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### Borrowing-Lending Interest Rate Differentials and the Time Allocation of Consumption

Thayer H. Watkins

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**BORROWING-LENDING INTEREST RATE DIFFERENTIALS  
AND THE TIME ALLOCATION OF CONSUMPTION**

**Thayer Watkins**

**August 27, 1969**

BORROWING-LENDING INTEREST RATE DIFFERENTIALS  
AND THE TIME ALLOCATION OF CONSUMPTION

by

Thayer Watkins\*

I. Introduction

Consumers allocate their expenditures over time by borrowing or saving. They would do this even if there were no interest paid, but if there is interest paid (or charged) on accumulated assets (or debt), then this affects the tradeoff between consumption at different points in time. If the interest rate is independent of the level of assets, then one can construct a price ratio for consumption at different points in time which is independent of the levels of consumption. The resulting allocation problem is formally equivalent to the static case of allocating consumption between different goods. However, when the interest rate is not independent of asset levels, then one cannot construct the appropriate constant price ratios. The problem of prime significance is the case in which the interest rate on debt is greater than that on assets, because this is an important consideration for a consumer if he is contemplating going into debt to finance present consumption. The special problems involved in borrowing to finance the purchase of consumer durables are not analyzed in this paper.

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The time allocation of consumption is of special importance to other areas of economic theory. In the Keynesian system the aggregate saving of households (the non-spending of accrued rights to production) occupies a central position although recent thought has emphasized other sources of saving; e.g., retained earnings and depreciation. The aggregate or macro-relations between savings and income have generally been based theoretically on the corresponding micro-relations. The demonstrated usefulness of empirical Keynesian models would not be diminished if it were found in cross-section studies that current income is wholly inadequate to explain current saving. However, policies derived from the macroeconomics models may be called into question because of doubts concerning the specification of the micro-relations.

After a period of unqualified acceptance of current income as the major determinant of consumption expenditures, there has been an equally unqualified de-emphasis of current income in favor of some more long run concept such as permanent income. There is no question but what the mechanism for the determination of consumption can be very complex. For purpose of model building, simplifications must be found. Simplifications can arise in two directions. By abstracting from the complications and market imperfections, one can obtain reasonably simple analytical results about how consumers should behave over time. The other direction of simplification involves an emphasis of the complications and imperfections to the extent that all but the simplest consumer behavior is precluded. Most work on the theory of the consumption function has emphasized the first type of simplification, whereas the real basis of the relationship

between current consumption and current income may lie in the imperfections of the capital market.

This paper examines the character of optimal consumption programs when the interest rate on debt is greater than the interest rate on assets, utilizing a model involving firm expectations of future income and prices. The complexity of the problem for decision-making under certainty justifies the analysis of this case prior to any consideration of uncertainty.

The analysis of the allocation of consumption under a differential in the borrowing and lending rate was given by Irving Fisher [1]. Assuming only two time periods, the situation is as illustrate in Figure 1. The individual's preferences are given by the set of indifference curves. The point  $(y_1, y_2)$  is the real wage income received in periods 1 and 2, respectively. Assuming a drastic margin between the interest rates on debt and savings, the feasible efficient consumption bundles are represented by a broken line. In this example, the consumer chooses neither to borrow nor to lend. Given other preferences, the optimal choice may involve either borrowing or lending. The implicit interest rate for the consumer can be determined from the marginal rate of substitution between present and future consumption. The alternatives facing the consumer who cannot borrow are the same as those available when the interest rate on debt is infinite.

If expansion paths of the bundles involving the constant marginal rates of substitution corresponding to the two interest rates are traced out as in Figure 2, the commodity space is divided into three regions.

The consumer's decision as to borrowing, saving or doing neither depends upon the time stream of real income. The amount of borrowing or saving depends upon the levels of income for the two periods or where the point  $(y_1, y_2)$  falls. There is a region within which income can vary without affecting the optimality of the choice of consumption equal to income (zero saving).

For this type consumer problem, irrespective of the consumption program is chosen, there will be an implicit interest rate that would have led the consumer to make the same decision. In Figure 1 the individual chooses consumption levels equal to  $y_1$  and  $y_2$ . His marginal rate substitution is given by the slope of the dotted line. Had the individual been required to choose from the consumption bundles on or below the dotted line, his choice would be the same. The set on or under the dotted line is the budget set for some interest rate and present value of income or wealth. This interest rate will be referred to as the implicit interest rate for the individual. The rate of subjective time preference is defined for special sets of preferences (or utility functions) and in general will not be the same as the implicit rate. The implicit interest rate is the interest rate which can be used to explain choice in the face of market imperfections as though the choice was made under conditions of perfect markets. The wealth variable in the explanation must be determined from the income stream and the implicit interest rate and generally cannot be known prior to the individual's choice.

It should be noted that there is an important relationship between the consumer choice from budget sets and consumer choice involving realistic complications. If the choice is from a convex set, then there is a budget set such that the choice can be interpreted as having been selected from that budget. Thus a theorist may say, "Let us assume that there exist perfect markets and see whether consumer behavior can be explained as though there were perfect markets," and ex post it may be possible to find a set of market variables which explain the choice. But it should be kept in mind that the success of such activity implies no more than the convexity of the set of alternatives and is irrelevant for ex ante prediction. Thus, if the imperfections of the market for human capital were emphasized and the need arose to prove that the market for human capital is not perfect, it may be impossible on the basis of ex post analysis of consumer behavior to prove that a perfect human capital market did not exist.

## 2. The Model

For the analysis, it is mathematically convenient to use a model involving continuous time rather than discrete time. This aspect of the model should be given a proper interpretation. The continuous time model should be thought of as an approximation to the discrete time model rather than vice versa. The discrete time model quite properly allows the details of the allocation of consumption within the time interval to be ignored. It would be senseless to deal with all the economically trivial decisions involved in a model which truly deals with continuous time. The justification for using linear functionals to describe consumer preferences depends partly upon this aspect of the model. It is assumed that the consumer preferences for consumption programs can be described by a utility functional of the form

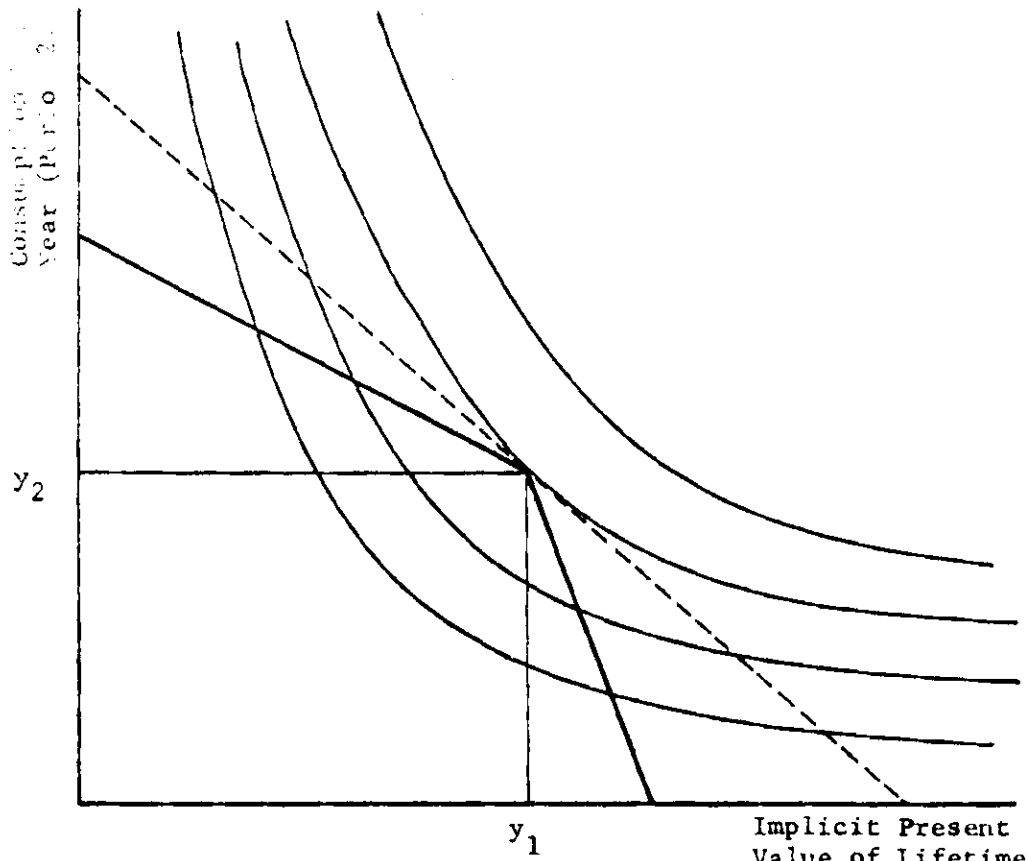


Figure 1

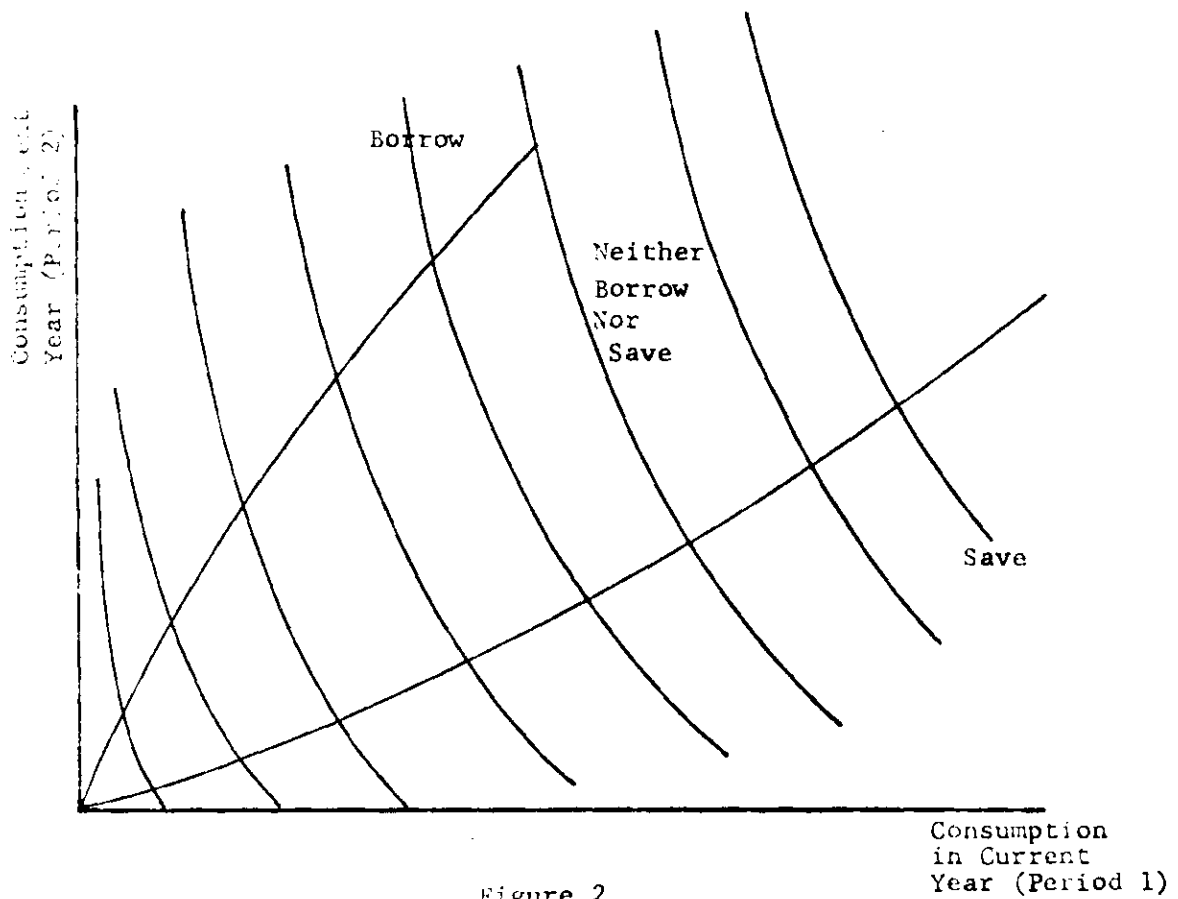


Figure 2



$$(1) \quad U = \int_0^T u(c(t))g(t)dt$$

where  $u(\cdot)$  is the instantaneous utility function,  $g(t)$  is the subjective time discount function, and  $c(t)$  is the rate of consumption at time  $t$ . In this formulation,  $c(t)$  is the rate of consumption defined at an instant; however properly interpreted it is the average rate of consumption over some time interval. This interpretation means the erratic fluctuations of day-to-day consumption are to be ignored while time is maintained as continuous rather than discrete. The length of the life span is  $T$ .

The constraint on consumption decisions is that assets at death, time  $T$ , must be non-negative. Let  $A(t)$  be the real level of assets at age  $t$ . The rate of change of real assets is

$$(2) \quad \frac{dA(t)}{dt} = y(t) + I(A(t), t) - c(t)$$

where  $y(t)$  is real wage income (also an average rate) and  $I(A(t), t)$  is the rate of interest income received at time  $t$  on assets  $A(t)$ . This very general case will be considered now and later special restrictions on the nature of this interest function will be imposed. The constraint on consumer decisions is that  $A(T) \geq 0$ , but for the model under analysis this may be taken as

$$(3) \quad A(T) = 0.$$

The model being considered leaves out all complication except the dependence of the interest rate on the level of assets. The other elaborations of the model of consumer behavior are interesting and important in

their own right such as bequest motives, changing family size, uncertainty, etc., but it is essential to cover the effect of interest rate differentials separate from the other complications. See Yaari, [4, 5, 6].

The problem of the time allocation of consumption under credit restrictions reduces to finding consumption programs which maximize utility as given by (1) subject to the constraints given by (2) and (3). One might consider the constraints as  $A(0) = A_0$  and  $A(T) = A_T$ .

The indirect utility function for this problem would be

$$U = (A_0, A_T; I(A(t), t) y(t)),$$

i.e. a function of initial and final assets and a functional of the interest function and income stream. In simpler problems, the maximum utility that could be achieved would depend only upon the difference in the asset levels. However, in the problem under analysis, the interest rate depends upon the absolute level of assets so the simplification is not valid. By stating the problem in this fashion, one may deal with

$$\frac{\partial U}{\partial A_T}$$

as well as  $\frac{\partial U}{\partial A_0}$  even when  $A_T = 0$ .

### 3. Existence and Uniqueness of Solutions

There are a few formal questions about the existence and uniqueness of a solution to the optimization problem being considered. The analysis is straight forward and involves little of economic interest. The Weierstrass theorem states that a maximum is attained within a compact set. Compactness in this case is equivalent to boundedness and closedness. Yaari [4] covers this area rigorously.

The only possible difficulty is the restriction of consumption programs to piece-wise continuous functions. It is desirable to limit consideration to piece-wise continuous functions in order to make use of certain mathematical techniques. However, the set of piece-wise continuous functions is not compact, which means existence of optimal solutions may be in doubt. On the other hand, if the rate of consumption is to be interpreted as the average rate of consumption over some period of time (as it should be for economic analysis), then piece-wise continuous functions constitute too inclusive a set.

### 3.1 Convexity of the Choice Set

Let  $c_1(t)$  and  $c_2(t)$  be two consumption programs and  $A_1(t)$  and  $A_2(t)$  be their corresponding asset patterns for the same income stream. The choice set is convex if when two consumption programs are feasible; that is to say,

$$A_1(0) = A_0 \quad \text{and} \quad A_2(0) = A_0$$

and

$$A_1(T) \geq 0 \quad \text{and} \quad A_2(T) \geq 0 ,$$

then  $c_3(t) = ac_1(t) + (1-a)c_2(t)$  is also feasible for any value of  $a$  from 0 to 1 .

Let  $A_3(t)$  be the asset pattern generated from the income stream and the consumption program. Furthermore, let

$$A_4(t) = aA_1(t) + (1-a)A_2(t) .$$

From these definitions, it follows that

$$\frac{dA_3}{dt} = y(t) - p(t)c_3(t) + I(A_3(t), t)$$

and 
$$\frac{dA_4}{dt} = y(t) - p(t)c_3(t) + aI(A_1(t), t) + (1-a)I(A_2(t), t) .$$

The interest function  $I(A, t)$  is assumed to be concave; i.e.

$$aI(A_1, t) + (1-a)I(A_2, t) \leq I(aA_1 + (1-a)A_2, t)$$

for all  $A_1, A_2$  and  $t$  and all values of  $a$  from 0 to 1. This implies that if

$$A_3(t) \geq A_4(t)$$

then 
$$\frac{dA_3}{dt} \geq \frac{dA_4}{dt}$$

and consequently, if  $A_3(v) \geq A_4(v)$

then  $A_3(u) \geq A_4(u)$  for all  $u \geq v$ .

Since  $A_1(0) = A_2(0) = A_3(0) = A_4(0)$ , it follows that

$$A_3(t) \geq aA_1(t) + (1-a)A_2(t) \text{ for all } t$$

and therefore

$$A_3(t) \geq \min [A_1(t), A_2(t)]$$
 for all  $t$ . Thus if  $c_1(t)$  and  $c_2(t)$  are both feasible, then  $ac_1(t) + (1-a)c_2(t)$  will also be feasible for any value of  $a$  from 0 to 1. The same proof would hold if the constraint on terminal assets were replaced by a more general restriction such as that assets of each age must exceed some given level.

### 3.2 Closedness of the Choice Set

Let  $c_i(t)$ ,  $i=1, 2, \dots$ , be a sequence of feasible consumption programs; i.e.

$$A_i(T) \geq 0 \text{ for all } i, \text{ and let}$$

$$c(t) = \lim_i c_i(t)$$

exist. For the choice set to be closed, it is required that the limit consumption program  $c(t)$  also be feasible. The method of proof is by contradiction. Assume that the asset pattern,  $A(t)$ , corresponding to the limit program violates the condition for feasibility; i.e.,

$$A(T) < 0 .$$

Because of the definition of the limit program, there will be some consumption program in the sequence that is so close to the limit sequence that the terminal assets for this program would also be negative contrary to the assumption that all programs in the sequence are feasible. The statement

$$\lim_{i \rightarrow \infty} \{c_i(t)\} = c(t)$$

means that for any positive  $\epsilon$  there exists an  $N$  such that for all  $n$  greater than  $N$

$$\sup_t |c(t) - c_n(t)| \leq \epsilon$$

It is expedient at this point to place a more severe restriction on the interest function; i.e.

$$\sup_{A_1, A_2, t} \frac{I(A_1, t) - I(A_2, t)}{A_1 - A_2} = R < \infty$$

This condition will be satisfied by all interest function about which one might be concerned.

From the definitions it follows that

$$\frac{d(A_n(t) - A(t))}{dt} = c(t) - c_n(t) + I(A_n(t), t) - I(A(t), t)$$

and thus

$$\left| \frac{d(A_n(t) - A(t))}{dt} \right| \leq \sup_t |c(t) - c_n(t)| + R|A_n(t) - A(t)|$$

At worst (as far as the validity of the theorem is concerned)  $A_n(t)$  is always greater than  $A(t)$  for  $t$  between 0 and  $T$ . Of course,  $A_n(0) = A(0)$ . It therefore follows that

$$A_n(T) \leq A(T) + \sup_t |c(t) - c_n(t)| \left( \frac{e^{RT} - 1}{R} \right)$$

and hence by choosing  $\epsilon$  arbitrarily small the values of  $A_n(T)$  will be arbitrarily close to  $A(T)$  for some value of  $n$ . Therefore if all the consumption programs are feasible then so is the limit program. Therefore the set of consumption programs is closed.

### 3.3 Uniqueness

As is the usual ploy in problems of this type, the objective function is chosen to be strictly concave such that if an optimum exists it will be unique. More particularly, necessary conditions for an optimum are derived and there is only one program which satisfies the set of necessary conditions. In this case, strict concavity of the utility function means that the instantaneous marginal utility function is strictly decreasing.

### 3.4 Boundedness

In a model involving discrete time the constraint would assure the boundedness of the choice set of consumption programs. One of the disadvantages of the continuous time model is that the choice set is not bounded in the metric which has been used thus far in this paper; i.e., the distance between two consumption programs is the supremum of the absolute value of their difference. The major source of the unboundedness is the existence of the programs such as that involving deferred consumption until the last instant and then all assets being spent in one gigantic splurge. A more convenient metric might be found such the above type program is not infinitely distant from programs of relatively uniform consumption throughout the life span.

Even though the choice set might not be bounded this fact does not, generally, create any difficulties because of the convexity of the preference relations which result in the rejection of the extreme, non-uniform consumption programs. Thus boundedness might be handled by dealing with the intersection of the original choice set and the set of programs which are at least as good as a program of constant consumption.

### 4. The Determination of Optimal Consumption Programs

It is expedient to begin the analysis on a mathematically simplified version of the problem. Assume that the interest function has a continuous first derivative with respect to  $A$ . The following necessary condition for an optimal consumption program can be derived when  $\frac{\partial I}{\partial A}$  is continuous using Pontryagin's maximal principle:

$$(4) \quad u'(c(t))g(t) = \lambda e + \int_t^T \left( \frac{\partial I}{\partial A} \right) ds$$

where  $\lambda$  is a positive constant. When  $I(A(t), t) = i(t)A(t)$ , this reduces to the usual condition. Condition (4) also implies that if all derivatives are defined, then

$$\frac{u''(c(t))}{u'(c(t))} \dot{c}(t) = - \left[ \frac{\dot{g}(t)}{g(t)} + \left( \frac{\partial I}{\partial A} \right) \right]$$

which is a generalization of the condition derived by Yaari [4]. Note that in assuming  $\frac{\partial I}{\partial A}$  to be continuous, the simple case of different interest rates on borrowing and lending have been eliminated. Later this failing will be remedied.

The condition for an optimal program (4) may be combined with the differential equation (2) to give an integro-differential equation which may, in principle, be solved for any value of  $\lambda$  providing the marginal utility function has an inverse. If we find a value of  $\lambda$  such that  $A(0) = A_0$ , the initial level of assets and  $A(T) = 0$ , then we have found the optimal consumption program. Unfortunately, it is not feasible to solve the integro-differential equation analytically even for the

simplest cases when  $\frac{\partial^2 I}{\partial A^2} \neq 0$ . This is no great loss if one can construct



diagrams that aid in the interpretation of characteristics of consumption programs. The fact that the condition for the optimal program involves a "discount factor"

$$e^{-\lambda \int_t^T ds}$$

which discounts unit expenditures forward to time  $T$  rather than back to time  $0$  as is the usual case means that one must work backwards (from  $t=T$  back to  $t=0$ ) in the analysis. This unavoidably makes certain points of the exposition awkward and unnatural. In this optimization problem the procedure is as follows. A value for the parameter  $\lambda$  is chosen, consumption is then determined from (4) and the differential equation is solved backwards for the initial value of assets. If the initial level of assets is different from the value given in the problem, a different value of the parameter is chosen. The parameter  $\lambda$  may be interpreted as the marginal utility of terminal assets  $\left( \frac{\partial U}{\partial A_T} \right)_{A_T=0}$ . The same

condition and procedure would apply if the constraint involved  $A(T) = A_T$ .

For a set of values of  $\lambda$  the time pattern of asset levels associated with the optimal consumption programs might be of the form shown in Figure 3.

This one parameter family of curves has the property that no two cross if the instantaneous marginal utility function  $u'(c)$  is monotonically decreasing. The one curve that is equal to the initial assets  $A_0$  at  $t=0$  determines the value of  $\lambda$  that is sought. From  $\lambda$  and condition (4) the optimal consumption program is completely determined. However one should not automatically assume that for any value of initial assets

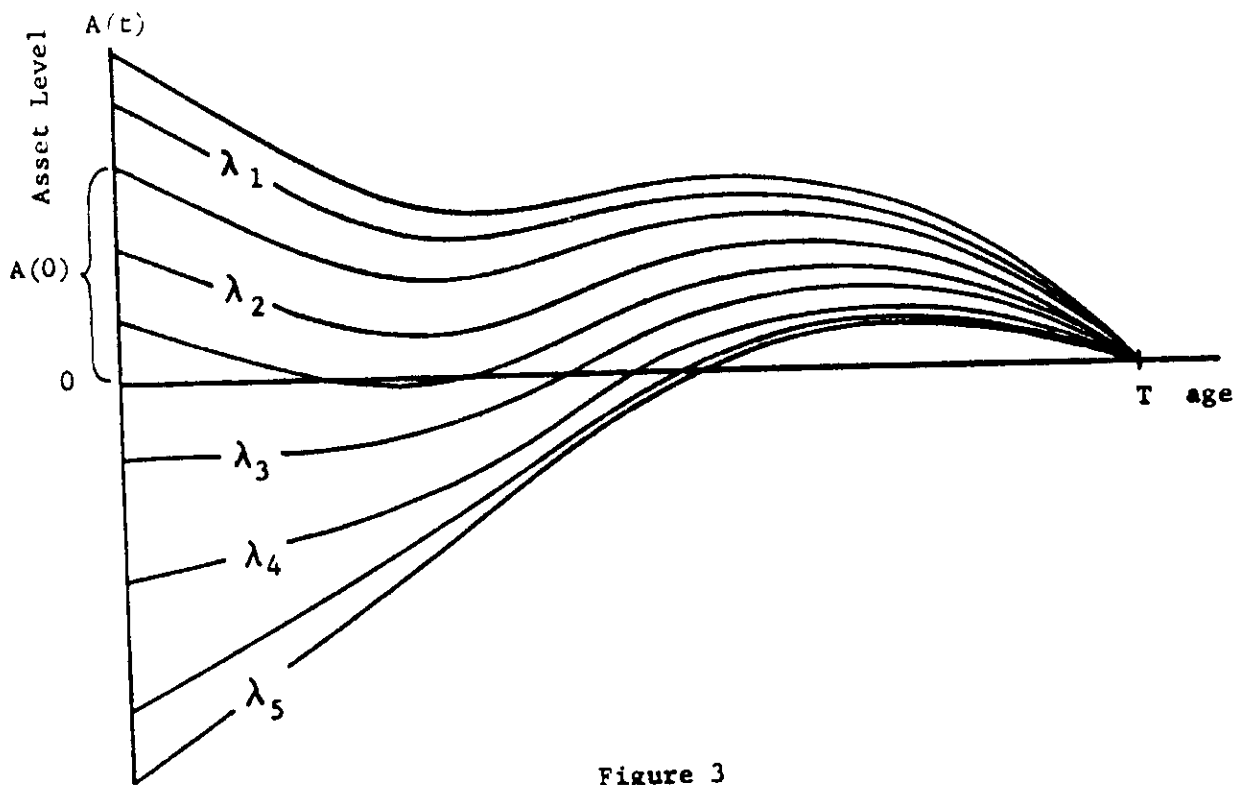


Figure 3

there will be some value of  $\lambda$  such that the solution of the integro-differential equation satisfies that condition at time equal zero. All that is known is that given any value of  $\lambda$  there will be a solution having some value at time zero. There may or may not be values corresponding to each level of initial assets. Figures 4 and 5 show examples of the two cases. It turns out that when the interest function has continuous derivatives then the relationship between initial asset level and the parameter is continuous.

Now consider a differential in the borrowing and lending interest rates. In particular, suppose

$$(6) \quad I(A, t) = \begin{cases} i_1 A & \text{if } A > 0 \\ 0 & \text{if } A = 0 \\ i_2 A & \text{if } A < 0 \end{cases}$$

where  $i_1$  is the lending interest rate and  $i_2$  is the borrowing rate,  $i_2 \geq i_1$ . We assume that these rates are constant over time. It would not be much more difficult to consider the case of time varying interest rates, but the results are not changed in any essential manner and therefore we deal only with the simplest case. The interest function given in (6) means that

$$\frac{\partial I}{\partial A} = \begin{cases} i_1 & \text{if } A > 0 \\ i_2 & \text{if } A < 0 \end{cases}$$

but the derivative does not exist for  $A = 0$ . The methods of optimal control theory do not apply to this case without modification. Gamkrelidze

[2] derived auxiliary conditions which would apply to this case; however, it seems better in this instance to use less systematic techniques.

Although the interest function specified in (6) does not have the properties required to make Pontryagin's principle applicable, if assets were entirely positive (or negative) or some time interval, then the conditions given in (4) would have to be satisfied by an optimal consumption program. Even over a time interval during which the asset level changed sign and the asset level were zero for an instant, the same condition would have to be satisfied. If the condition were not satisfied, then the rate of consumption would change discontinuously at the instant the asset level is zero, and there would be a reallocation of consumption in some small interval about the above mentioned instant which would increase utility. The value used for  $\frac{\partial I}{\partial A}$  at the instant at which the level of assets is zero will not affect the integral in (4).

There is no ambiguity in the determination of consumption from condition (4) except in one situation. Even when  $A(t) = 0$ , there is no

problem since only  $\int_t^T \left( \frac{\partial I}{\partial A} \right) ds$  is involved which is unaffected by  $\left( \frac{\partial I}{\partial A} \right)_t$ .

Once  $c(t)$  is determined and consequently  $\frac{dA(t)}{dt}$ , the level of assets will, in general, change to a level such that  $\frac{\partial I}{\partial A}$  is unambiguously determined. However, if the level of consumption determined from (4) happens to be equal to  $y(t)$  and hence  $\frac{dA(t)}{dt} = 0$ , then a special condition

develops. There are three choices that might be made (1) set  $\left(\frac{\partial I}{\partial A}\right)_0 = i_1$ , (2) set  $\left(\frac{\partial I}{\partial A}\right)_0 = i_2$ , and (3) set  $\left(\frac{\partial I}{\partial A}\right)_0 = \frac{\dot{g}(t)}{g(t)} + \frac{u'(y(t)) \dot{y}(t)}{u''(y(t)) y(t)}$ .

The latter condition would maintain the validity of (5) by tautology and the asset level would be maintained at zero. All three choices might seem reasonable. In order to decide among the three, it is necessary to go back to the simplest case.

When there is a difference between the borrowing and lending rate, the individual is confronted with two interest rates when he has zero assets. From the Fisher diagram, one knows that the condition under which an individual will just exactly consume his income is signaled by the following response. When confronted with the borrowing rate, the individual wishes to lend, and when confronted with the lending rate, he wishes to borrow. There will be an implicit interest which will induce zero borrowing and lending. The third choice is that implicit interest rate.

When the relationship between the initial level of assets and the Lagrangian parameter is continuous, there will be a uniquely determined value for the Lagrangian parameter for each given level of initial assets. When there is an interest rate differential, the above relationship may not be continuous and for some values of the parameter an infinitesimal change results in a finite change in the initial value of assets. Figures 5 and 6 illustrate this case. This phenomenon occurs because there is a time at which assets are zero and the consumer chooses to exactly consume his income.

It has been stated that condition given in (4) must hold for an optimal consumption program providing the level of assets is not identically zero over some finite (measurable) interval. This latter provision does not seem to involve major difficulties. One might ask why the optimal consumption program could not be found by the same procedure as before. Since the direction of solution is backwards, it is awkward to say that the curve "branches" at the point of tangency. This is analogous to a beam of light that is bent upon entering some media such that at a point of tangency with the surface either of two paths could be followed.

While maintaining the same value for the parameter  $\lambda$ , another parameter is considered, the time at (or after which) the level of assets is equal to zero. During the interval over which net worth is zero, the implicit interest rate is accumulated in the discount factor. This time parameter can vary over an interval such that the implicit interest rate is between the two interest rates. Variation in this parameter also results in a nested family of curves (see Figure 7) and there are two branches of the curve for each value of the parameter. Each branch corresponds to the use of one of the market interest rates for zero net worth. Some of the branches may be tangent to the horizon axis at earlier times. In this case, there would be a nested family of curves within the nested family of curves. Once this procedure had been carried out for enough values of the parameters, one could pick the appropriate values for any initial assets.

The diagrams presented in this study are not the results of

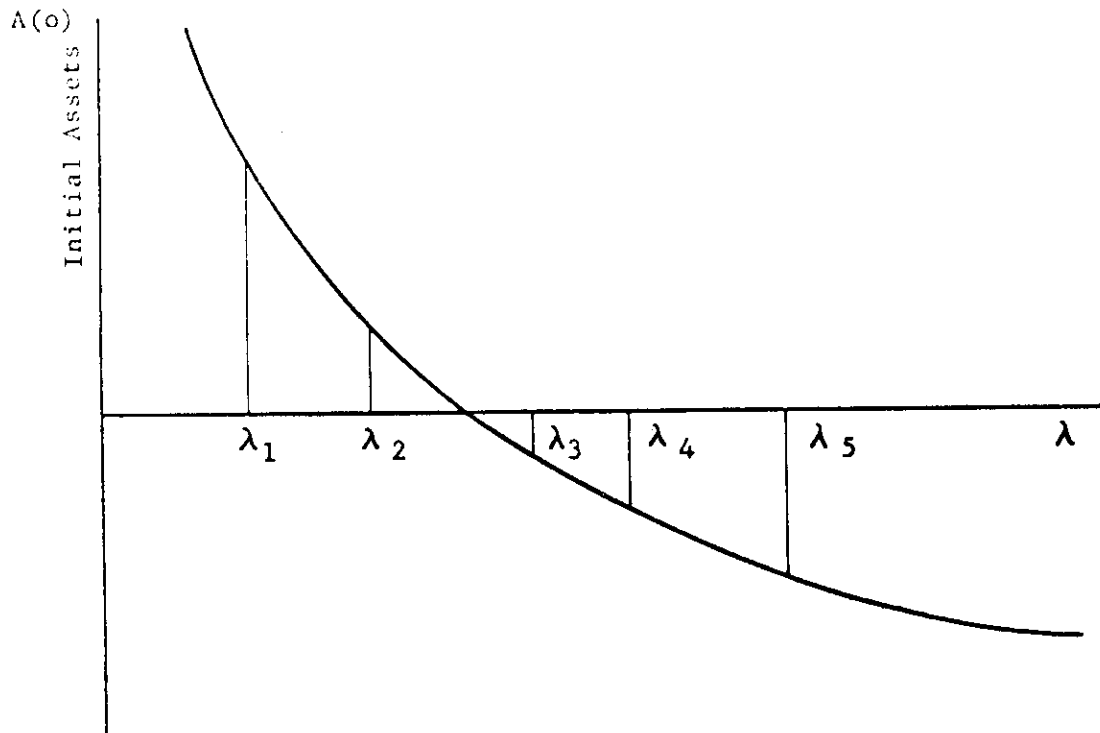


Figure 4

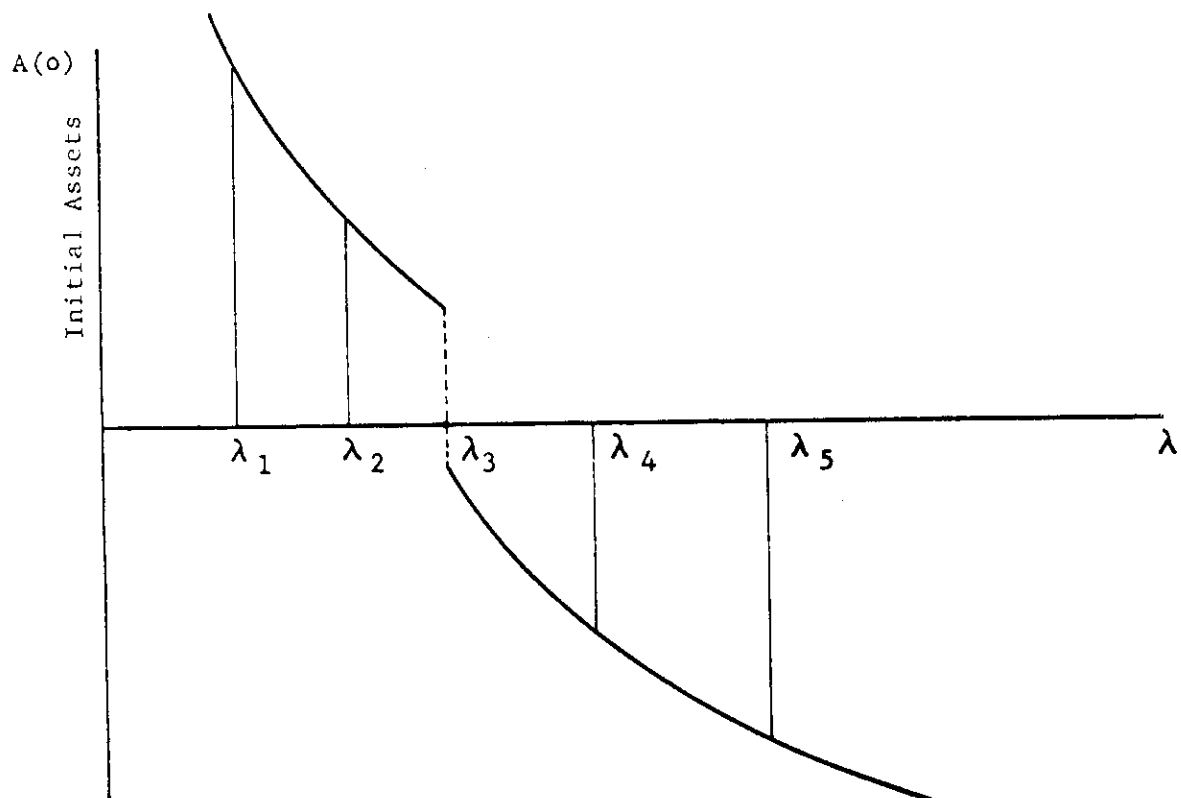


Figure 5

actual computation and may therefore involve some inconsistencies. They are only to illustrate the general characteristics of the asset patterns associated with optimal consumption programs.

Logically, it is only necessary to show that this phenomenon can occur and this can be accomplished by displaying one numerical example. Consider the case of logarithmic utility and zero time preference over a remaining life span of twenty years and the following income stream

$$y(t) = \begin{cases} .368e^{-.1t} & 0 \leq t \leq 15 \\ 0.0 & 15 \leq t \leq 20 \end{cases}$$

The necessary condition for an optimal consumption program is

$$\frac{1}{c(t)} = \lambda e^{-\int_t^{20} \frac{\partial I}{\partial A} dt}$$

When the lending and borrowing interest rates are five and fifteen per cent per year, there are two programs corresponding to a value of  $\lambda$  of .607. The optimal consumption program for an individual with initial assets of 1.54 is

$$c(t) = .607e^{.05t}$$

and for an individual starting with initial assets of -7.83 the optimal program is

$$c(t) = \begin{cases} .135e^{.15t} & 0 \leq t \leq 10 \\ .607e^{.05t} & 10 \leq t \leq 20 \end{cases}$$

Assets reach a zero level at  $t=10$  for both individuals and the marginal utility of terminal assets is .607 for both, although the marginal utility



of initial assets is 1.65 for the first individual and 4.48 for the second individual. When initial assets are between -7.83 and 1.54, the optimal consumption program involves a period just prior to year 10 in which consumption is exactly equal to income and in each case the marginal utility of terminal assets is equal to .607.

A previous study (Watkins [3]) on the influence of credit restrictions on consumption stated results in terms of a multiplier function. The value of that function at each instant can be related to the marginal utility of assets available at that instant. It would have been better to have phrased the results such that each optimal consumption program is associated with one constant valued Lagrangian parameter.

It would be desirable to have a programmable algorithm for finding the optimal consumption program for one set of given conditions without solving the equations for a large number of different values. Since the discontinuity of a function occurs at a precise value of  $\lambda$ , there may be serious difficulties in using digital computers, and a program has not yet become available for this optimization problem. However, the following procedure should work once the value of  $\lambda$  has been found. The procedure consists in considering two branches from this time back. The branches differ in the choice of the applicable interest rate when assets are zero (or at any other asset level such that the interest rate is ambiguous). The two branches are defined symmetrically. One branch comes from setting the interest rate at its maximum at the branch point and setting it at its minimum when any other ambiguity arises in the solution. This will be

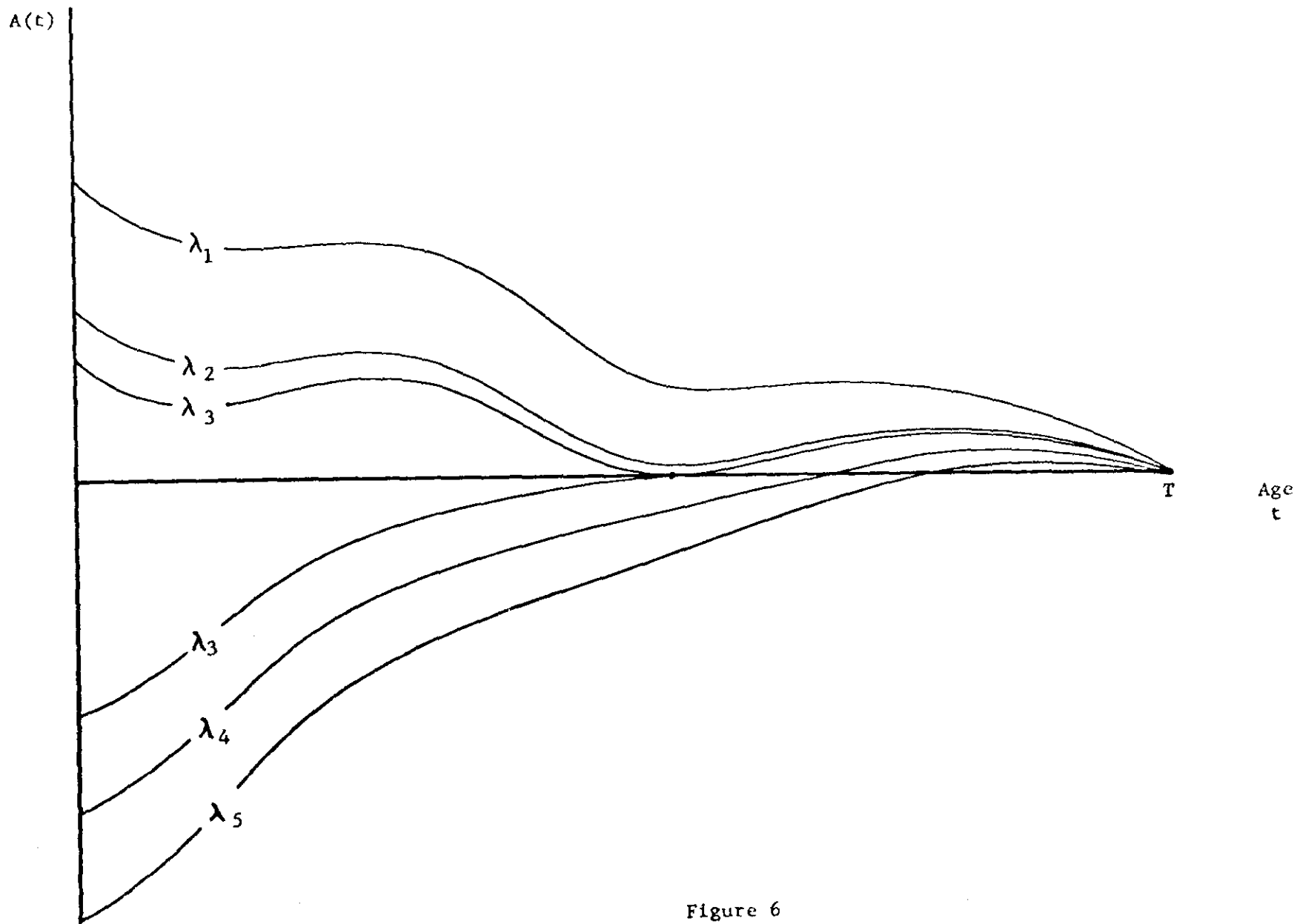


Figure 6

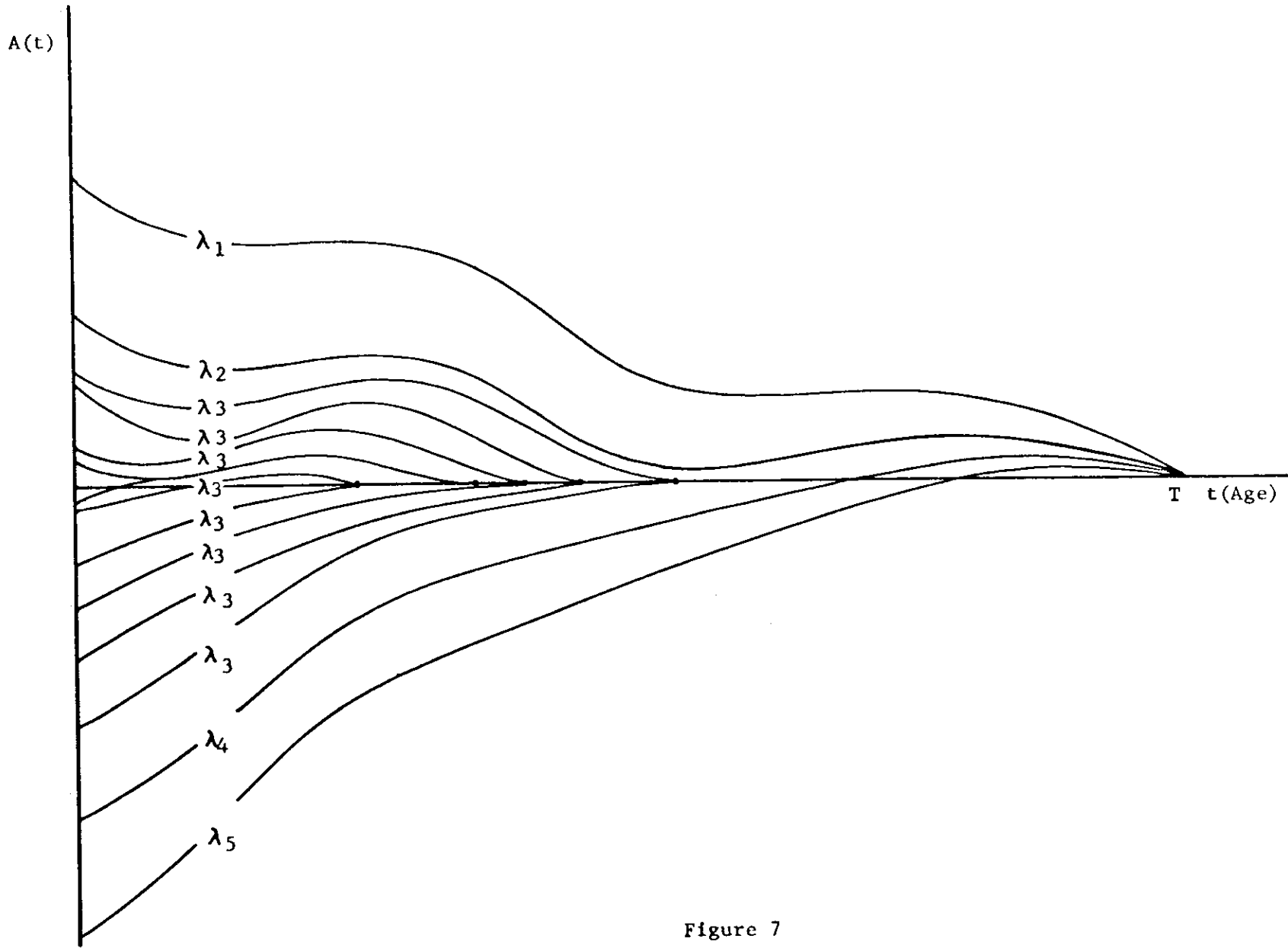


Figure 7

the high value of initial assets. The other branch (which produces the low value of initial assets) arises from setting the interest rate at its minimum at the branch point and its maximum at any other points of ambiguity.

If the given initial value of assets is between the high and low values found, then the operative interest rate chosen is the one which will justify consumption expenditures exactly equal to income. The above step is to be repeated.

If the given initial assets are not between the high and low values, then the branch is chosen which is closer and the solution follows that branch to the time at which assets are zero and the consumption expenditure is equal to wage income.

This procedure brackets the given initial assets within successively narrower limits and ultimately gives a solution satisfying the optimality conditions as well as the boundary conditions. The relationship between initial assets and the Lagrangian parameter should be thought of as a correspondence rather than a discontinuous function.

A proof that the algorithm will determine a unique consumption program would depend upon showing that the asset-time functions for two different values of the Lagrangian multiplier cannot cross and neither can two asset-time curves which differ only by the time at which assets are zero.

## 5. Applications

The applications of the above analysis fall into two groups. The first group consists of comparative statics involving changes in the income stream, the life span and the interest rates. The purpose of these appli-

cations is to illustrate the influence of windfalls, uncertainty of life span and so forth on present consumption. The second group involves the construction of various types of information which might be used in empirical savings studies; e.g. scatter diagrams of current consumption and current income. The purpose of these applications is to determine the influence that credit restrictions might have on the results of empirical analysis. The illustrations have been set up to show the phenomenon previously described. Therefore, it must be kept in mind that the illustrations are to show what may happen rather than what must happen. The effects dealt with in the conventional analysis of the time allocation of consumption are well known so that little attention will be given to them.

The effect of an unanticipated windfall can be seen by examining Figure 8. Suppose at time 1 equal unexpected windfalls become available to two individuals with identical tastes and income streams but differing in the level of initial assets. Both find they suddenly have more wealth than they expected. For the richer individual, the windfall changed his consumption program over his remaining life span. However, for the poorer individual the consumption pattern is changed only over the interval from  $t_1$  to  $t_2$ . Also the marginal utility of the terminal assets is decreased for the richer individual, but for the poorer individual the marginal utility of terminal assets has (for the example given) the same value as before. The marginal utility of initial assets fell as a result of the windfall.

The influence of a windfall which is anticipated but not now available is more difficult to illustrate because the optimization problem has to

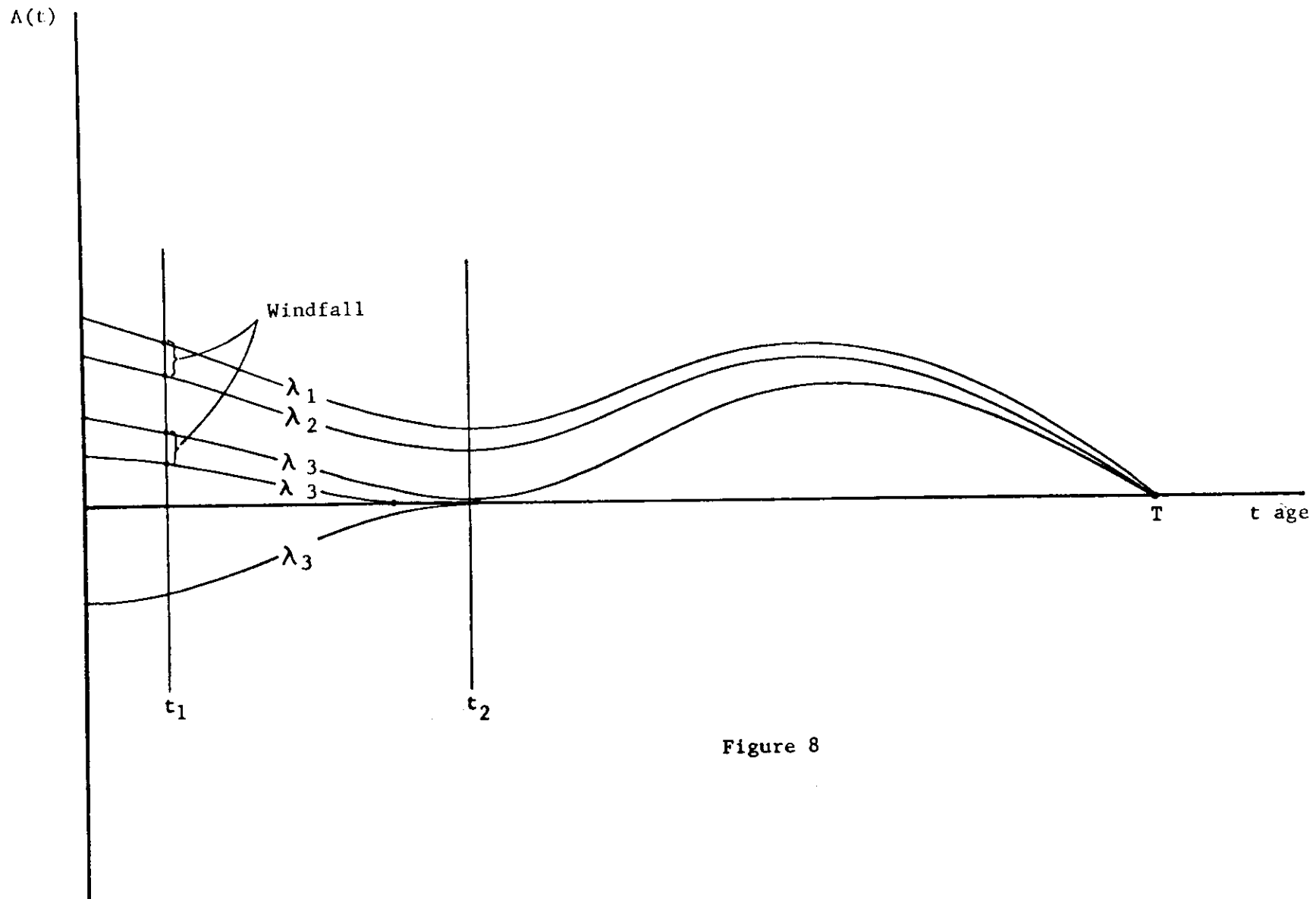


Figure 8

be entirely re-solved. For the sufficiently wealthy, the windfall may alter consumption over the entire life span, but for those who periodically go through times of zero assets or who choose consumption equal to income, the influence is of much more limited duration.

If two optimization problems differing only in the length of the life span were analyzed and the diagrams showing the time path of the assets were constructed, there would be changes but not in the functions which go to zero with zero slope. For those consumption programs which involve zero assets in the immediate future, variation in circumstances beyond the immediate future may leave them unchanged. The variation in circumstances may affect the time at which the individual chooses to begin saving for retirement but not how he consumes his presently available resources. More generally, any effect may be limited entirely to one interval over which the asset level is non-zero. Once the consumer chooses a period of zero assets, his future choices will be unaffected by factors which only influence the time at which the level of assets are to become zero. Thus the present consumption decision may not be affected for some individuals by small variation in the length of the life span or by uncertainty in the life span.

Although the examples only show what could happen, the results are similar enough to casual observations of people's behavior that what in the diagrams seems to be the odd case may be the appropriate case for the majority of people.

Probably the most important consequence of the market imperfection of the differential in the interest rate is the distortion it would produce

in occupation choice. In the model wage income is taken as given, but in a more general situation the choice of training, occupation and hence the income stream would be included. The choice of occupation is in important respects a matter of the allocation of a major resource, but if it is not possible to borrow against future returns, then entirely subjective factors such as impatience or time preference may dictate the choice. This is no more rational than to have land use assigned on the basis of some arbitrary characteristic of the owner of the land.

The final application of the results of the analysis is to deduce the appearance of the scatter diagram of current consumption versus current income for a population of identical individuals differing only in age. Suppose non-wage income begins at a low level and increases to a maximum just before retirement. In early years the combination of low income and impatience may induce the individual to borrow even at the higher interest rate on debt. Later the higher interest rate on debt provides the incentive for paying off the debt, but the lending interest rate may not be sufficiently high to induce the individual to go beyond the repayment of debt to the accumulation of assets. For a period of time the optimal action may be to maintain zero assets and adjust consumption spending to current income. Later the prospect of reduced wage income induces the individual to save for retirement and in retirement assets as well as the interest on assets are used to finance consumption spending. It is easier to construct the plot of current consumption versus current wage income rather than including current interest to begin with. In Figure 9 this plot is shown as a dotted line. The first phase involving debt has consumption greater than income, but later as income grows the debt is paid off and consumption is below



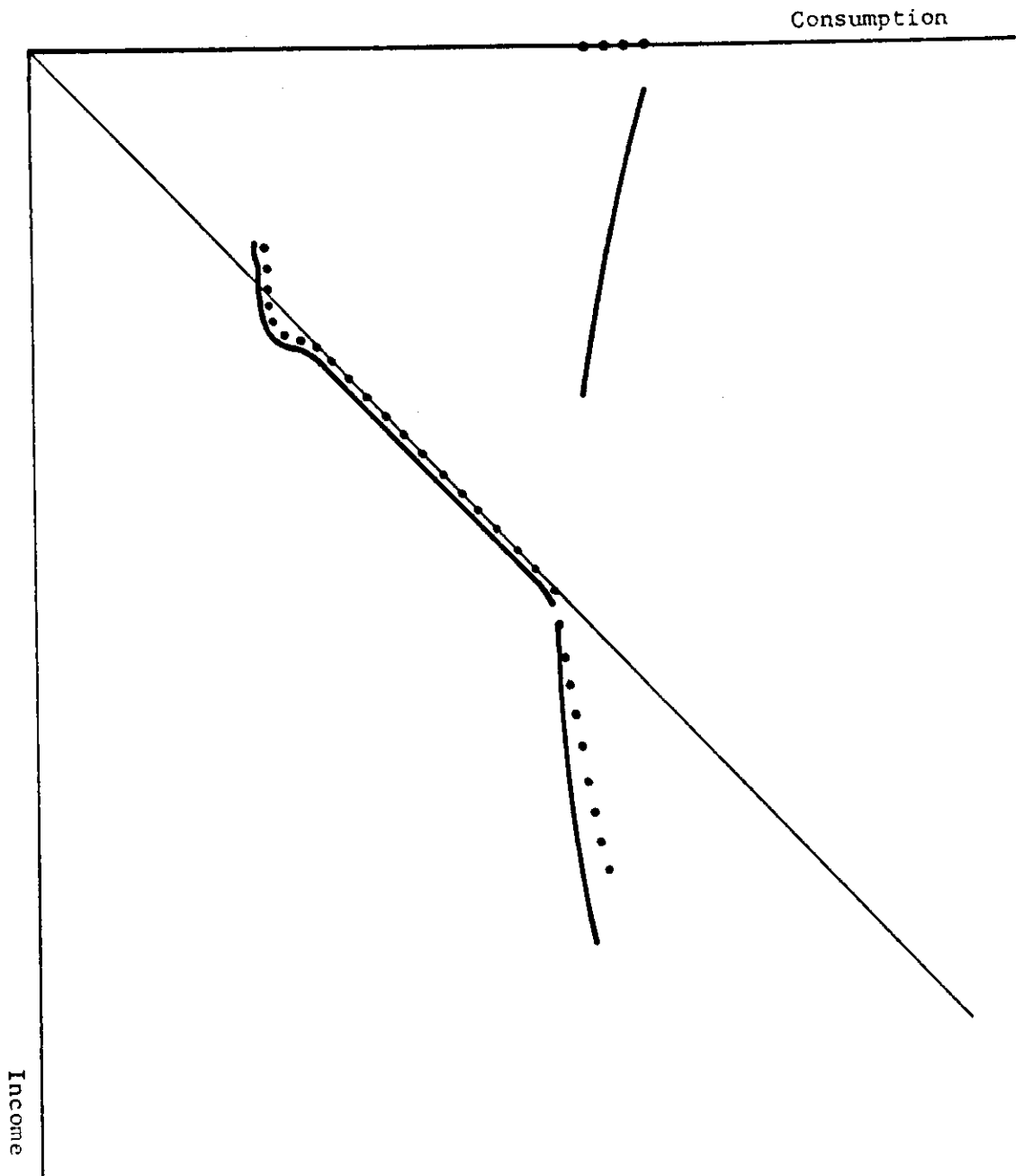


Figure 9

income. After this phase comes a period during which consumption is equal to income. Finally, there is a period of saving for retirement and consumption is again below current income. However, when current wage income drops to zero, consumption is maintained and may increase or decrease during retirement. When interest income is added onto wage income, the plot becomes that shown in Figure 9 by the solid line. During periods of debt, it would be more consistent to deduct interest payments, but this is not the usual practice in consumption function studies. A regression analysis of data would give a marginal propensity to consume of something less than one. The statistical result may or may not have any relationship to the affect that a temporary tax cut might have on aggregate consumption. The marginal propensity to consume of the portion of the population with zero assets would be very close to unity and the marginal propensity to consume of the rest of the population (both those with negative assets as well as positive) may be very low. The aggregate marginal propensity to consume depends, of course, on the relative number in each group.

## 6. Conclusions

In the simple model analyzed, the effect of a differential between the borrowing and lending interest rate is to make the optimal consumption equal to income for a range of situations. For situations in which maintaining zero assets for the immediate future is optimal, a change in the circumstances of consumer choice outside of the immediate future may leave present decisions unaffected. On the other hand, if consumption equal to current income is optimal, a change in current income through a temporary

tax change may have an immediate effect on consumption. In other words, the market imperfection of the borrowing-lending rate differential may make a consumer's present action unresponsive to variation in some factors such as future income, life span, or the interest rate themselves and "overly responsive" to variation in other factors such as current income.

The model made no provision for consumer durable purchases or various types of uncertainty so the analysis and conclusions are not definite. However, there seem to be clear evidence that various realistic complications of consumer decision-making need to be incorporated into economic thought and analysis. In particular, the basis for any relationship between consumption and current income is fundamentally a consequence of capital market imperfections. Although consumer reaction to market imperfection can be explained, after the fact, in terms of reaction to implicit interest rates, the prediction of consumer reaction requires the explicit recognition of market imperfection such as differentials in interest rates.

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