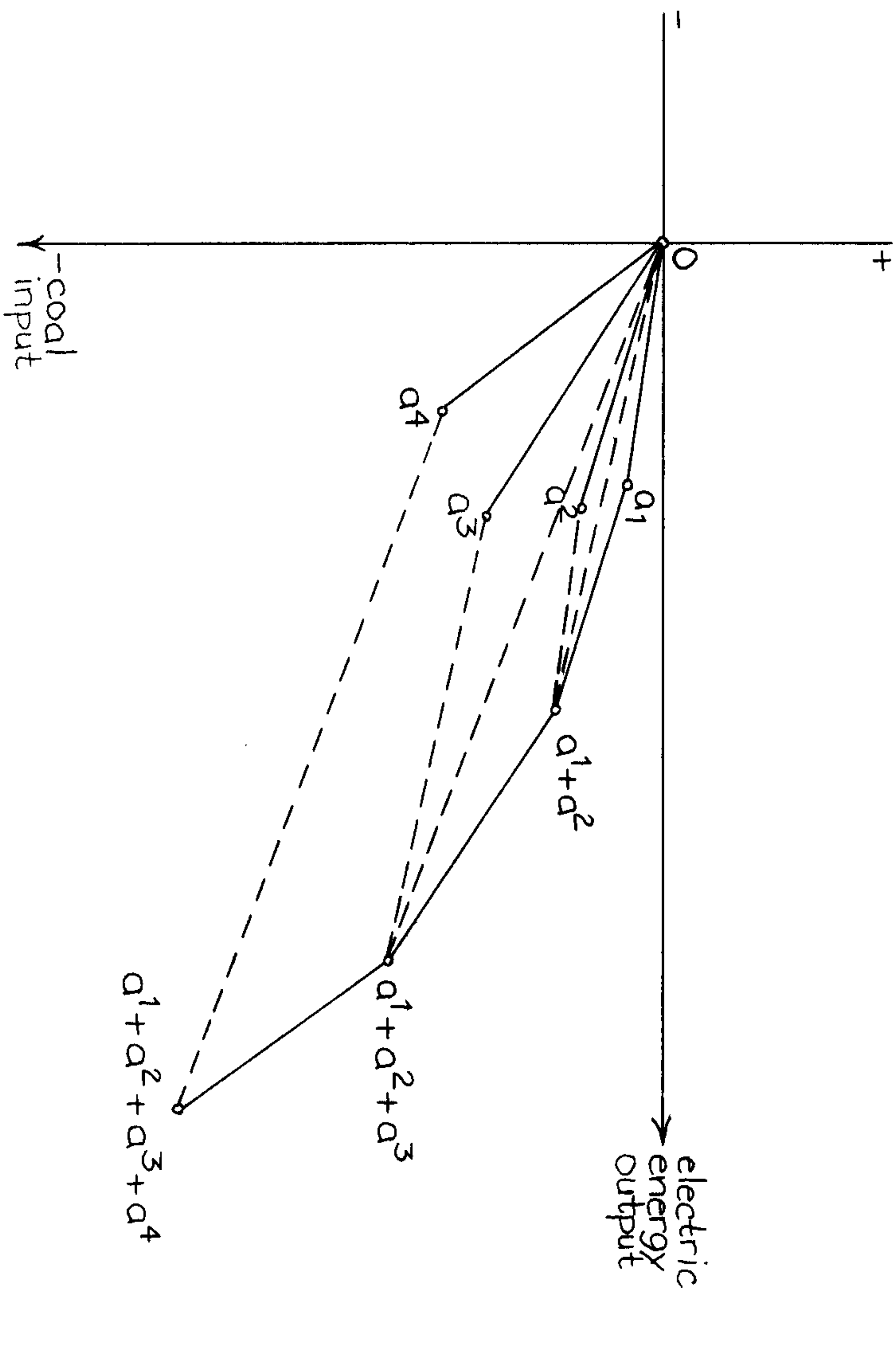
FIGURE 2.1



-coal
input
+
O
a1
a2
a3
a4
a1+a2
a1+a2+a3
a1+a2+a3+a4
electric
energy
output

FIGURE 2.2

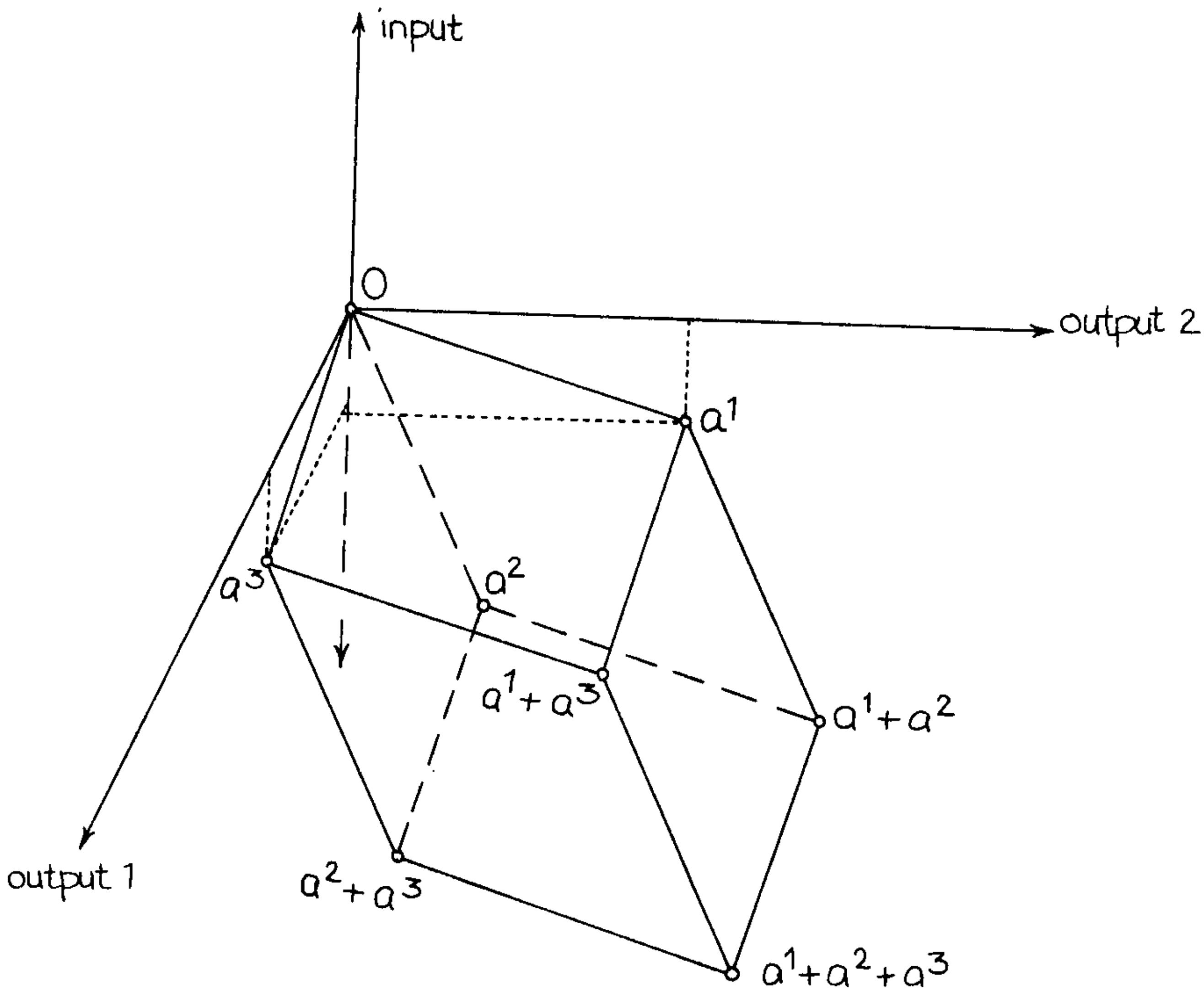


FIGURE 3.1

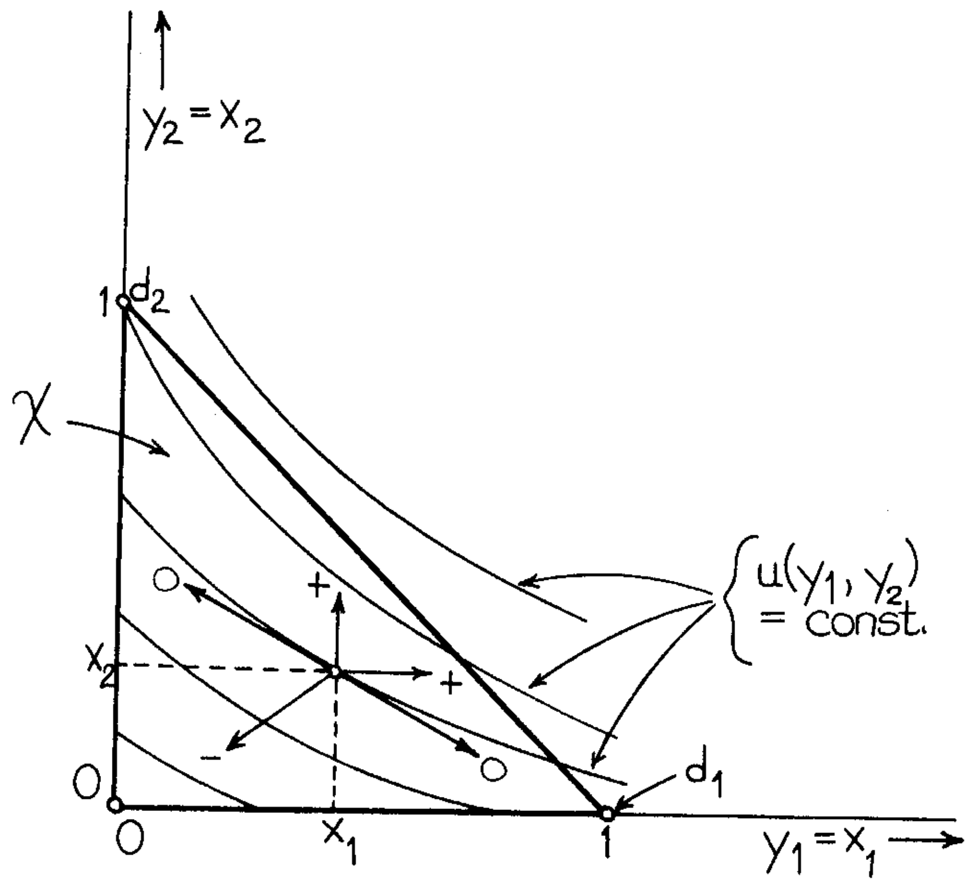


FIGURE 3.2

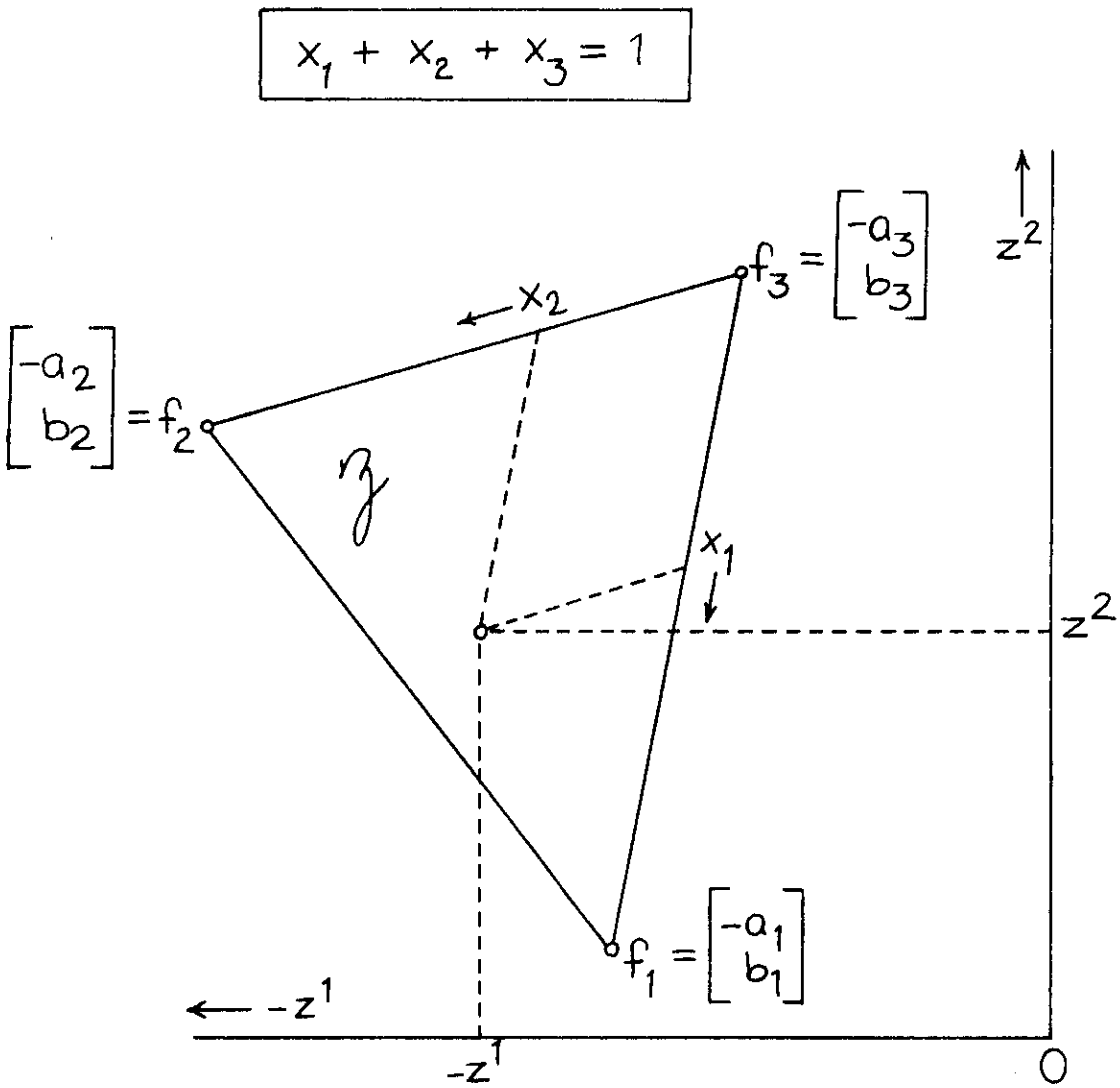
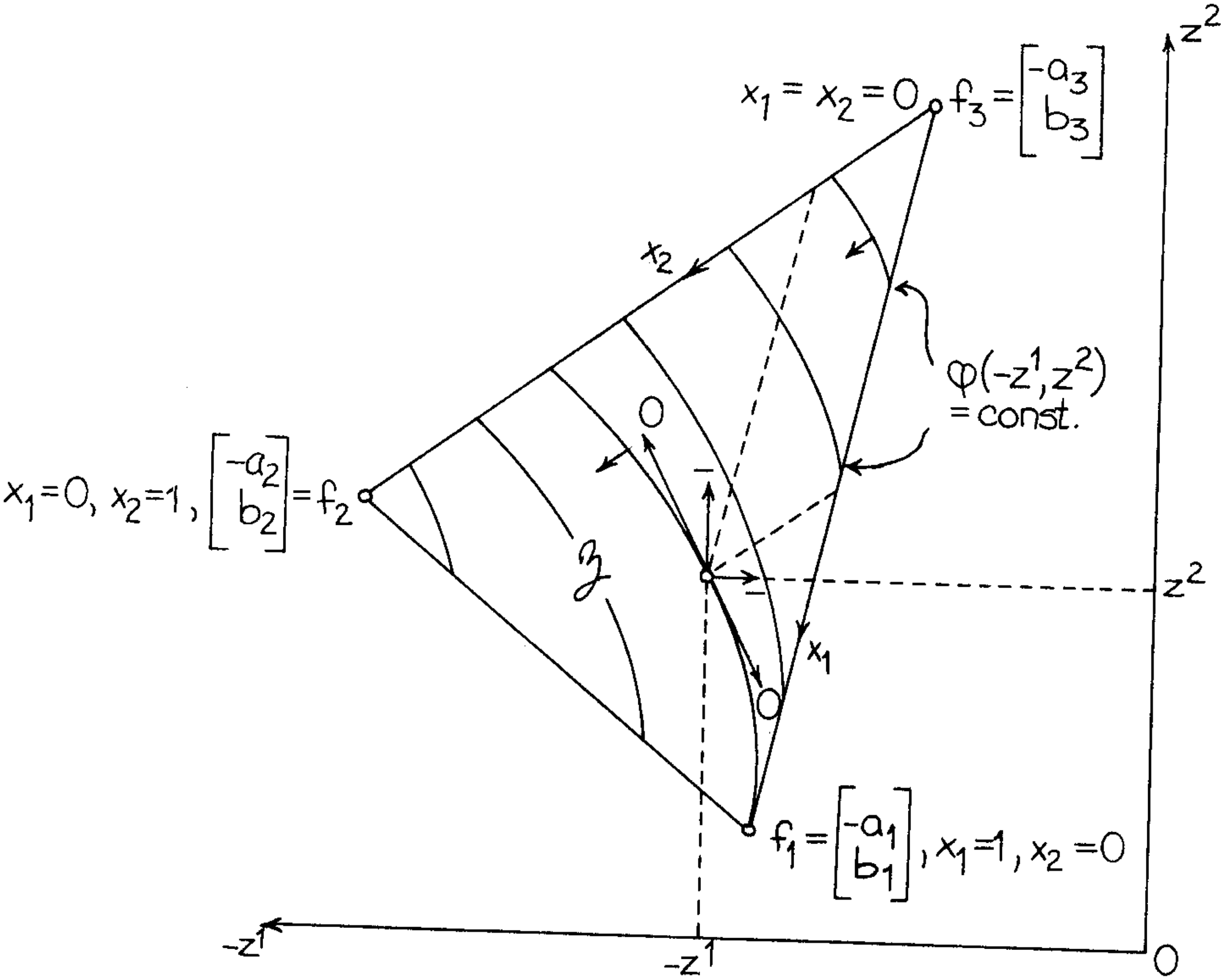

$x_1 + x_2 + x_3 = 1$
$f_3 = \begin{bmatrix} -a_3 \\ b_3 \end{bmatrix}$
z^2
x_2
$\begin{bmatrix} -a_2 \\ b_2 \end{bmatrix} = f_2$
x_1
z^2
$f_1 = \begin{bmatrix} -a_1 \\ b_1 \end{bmatrix}$
$-z^1$
$-z^1$
0

FIGURE 3.3



$x_1 = x_2 = 0$, $f_3 = \begin{bmatrix} -a_3 \\ b_3 \end{bmatrix}$
z^2
x_2
$\varphi(-z^1, z^2)$ = const.
O
$x_1 = 0$, $x_2 = 1$, $\begin{bmatrix} -a_2 \\ b_2 \end{bmatrix} = f_2$
z^2
x_1
O
$f_1 = \begin{bmatrix} -a_1 \\ b_1 \end{bmatrix}$, $x_1 = 1$, $x_2 = 0$
$-z^1$
$-z^1$
O

FIGURE 3.4

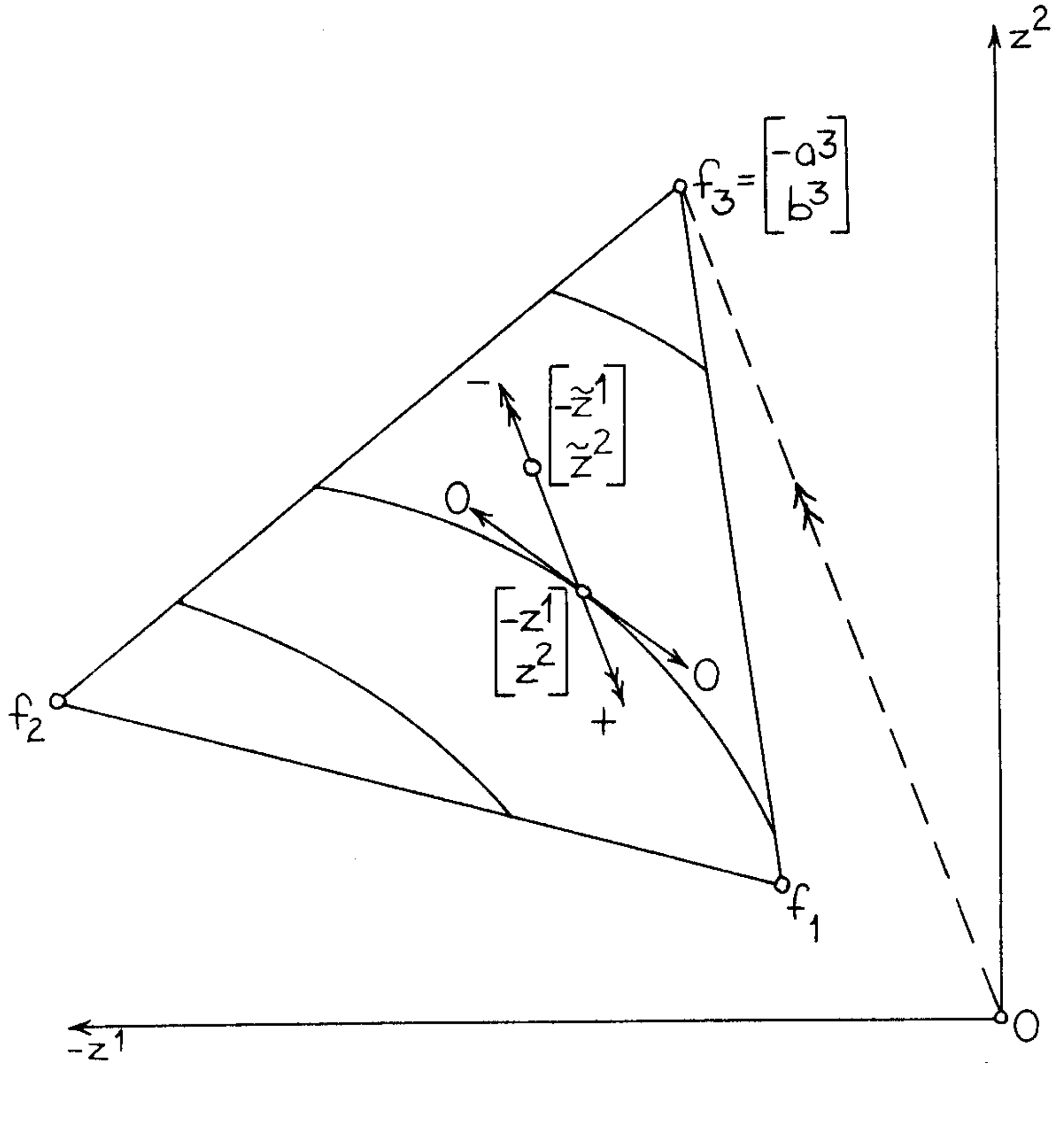


FIGURE 3.5

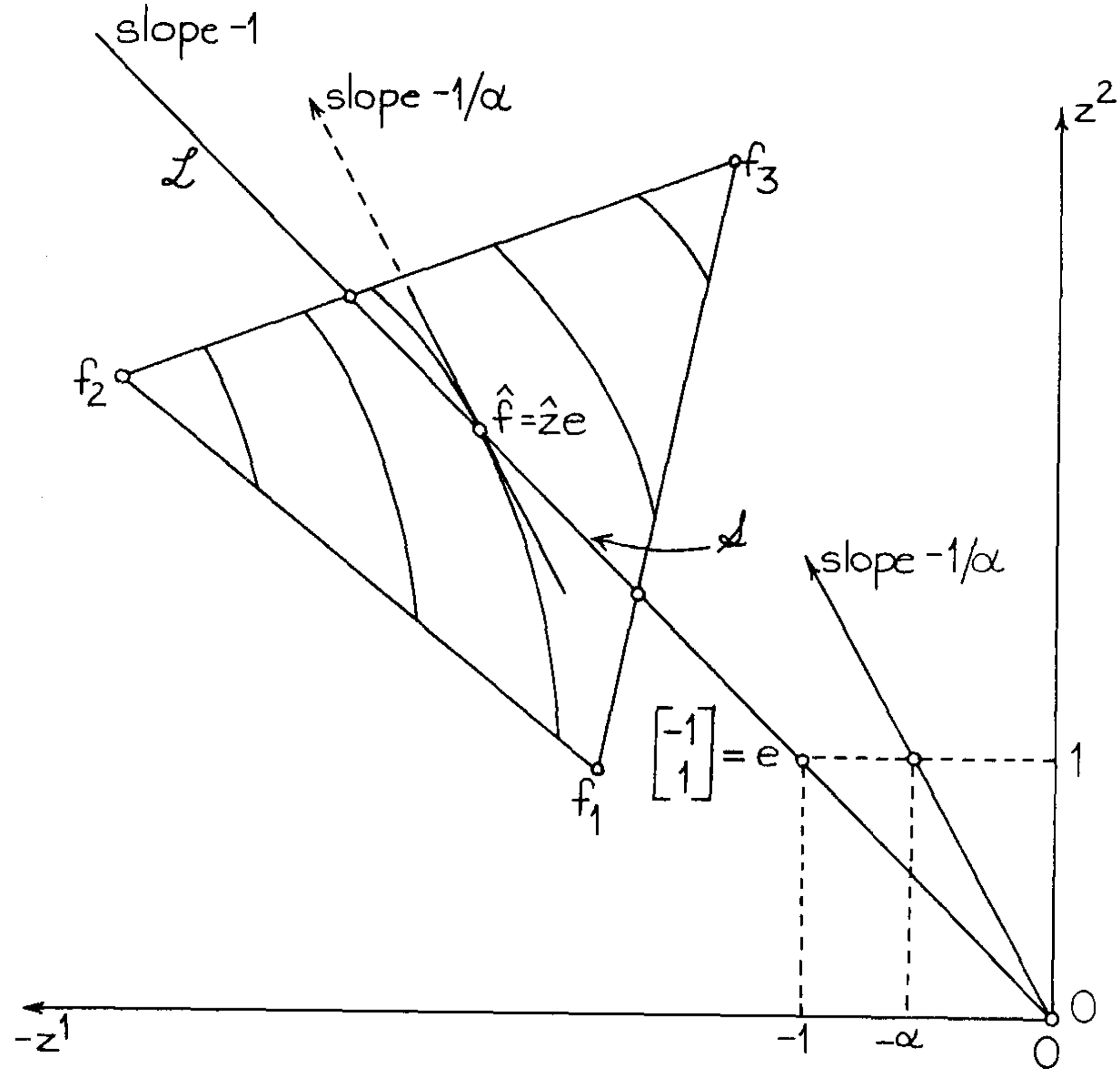


FIGURE 4.1

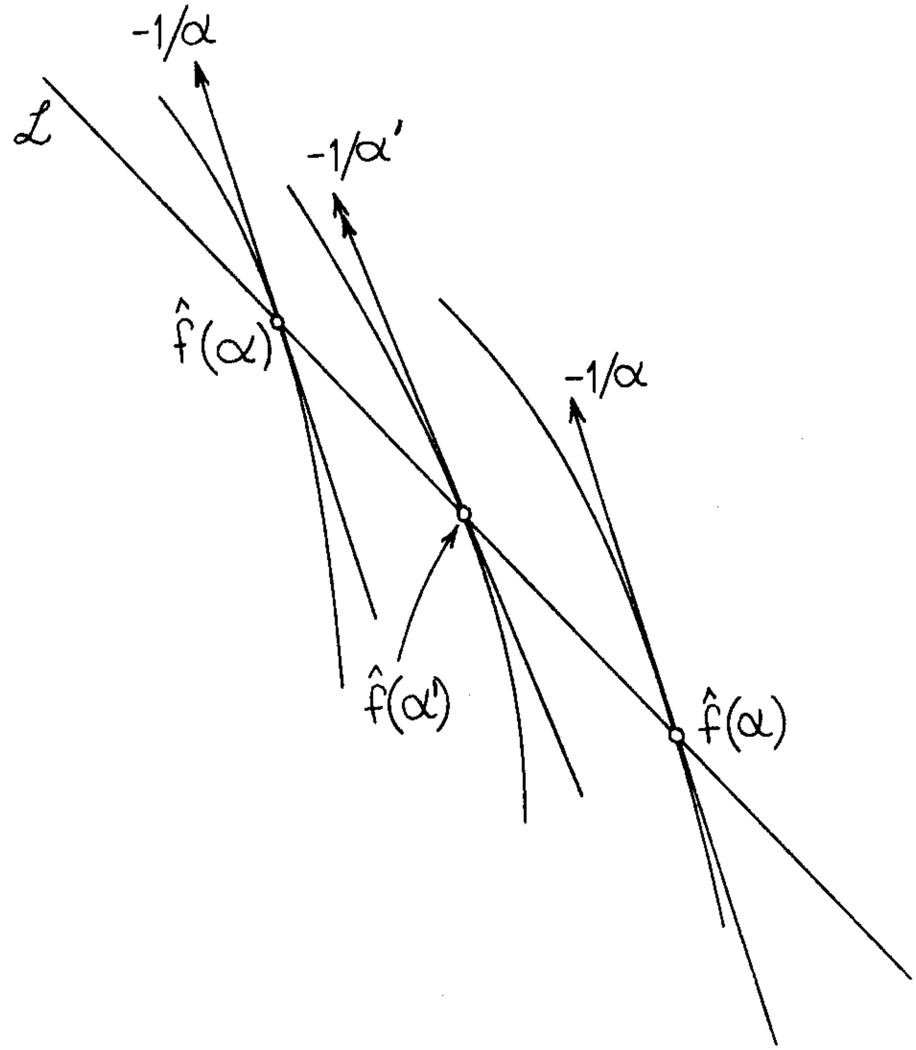


FIGURE 4.2

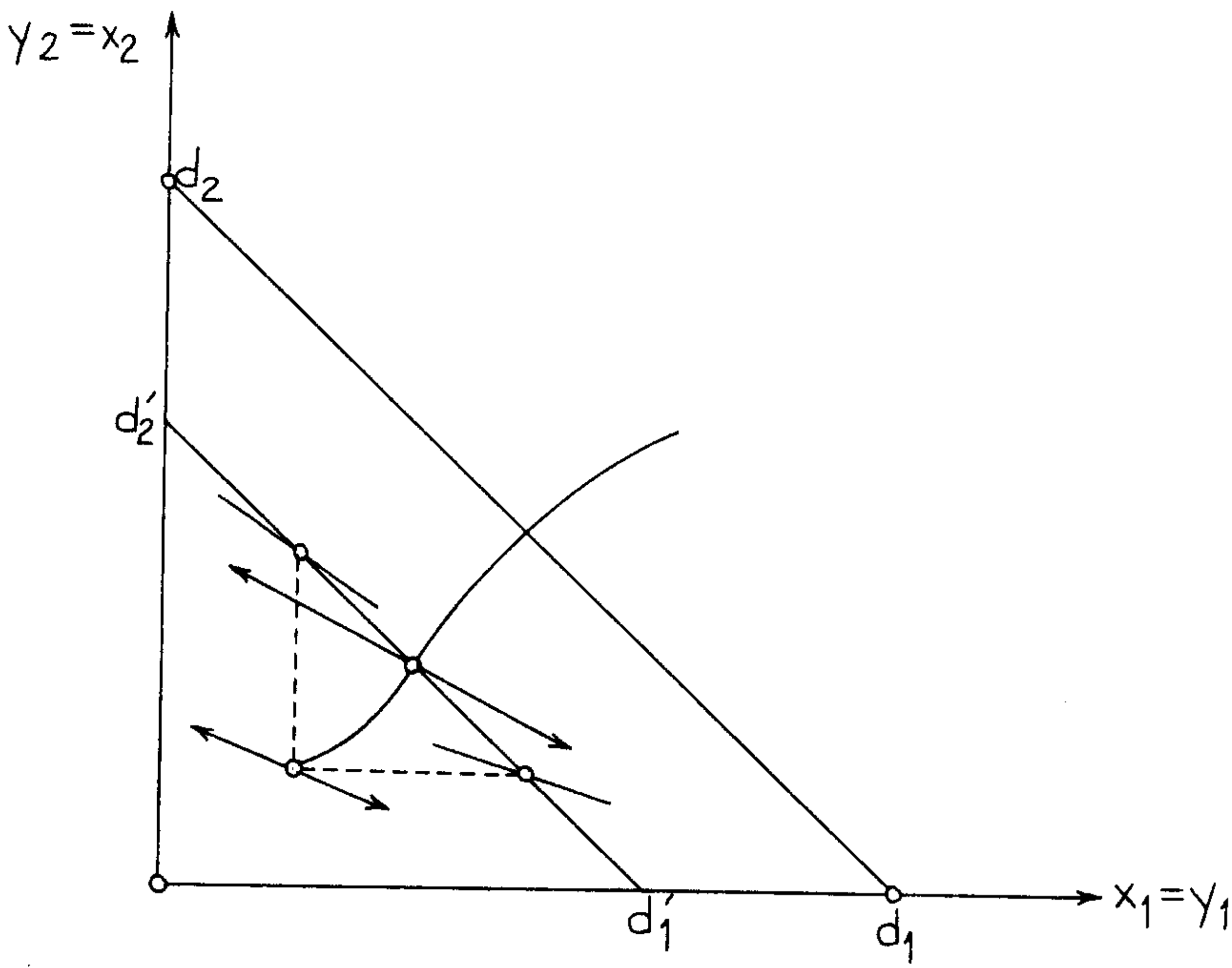


FIGURE 4.3

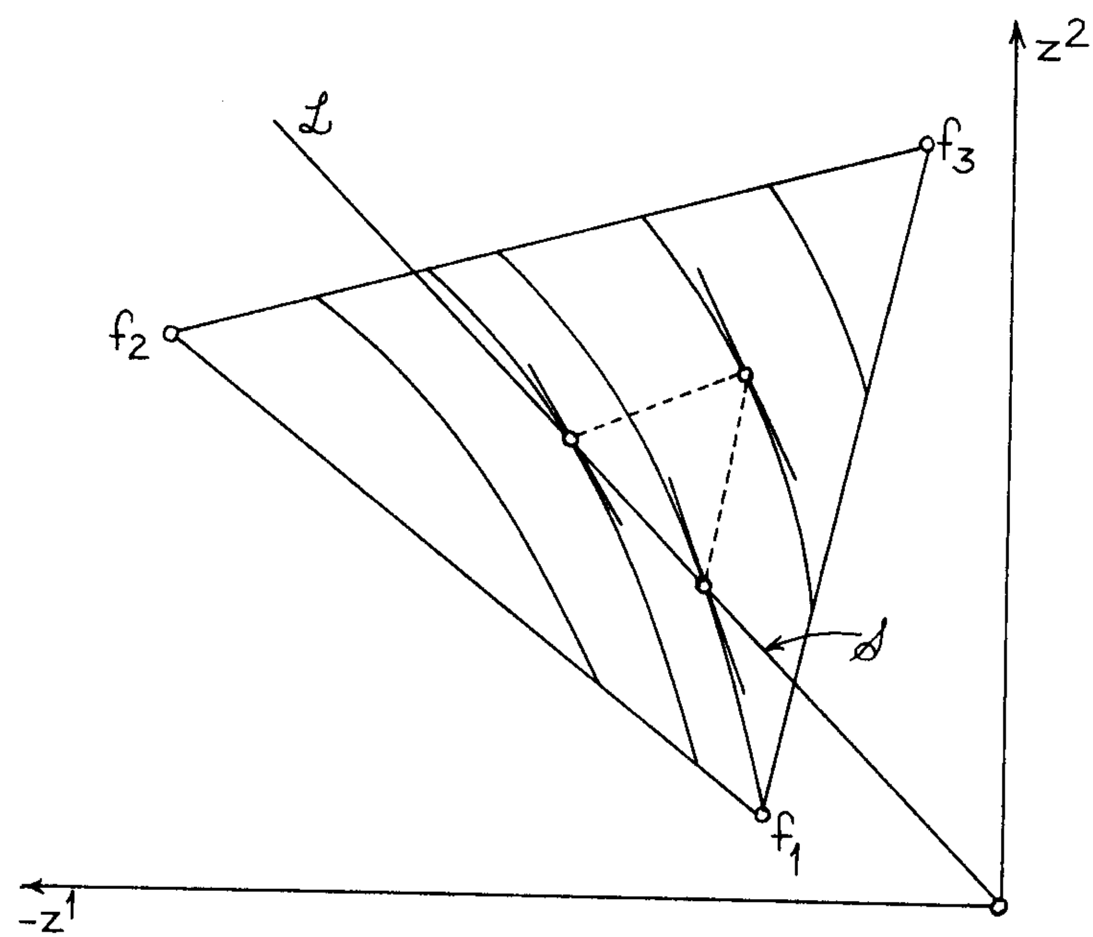


FIGURE 4.4

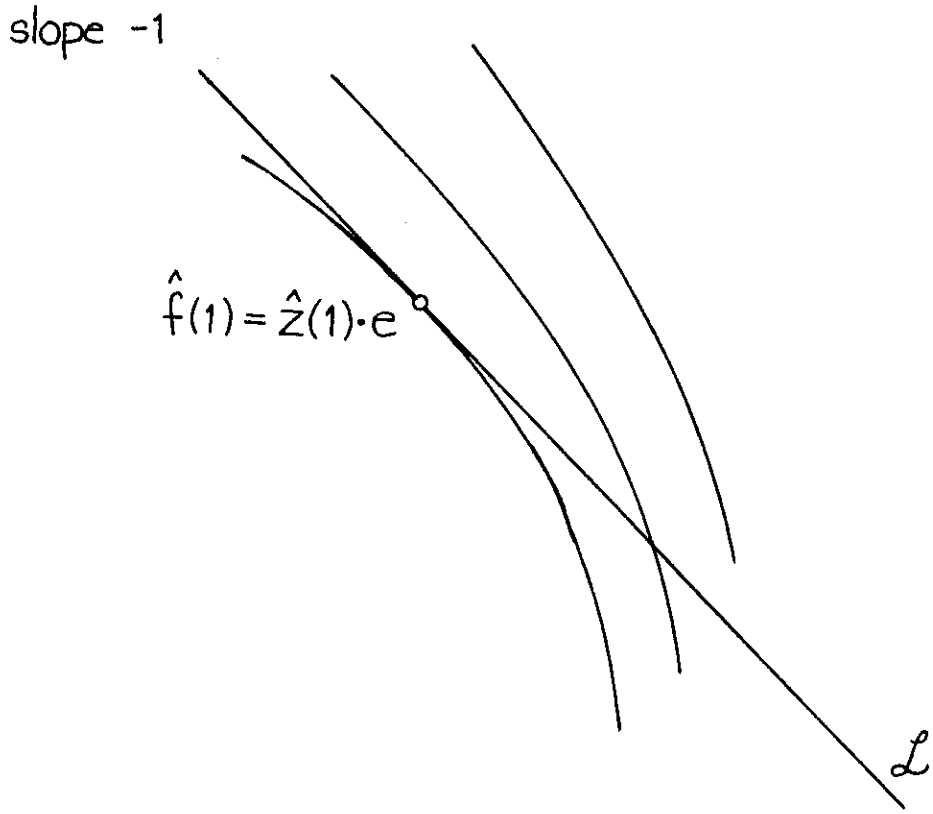


FIGURE 4.5

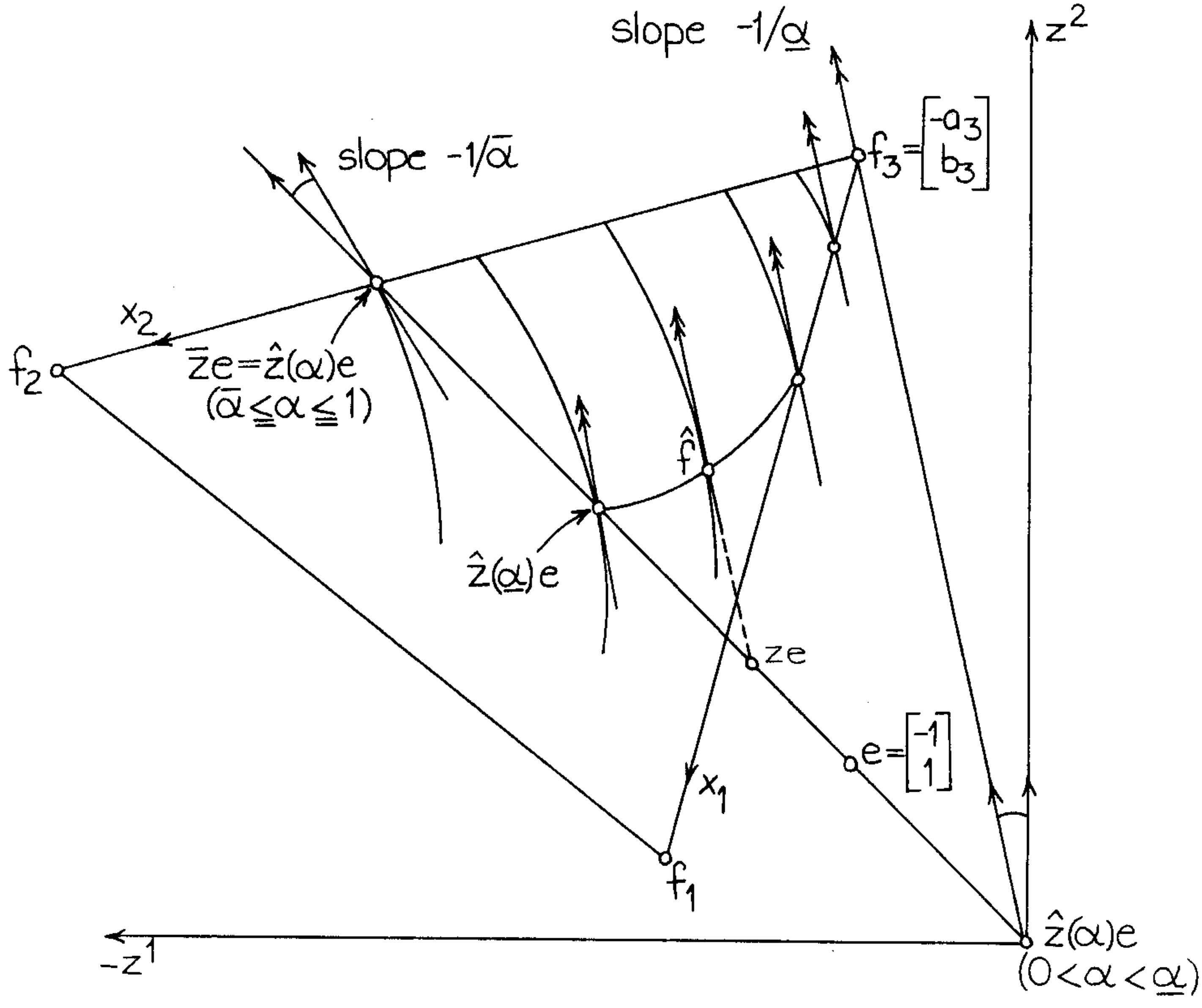

slope -1/α̲
z²
slope -1/ᾱ
f₃=[-a₃ b₃]
x₂
f₂
z̄e=ẑ(α)e
(ᾱ≦α≦1)
f̂
ẑ(α̲)e
ze
e=[-1 1]
x₁
f₁
-z¹
ẑ(α)e
(0<α<α̲)

FIGURE 4.6

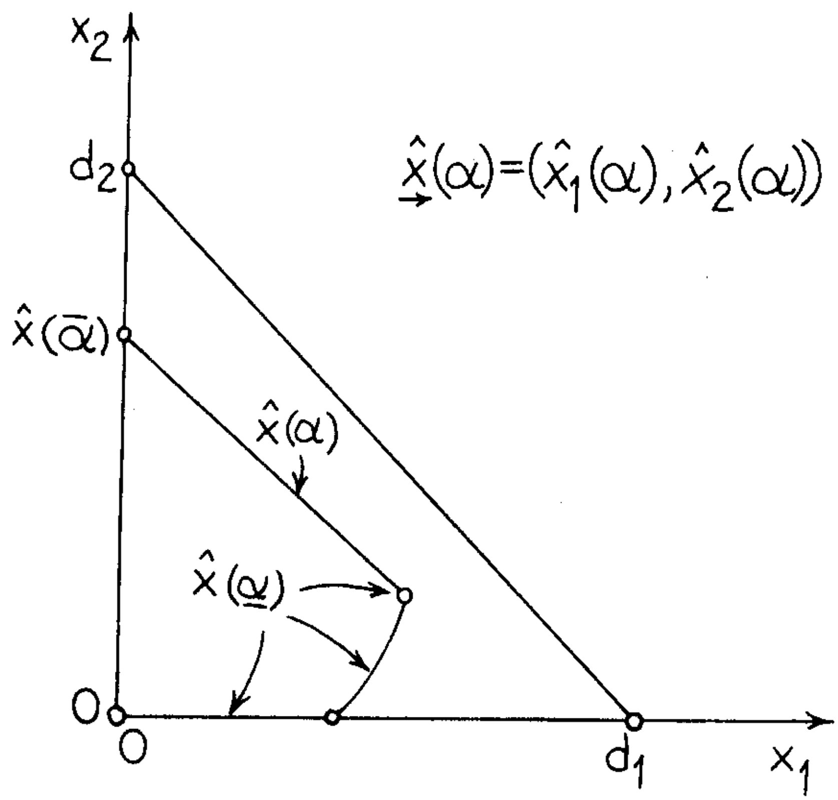

x_2
d_2
$\hat{\underline{x}}(\alpha)=(\hat{x}_1(\alpha),\hat{x}_2(\alpha))$
$\hat{x}(\bar{\alpha})$
$\hat{x}(\alpha)$
$\hat{x}(\underline{\alpha})$
0
0
d_1
x_1

FIGURE 5.1

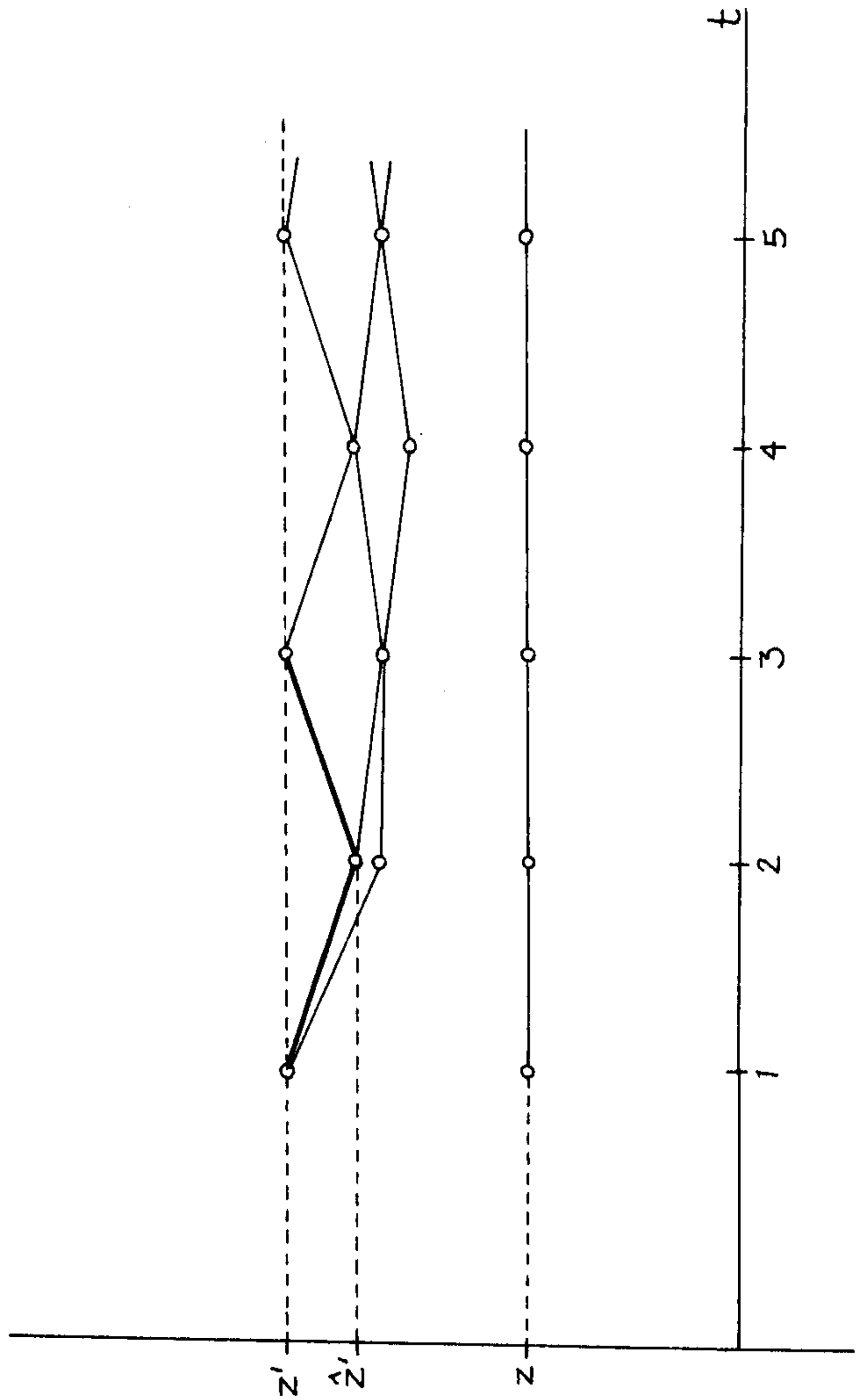


FIGURE 5.2