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The Cores of Large Standard Exchange Economies

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by

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I. Introduction

An exchange economy consists of a set of traders each of whom is characterized by an initial endowment and a preference relation. In addition, one usually assumes that the set of traders is finite. Edgeworth's conjecture that as the number of traders in an exchange economy increases, the core approaches the set of competitive equilibria has been formalized in two disparate ways by mathematical economists.

One approach has been to talk about a sequence of economies growing without bound and to look at the relationship between the core and the set of competitive equilibria for very large economies. This was the method of Debreu-Scarf [5].

The other approach has been to consider an exchange economy having an infinite number of traders, to define the notions of core and competitive equilibrium in this economy, and to show the equivalence between these two concepts. Aumann's work on continuous economies [2] has been of this nature.

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Here we report the results obtained by a new method for the resolution of Edgeworth's conjecture, based on nonstandard analysis, which synthesizes the asymptotic method of Debreu-Scarf and the infinite method of Aumann. We have shown in [3] that within nonstandard analysis the concept of the core and competitive equilibrium are the same. As a consequence of this theorem we have derived a number of asymptotic results concerned with infinite families of standard (finite) exchange economies.

In Section II we give a brief introduction to the essentials of nonstandard analysis; in Section III we describe the economic model and state our theorem concerning Edgeworth's conjecture; in Section IV, we give an asymptotic result or limit theorem for the cores of large standard economies.

II. Nonstandard Analysis

Let R be the system of real numbers. Any statement about R (involving individuals, subsets of R, functions on R, relations on R, sets of relations on R, etc.) can be expressed in a formal language which includes: names for all these entities; connectives: $\neg$, $\wedge$, $\vee$, $\Longrightarrow$; variables for entities of different types; quantifiers $\exists$ and $\forall$ over different types of variables.

Let K be the set of all statements in this formalized language which are true for R. Then it can be shown that there exists: a proper extension, $^*R$, of R; a designated subset of the set of all subsets of $^*R$; a designated subset of the set of all functions from $^*R$ to $^*R$; a designated subset of the set of all relations on $^*R$; etc., such that the following holds. Every statement which is true in R remains true in $^*R$ provided we reinterpret the existential quantifiers which occur in that statement as follows: "there exists" a set shall mean "there exists" an internal set, "there exists" a family of relations, shall mean "there exists" an internal family of relations, similarly for all other types of entities. Here internal means an element of the appropriate designated sets of entities. (Note that this qualification does not apply to individuals.)

We shall call: the elements of $^*R$, real numbers; the elements of R, standard real numbers; and the elements of $^*R$ which do not belong to R, nonstandard real numbers. As a proper extension of R, $^*R$ must be a nonarchimedean field, that is, it contains numbers whose absolute values are greater than all standard real numbers, which will be called infinite numbers. All other numbers will be called finite. The reciprocals of infinite numbers together with zero are said to be infinitesimals or infinitely small numbers. Every finite number, x, is infinitely close to a unique standard real number, ^{o}x, which is called the standard part of x. The monad of a number, $\mu(x)$, is the set of all numbers which are infinitely close to x.

$^*R$ also contains a set of numbers, denoted as $^*N$, which has the same properties as N (the set of natural numbers) in the sense that any statement true about N is true about $^*N$ when reinterpreted in terms of internal entities. $^*N$ is a proper extension of N. We shall call: the elements of $^*N$, natural numbers; the elements of N, finite natural numbers; and the elements of $^*N$ which do not belong to N, infinite natural numbers. An internal set which has ω elements, where $\omega \in {}^*N$, is said to be starfinite.

A structure $^*R$ of the required kind may be constructed as an ultrapower of R over the natural numbers. That is to say $^*R$ consists of all sequences of real numbers; equality between such sequences as well as any other kind of relation is defined with respect to a free ultrafilter in the Boolean algebra $\mathcal{P}(N)$, the power set of the natural numbers. Similarly internal sets, internal relations, etc., can be identified with sequences of sets, relations, etc. again reduced with respect to the ultrafilter. Compare references [6], [7], [8].

We shall not use the construction just mentioned, but we assume that the following particular property of the $^*R$ just constructed is satisfied. Let f be a function with domain N whose range values are numbers (internal sets of numbers) of $^*R$. Then there exists an internal sequence of numbers (internal sets of numbers) of $^*R$, such that the values of the sequence agree with the values of f on all finite j. That is, if $f : N \to {}^*R$, then there exists an internal $g : {}^*N \to {}^*R$ such that for all $j \in N$, $g(j) = f(j)$.

$^*R_n$ is the n-fold cartesian produce of $^*R$ and $^*\Omega_n$ is the positive orthant of $^*R_n$. Let $\bar{x}$, $\bar{y}$ be vectors in $^*R_n$. The <u>monad of</u> $\bar{x}$, $\mu(\bar{x})$ is the set of points whose distance from $\bar{x}$ is an infinitesimal. If $\bar{y} \in \mu(\bar{x})$, we shall write $\bar{x} \simeq \bar{y}$, $\bar{x} \geq \bar{y}$ means $x_i \geq y_i$ for all i; $\bar{x} > \bar{y}$ means $x_i \geq y_i$ for all i and $x_i > y_i$ for some i; $\bar{x} \gg \bar{y}$ means $x_i > y_i$ for all i. $\bar{x} \gtrsim \bar{y}$ means that $\bar{x} \geq \bar{y}$ or $\bar{x} \simeq \bar{y}$. $\bar{x} \not\gtrsim \bar{y}$ means that x_i is greater than y_i by a finite amount for all i.

III. Economic Model and Edgeworth's Conjecture

Let T be an initial segment of ${}^*N$, where $|T|$, the number of elements in T, is ω some infinite natural number. That is, $T = \{1, 2, \ldots, \omega\}$ and $\omega \in {}^*N - N$. T is to be interpreted as the set of traders in the economy. If S is any internal subset of T, then $|S|$ will denote the number of elements in S.

A nonstandard exchange economy, $\mathcal{E}$, consists of a pair of functions I and P, where $I : T \to {}^*\Omega_n$ and $P : T \to {}^*\Omega_n \times {}^*\Omega_n$. Denoting the functions I and P respectively as $\{\bar{x}_t\}_{t=1}^{\omega}$ and $\{\succ_t\}_{t=1}^{\omega}$, $\langle I(t), \succ_t \rangle =$ $\mathcal{E} = \langle \{\bar{x}_t\}_{t=1}^{\omega}, \{\succ_t\}_{t=1}^{\omega} \rangle$. $\bar{x}_t$ is the initial endowment of the t^{th} trader and $\succ_t$ is his preference relation over ${}^*\Omega_n$, the space of commodities. The nonstandard exchange economies which we will consider are assumed to have the following properties:

(i) The function indexing the initial endowments, $I(t)$, is internal.

(ii) $I(t)$ is standardly bounded, i.e. there exists a standard vector $\bar{r}_0$ such that for all t, $I(t) \le \bar{r}_0$.

(iii) $\frac{1}{\omega} \sum_{1}^{\omega} I(t) \not\approx \bar{0}$.

(iv) The relation, Q, where $Q = \{\langle t, \succ_t \rangle | t \in T, \succ_t \subseteq {}^*\Omega_n \times {}^*\Omega_n\}$ is internal. For all t,

(α) $\succ_t$ is irreflexive, i.e. if $\bar{x} \succ_t \bar{y}$ then $\bar{x} \neq \bar{y}$

(β) If $\bar{x} > \bar{y}$ then $\bar{x} \succ_t \bar{y}$

(γ) If $\bar{x} \not\approx \bar{y}$ and $\bar{x} \succ_t \bar{y}$ then there exists a standard $\delta > 0$ such that if $\bar{z} \in S(\bar{x}, \delta)$, open ball with center $\bar{x}$ and radius δ, then $\bar{z} \succ_t \bar{y}$.

Equivalently, if $\bar{x} \not\leq \bar{y}$ and $\bar{x} \succ_t \bar{y}$ and $\bar{z} \in \mu(\bar{x})$ then $\bar{z} \succ_t \bar{y}$. It is shown in [3] that these assumptions are consistent.

An <u>assignment</u> Y is an internal function $Y(t)$ from T, the set of traders, into ${}^*\Omega_n$.

An <u>allocation or final allocation</u> is a standardly bounded assignment $Y(t)$ from the set of traders, T, into ${}^*\Omega_n$ such that $\frac{1}{\omega} \sum_{t=1}^{\omega} Y(t) \simeq \frac{1}{\omega} \sum_{t=1}^{\omega} I(t)$.

(iv) implies that for all internal $X, Y \in {}^*\Omega_n^T$, where T is the set of traders, that

(v) $\{t | X(t) \succ_t Y(t)\}$ is an internal set of traders.

A <u>coalition,</u> S, is defined as an internal set of traders. It is said to be negligible if $|S|/\omega \simeq 0$. Note that if S is negligible then for all allocations $X(t)$, $\frac{1}{\omega} \sum_{t \in S} X(t) \simeq \bar{0}$.

A coalition, S, is <u>feasible</u> with respect to an allocation Y if $\frac{1}{\omega} \sum_{t \in S} Y(t) \simeq \frac{1}{\omega} \sum_{t \in S} I(t)$.

An allocation Y <u>dominates</u> an allocation X via a coalition S if S is feasible with respect to Y and if for all $t \in S$, $X(t) \not\geq Y(t)$ and $Y(t) \succ_t X(t)$.

The <u>core</u> is defined as the set of all allocations X which are not dominated by any allocation Y via any non-negligible coalition.

A <u>price vector,</u> $\bar{p}$, is finite nonstandard vector in ${}^*\Omega_n$ such that $\bar{p} \gneqq \bar{0}$.

The t^{th} trader <u>budget set,</u> $B_{\bar{p}}(t)$, is $\{\bar{x} \in {}^*\Omega_n | \bar{p} \cdot \bar{x} \lesssim \bar{p} \cdot I(t)\}$.

$\bar{y}$ is said to be <u>maximal in</u> $B_{\bar{p}}(t)$ if $\bar{y} \in B_{\bar{p}}(t)$ and there does not exist an $\bar{x} \in B_{\bar{p}}(t)$ such that $\bar{x} \not\leq \bar{y}$ and $\bar{x} >_t \bar{y}$.

A <u>competitive equilibrium</u> is defined as a pair $\langle \bar{p}, X \rangle$, where $\bar{p}$ is a price vector and X an allocation such that $X(t)$ is maximal in $B_{\bar{p}}(t)$ for almost all the traders. That is, if $K = \{t \mid X(t)$ is maximal in $B_{\bar{p}}(t)\}$ then $|K|/\omega \simeq 1$.

<u>Theorem 1</u>. If $\mathcal{E}$ is a nonstandard exchange economy satisfying the above assumptions, then an allocation X is in the core of $\mathcal{E}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle \bar{p}, X \rangle$ is a competitive equilibrium of $\mathcal{E}$.

The proof of this theorem is given in [3].

IV. <u>Limit Theorem</u>

A standard exchange economy $\mathcal{E}$ of size m consists of m traders, where m is a standard natural number, whose initial endowments and preferences are restricted to the standard commodity space Ω_n . Ω_n is the positive orthant of R_n , the n-fold cartesian product of R . Let

$= \langle I(t), >_t \rangle$ where for all t , $I(t) \in \Omega_n$ and $>_t \in \Omega_n \times \Omega_n$, $t \in \mathcal{E}$ will refer to the t^{th} traders endowment and preference relation.

A <u>competitive equilibrium</u> for $\mathcal{E}$ is a pair $\langle \bar{p}, X \rangle$ such that: $\bar{p} \in \Omega_n$; $\sum_{t=1}^{m} X(t) = \sum_{t=1}^{m} I(t)$; for each t , $X(t) \in \Omega_n$ and $\bar{p} \cdot X(t) \leq \bar{p} \cdot I(t)$; there does not exist a $\bar{y} \in \Omega_n$ where $\bar{p} \cdot \bar{y} \leq \bar{p} \cdot I(t)$ and $\bar{y} >_t X(t)$. $\bar{p}$ is called a <u>price vector</u> and $X(t)$ a <u>competitive allocation</u>.

$X(t)$ is an <u>allocation</u> if for each t , $X(t) \in \Omega_n$ and

$\sum_{t=1}^{m} X(t) = \sum_{t=1}^{m} I(t)$. An allocation $X(t)$ is blocked by an allocation $Z(t)$ if there exists a coalition of traders, S , such that $\sum_{t \in S} Z(t) = \sum_{t \in S} I(t)$ and for all $t \in S$, $Z(t) >_t Y(t)$. The core of $\mathcal{E}$ is the set of unblocked allocations.

Let $\mathcal{G} = \{\mathcal{E}_i\}_{i \in \vartheta}$ be an unbounded family of standard exchange economies, i.e. for each $k \in N$, there exists $i \in \vartheta$ such that $|\mathcal{E}_i|$, the number of traders in the i^{th} economy, is greater than k . Suppose $\mathcal{G}$ satisfies the following conditions:

(1) The initial endowments of the traders in $\mathcal{G}$ are uniformly bounded from above.

(2) The initial endowments of the traders in $\mathcal{G}$ are uniformly bounded away from zero in each commodity.

(3) Each trader's preference relation is "irreflexive," "continuous," and "strongly monotonic." $>_t$ is irreflexive if for all $\bar{x} \in \Omega_n$, $\bar{x} \not>_t \bar{x}$. $>_t$ is continuous if for all $\bar{x}, \bar{y} \in \Omega_n$ the sets $\{\bar{z} \in \Omega_n | \bar{z} >_t \bar{x}\}$ and $\{\bar{z} \in \Omega_n | \bar{y} >_t \bar{z}\}$ are open sets in R^m . $>_t$ is strongly monotonic if $\bar{x} > \bar{y}$ implies that $\bar{x} >_t \bar{y}$.

(4) The family of all trader's preference relations in $\mathcal{G}$ is equicontinuous on Ω_n .

A family, $\mathcal{F}$, of preference relations is said to be equicontinuous on Ω_n if $(\forall \epsilon > 0)(\exists \delta > 0)(\forall t \in \mathcal{F})(\forall \bar{x}_t, \bar{y}_t \in \Omega_n)[|\bar{x}_t - \bar{y}_t| \geq \epsilon \wedge \bar{x}_t >_t \bar{y}_t \Longrightarrow (\forall \bar{w} \in S(\bar{x}_t; \delta))\bar{w} >_t \bar{y}_t]$. An example of such a family is the set of preference relations $>_f$ defined by a family of equicontinuous utility functions, where $\bar{x} >_f \bar{y}$ if and only if $f(\bar{x}) > f(\bar{y})$.

Given a standard exchange economy $\mathcal{E} = \langle I(t), \succ_t \rangle$, an allocation $X(t)$ for $\mathcal{E}$, and a price vector $\bar{p}$ in Ω_n, we define the following sets for each positive real number δ:

$$E^{\bar{p}}_{\delta}(X) = \{t \in \mathcal{E} \mid \bar{p} \cdot X(t) - \bar{p} \cdot I(t) \geq \delta\}$$

$$F^{\bar{p}}_{\delta}(X) = \{t \in \mathcal{E} \mid \exists \bar{y} \in \Omega_n) \bar{p} \cdot \bar{y} \leq \bar{p} \cdot I(t) \wedge \bar{y} \succ_t X(t) \wedge |\bar{y} - X(t)| > \delta\}$$

$$G^{\bar{p}}_{\delta}(X) = E^{\bar{p}}_{j}(X) \cup F^{\bar{p}}_{\delta}(X)$$

<u>Theorem 2</u>. Suppose $\mathcal{G}$ is an unbounded family of standard exchange economies satisfying the assumptions stated above. Then for every $\delta > 0$, there exists an $m \in N$ such that for all economies, $\mathcal{E}$, in $\mathcal{G}$ and for all allocations $X(t)$. If $|\mathcal{E}| > m$ and $X(t)$ is in the core of $\mathcal{E}$, then there exists a price vector $\bar{p}$, such that $|G^{\bar{p}}_{\delta}(X)|/|\mathcal{E}| < \delta$.

In order to prove Theorem 2, we need only note that the definition of the core in a standard exchange economy can also be applied to a nonstandard exchange economy. We shall call this the Q-core (quasi standard core). Suppose $\mathcal{E}' = \langle I(t), \succ_t \rangle$ is a nonstandard exchange economy satisfying all the assumptions of Section III. Let $\varphi = \{Y \in {}^*\Omega_n^T \mid Y$ be a standardly bounded assignment and $\sum_{t \in T} Y(t) \leq \sum_{t \in T} I(t)\}$. If $Z, Y \in \varphi$, then Z <u>Q-blocks</u> Y via a coalition $S \subseteq T$ if $\sum_{t \in S} Z(t) = \sum_{t \in S} I(t)$ and if for all $t \in S$, $Z(t) \succ_t Y(t)$. If $Y \in \varphi$ and Y is not Q-blocked by any $Z \in \varphi$, then Y is said to be in the <u>Q-core</u>.

Lemma: If $(\exists \delta > 0)(\forall t \in T)(\forall j)$ $I^j(t) \geq \delta$. Then the Q-core of $\mathcal{E}'$ is contained in the core of $\mathcal{E}$.

<u>Proof</u>: Suppose X is in the Q-core of $\mathcal{E}'$ and not in the core of $\mathcal{E}'$. Then there exists a coalition S and an allocation Y such that for all $t \in S$, $Y(t) \not\simeq X(t)$ and $Y(t) >_t X(t)$ also $\frac{1}{\omega} \sum_{t \in S} Y(t) \simeq \frac{1}{\omega} \sum_{t \in S} I(t)$, $|S|/\omega \not\simeq 0$. If $\frac{1}{\omega} \sum_{t \in S} Y(t) \leq \frac{1}{\omega} \sum_{t \in S} I(t)$, then $\sum_{t \in S} Y(t) \leq \sum_{t \in S} I(t)$. Hence the assignment $Z(t) = \begin{cases} Y(t), & t \in S \\ I(t), & \text{otherwise} \end{cases}$ is in φ and Z Q-blocks X, which contradicts the assumption that X is in the Q-core of $\mathcal{E}'$. So suppose for some j that $\frac{1}{\omega} \sum_{t \in S} Y^j(t) > \frac{1}{\omega} \sum_{t \in S} I^j(t)$. By assumption there exists a $\delta \gtrsim 0$ such that $\frac{1}{\omega} \sum_{t \in S} I^j(t) \geq \frac{|S|}{\omega} \delta$. Hence $\frac{1}{\omega} \sum_{t \in S} Y^j(t) \gtrsim 0$. Let $B_n = \{t \in S | Y^j(t) \geq \frac{1}{n}\}$ for all $n \in N$, then B_n is internal for all $n \in N$ and $B_n \subseteq B_{n+1}$. Suppose for all $n \in N$, $|B_n|/\omega \simeq 0$. Then $\exists \nu \in {}^*N - N$ such that $|B_\nu|/\omega \simeq 0$. This implies that $\frac{1}{\omega} \sum_{t \in S} Y^j(t) = \frac{1}{\omega} \sum_{t \in B_\nu} Y^j(t) + \frac{1}{\omega} \sum_{t \in S/B_\nu} Y^j(t) \simeq \frac{1}{\omega} \sum_{t \in S/B_\nu} Y^j(t) \leq \frac{1}{\nu} \frac{|S|}{\omega} \simeq 0$. This is true since allocations are standardly bounded. Therefore we have the contradiction that $0 \simeq \frac{1}{\omega} \sum_{t \in S} Y^j(t) \gtrsim 0$. Consequently there exists $n \in N$ such that $|B_n|/\omega \gtrsim 0$. Let $\alpha = |B_n|$, $\gamma = \frac{1}{\omega} \sum_{t \in S} Y^j(t) - \frac{1}{\omega} \sum_{t \in S} I^j(t)$, and for all $t \in B_n$, $\epsilon_t = \omega\gamma/\alpha$. Note that γ is a positive infinitesimal. We now define $Z^j(t) = \begin{cases} Y^j(t) - \epsilon_t & \text{for } t \in B_n \\ Y^j(t) & \text{for } t \in S/B_n \end{cases}$ for $i \neq j$ let $Z^i(t) = Y^i(t)$ for all $t \in S$. Finally let $Z(t) = I(t)$ for $t \in T/S$. Now $|B_n|/\omega \gtrsim 0$ implies that ω/α is finite, hence $\epsilon_t = \omega\gamma/\alpha$ is infinitesimal. Therefore for all $t \in S$, $Z(t) \simeq Y(t)$. Then by the continuity

of the preference relations, $Z(t) >_t X(t)$ for all $t \in S$. But $Z \in \varphi$, since $\frac{1}{\omega} \sum_{t \in S} Z^j(t) = \frac{1}{\omega} \sum_{t \in B_n} Z^j(t) + \frac{1}{\omega} \sum_{t \in S/B_n} Z^j(t) = \frac{1}{\omega} \sum_{t \in S} Y^j(t) - \gamma = \frac{1}{\omega} \sum_{t \in S} I^j(t)$,

i.e. $\sum_{t \in S} Z^j(t) = \sum_{t \in S} I^j(t)$. If there is more than one j for which $\frac{1}{\omega} \sum_{t \in S} Y^j(t) > \frac{1}{\omega} \sum_{t \in S} I^j(t)$, repeat the same construction.

Therefore we have demonstrated the existence of a $Z \in \varphi$ which Q-blocks X, contradicting the assumption that $X \in$ Q-core of $\mathcal{E}'$. Hence if X is in the Q-core of $\mathcal{E}'$, then X is the core of $\mathcal{E}'$.

The proof of Theorem 2 follows immediately. Suppose the theorem is false, then $(\exists \delta > 0)(\forall m \in N)(\exists \mathcal{E} \in \mathcal{Y})(\exists X)\{|\mathcal{E}| > m \wedge X(t) \in \text{Q-Core}(\mathcal{E})$ $(\forall \bar{p} \in \Omega_n)|G_\delta^{\bar{p}}(X)|/|\mathcal{E}| \geq \delta\}$. Hence by transfer the following sentence is a true statement about ${}^*\mathcal{Y}$, the nonstandard extension of $\mathcal{Y}$, for some positive real number δ: $(\forall m \in {}^*N)(\exists \mathcal{E} \in {}^*\mathcal{Y})(\forall X)\{|\mathcal{E}| > m \wedge X(t) \in \text{Q-Core}(\mathcal{E})$ $\wedge (\forall \bar{p} \in \Omega_n)|G_\delta^{\bar{p}}(X)|/|\mathcal{E}| \geq \delta\}$. Pick $\omega \in {}^*N - N$, then there exists an X in the Q-core of a nonstandard exchange economy $\mathcal{E}'$ which is not a competitive equilibrium in $\mathcal{E}'$. But by the lemma the Q-core is contained in the core and by Theorem 1 every core allocation is a competitive allocation. Hence we have a contradiction, and the proof of Theorem 2 is complete. Note that the assumptions on $\mathcal{Y}$, the unbounded family of standard exchange economies, are sufficient to guarantee that all of the nonstandard exchange economies in ${}^*\mathcal{Y}$ satisfy the conditions of Theorem 1.

We would like to note that although Theorem 2 is a consequence of Theorem 1, it is not "equivalent" to Theorem 1. A less intuitive result in terms of sequences of economies which is "equivalent" to Theorem 1 is given in [3].

An excellent discussion of the published literature pertaining to Edgeworth's conjecture is given in [1]. Edgeworth's original analysis of this problem may be found in [5].

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