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Objectives, Constraints and Outcomes in Optimal Growth Models

Tjalling C. Koopmans

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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IN OPTIMAL GROWTH MODELS

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by

Tjalling C. Koopmans*

1. *Introduction*

My purpose in this lecture is to report on a number of recent studies of optimal economic growth, to consider their results with you, and to examine possible directions of further research.

What are the reasons for giving thought to optimal economic growth? With regard to centrally planned economies, the answer is quite obvious. The planners have a very direct influence on the pace and character of economic growth, and may wish to have the benefit of economic thought in wielding that influence.

In the individual enterprise economies of the present day, the main determinants of saving and hence investment are the concern of individuals with their own support in old age, and with the economic opportunities of their children. Even in these economies, however, governments have a considerable influence on savings and on other aspects of economic growth. To that extent the same consideration applies here also.

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Finally, the growth economist will wish to evaluate and compare these two opposite forms of economic organization, and the many mixed and intermediate forms in the present-day world, from the points of view of intertemporal efficiency of allocation, and of the desirability of the rate of growth realized.

In view of the universal importance of the problem of optimal growth, it is unfortunate that, through limitations of language and of time for preparation, I can report today only on a few studies by economists writing in West European languages. There has in recent years been a remarkable upswing in the application of mathematical thinking to economic problems in the countries of Eastern Europe. I am hoping that our colleagues from these countries will contribute from their thought and experience in the discussion.

Meanwhile, I do wish to make the claim that the studies I will report on are not tied to any particular form of economic organization. Their postulates concern (a) production possibilities, and (b) intertemporal preferences regarding consumption. Technology is, indeed, universal. As to intertemporal preferences, there are, of course, important institutional differences in how these are arrived at and given effect to in different economic systems. However, in our *pre-institutional* type of analysis, we shall merely assume that such preferences are given, without inquiring how they are determined.

In all of the models considered it is assumed that the objective of economic growth depends exclusively on the path of *consumption* as foreseen for the future. That is, the capital stock is not regarded as an end in itself, or as a means to ends other than consumption. We have already taken a step away from reality by making this assumption. Or perhaps one might say we have moved a step ahead of reality. Specifically, a large and flexible capital stock has considerable importance for what is usually somewhat inadequately called "defense." I have been unable to find a more accurate term that does not prejudge the causes of and possible remedies for our costly international insecurity. So allow me to put it this way: the capital stock also helps to meet the cost of retaining all aspects of national sovereignty and power in a highly interdependent world. But in any case, we shall have to ignore this additional attribution of value to the capital stock.

The problem of allocation of resources to consumption in various future periods is, in principle, not different from other problems of allocating scarce resources to meet a variety of competing objectives. There are however some features specific to the optimal growth problem. One is that the future has no definite and foreseeable end. We acknowledge this fact by adopting an *infinite* time horizon.

Another common feature of the models we shall consider is that the allocation problems arising at successive points in time are of the same kind, differing from each other only in the amount of the available capital stock. This circumstance leads to the use of differential equations if one adopts a continuous time variable, or of difference equations if a discrete time variable is employed.

The models to be discussed are even more abstract and speculative than studies in economic theory usually are. The implicit premise is that a certain amount of conceptual analysis may help to clear the ground for subsequent more practical work.

2. Brief review of a few models of optimal aggregate economic growth.

Using the notations of Table 1, we show in Table 2 a few characteristics of each of five optimization models of aggregate economic growth, labeled (A) through (E). In each case, the following elements are specified:

(0) an initial capital stock K_o ,

(I) an optimality criterion,

(II) an assumption about population growth,

(III) an aggregate production function and an assumption about its change over time.

All our authors obtain what we may call <u>qualitative</u> results, such as conditions for the existence of an optimal path, and monotonicity and/or asymptotic properties of such paths. My survey is concerned with these qualitative results. In particular, column (IV) in Table 2 reports on the most interesting and informative conditions for existence. However, I must refer to the papers in question for full statements.

Inagaki and Mirrlees have also obtained highly interesting quantitative results, that is, computations of optimal paths in instructive special cases. I will not anticipate the presentation of these results by the respective authors.

(A) The table begins with Frank Ramsey [1928]. So does, to my knowledge, the explicit mathematical study of optimal economic growth. The year after the publication of his paper there came what has been called "the great depression" by some, "the great crisis of capitalism" by others. Had not these events intervened and deflected economists from following up Ramsey's powerful ideas, the optimal growth literature of the sixties might and indeed could have been written in the thirties.

As the objective to be maximized -- the optimality criterion -- Ramsey selected an integral over time of the utility flow $u(C_t)$ associated with the total consumption flow[1] C_t at time t. The function $u(C)$ was assumed increasing and strictly concave. The meaning of this is that a generation that can expect a high consumption level receives relatively less support in its "bid" for still more consumption than a generation that can look forward only to a moderate or low level of consumption. The marginal utility function $u'(C)$ thus represents a

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[1] For simplicity we omit another feature of Ramsey's model: that utility also depends on the amount of labor rendered by each person. We take that amount as given institutionally, and constant over time.

distributional weighting system between generations attaining different levels of consumption, much like a progressive tax system. (Distribution problems as between individuals living at the same time are ignored, however, by assuming equal distribution between contemporaries.)

The weights $u'(C)$ ignore the time at which a generation lives. On ethical grounds, Ramsey was opposed to applying in addition a discount factor like $e^{-\rho t}$, $0 < \rho$, to the utility flow, so as to favor generations in the near future over more remote ones. This admirable principle raises a technical problem: Without a discount factor, the integral over an infinite future will not converge for most of the paths to be compared. Instead of Ramsey's ingenious device for getting around this difficulty we shall describe an equivalent device, recently proposed[2] by von Weizsäcker [1965]. By the latter's definition of the optimality criterion, a consumption path C_t is declared better than another path C_t^* if there exists a T such that

$$\int_0^t u(C_\tau)d\tau > \int_0^t u(C_\tau^*)d\tau \quad \text{for all} \quad t \geqq T ,$$

that is, if, by the time T, the path C_t has <u>overtaken</u> the path C_t^* for good. Note that this statement involves only comparison of integrals for <u>finite</u> horizons! It is true that the <u>overtaking criterion</u> so defined does not choose between every conceivable pair of paths. However, the partial ordering defined by the criterion suffices for determining an optimal path in the circumstances assumed by Ramsey.

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2 A similar but not identical criterion was introduced and used by Malinvaud [1965].

These circumstances are a constant population and a constant technology. The latter is described by an aggregate production function $F(L,K)$ with constant returns to scale and positive but decreasing returns to each factor separately, that is, to aggregate labor L, or to aggregate capital K. The only policy decision required is to allocate at each time t the production flow $F(L_t,K_t)$ as between consumption C_t and the net increase $\dot{K}_t$ in (nondeteriorating) capital K_t. It turns out that, in either of two special cases of somewhat unpredictable occurrence, there exists a unique feasible path that ultimately overtakes every other feasible path, or hence that can properly be called the optimal path. The first case arises if utility $u(C)$ is foreknown to reach an absolute masimum $u(\overline{C})$ at some finite <u>bliss</u> level of consumption $\overline{C}$. The other case is that in which the production function $f(K)$ is foreknown to reach an absolute maximum $f(\overline{K})$ at some finite <u>saturation</u> level of the capital stock $\overline{K}$. In either case, along the optimal path, both consumption and capital grow monotonically until one or the other of them approaches its saturation level asymptotically. The other variable then approaches a corresponding asymptotic level that is determined from the production function (see entries (A,IV), (A,V) in Table 2).

(B) Thirty years went by before a generalization of Ramsey's study was made, independently and more or less simultaneously, in three studies by Cass, Koopmans, and Malinvaud, respectively, with considerable overlap in the results.[3] In the amalgam of their models to be

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[3] These studies were preceded by articles by Srinivasan [1964] and by Uzawa [1964] in which an integral over consumption flows, rather than over the utility levels thereof, was maximized.

discussed here, a <u>discount factor</u> $e^{-\rho' t}$ is introduced, without precluding the possibility that the <u>discount rate</u> ρ' is zero. The other departure from Ramsey consists in the introduction of exogenous exponential population growth, $L_t = e^{\lambda t}$, $\lambda > 0$ (equating population with labor force, and measuring each in units of the initial labor force $L_0 = 1$). This new assumption immediately raises a new ethical question: whether one should maximize, as in entry (B,I) of Table 2, an integral over discounted <u>per capita</u> utility $u(c_t)$, where $c_t = C_t/L_t$ is consumption per head, or an integral over a discounted sum $L_t u(c_t)$ of <u>individual</u> utilities, as in (D,I). While this is an important question of principle, there is no essential <u>mathematical</u> difference in the models as long as both population growth and the discounting formula are exponential. The only difference then is one of interpretation, in that, if we write $\rho' = \rho - \lambda$, then ρ' is the discount rate in model (B), ρ that in model (D).

The outcome of model (B) is the existence of a unique optimal path for both consumption and the capital stock, for any nonnegative[4] ρ' , regardless of whether or not there can be saturation with consumption or with capital. The reason is that

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[4] The "overtaking principle" again renders indispensible services in the case $\rho' = 0$.

now mere maintenance of any given level of per capita consumption requires continual net investment in order to maintain a constant ratio of the capital stock to the growing labor force. As a consequence, consumption per head cannot indefinitely remain at (or above) a level exceeding a highest sustainable level.

The famous golden rule path, discovered independently by Phelps [1961] and several others[5], presupposes an initial per-worker capital stock $k_o = \hat{k}$ that just allows the highest sustainable level of consumption per head to be attained at all times. That path is also uniquely optimal by the present criterion if $\rho' = 0$, and if the initial per-worker capital stock k_o happens to equal the number $\hat{k}$ defined in (B,V) for $\rho' = 0$. If $k_o \neq \hat{k}$, but still $\rho' = 0$, the unique optimal path will have both per capita consumption and capital per worker asymptotically approaching the golden rule path level, from below if $k_o < \hat{k}$, from above if $k_o > \hat{k}$.

If, on the other hand, $\rho' > 0$, the lower weights given to per capita utilities in the more remote future prevent the highest sustainable consumption per head from ever being approached. The optimal path now approaches lower asymptotic levels defined in entry (B,V) for $\rho' > 0$.

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[5] See papers by Allais, Desrousseaux, Joan Robinson, Swan, and von Weizsäcker, cited on p. 237 of Koopmans [1965].

The remaining case, $\rho' < 0$, is important in connection with the interpretation of $\rho' = \rho - \lambda$ as the net excess of a discount rate ρ applicable to individual utilities over the rate λ of population growth. If $\rho < \lambda$, no optimal path exists. What happens is that the circumstances favor a build-up of the capital stock per worker to a level in excess of that of the golden rule path (even though that is bad as a permanent policy), merely so as to make possible the consumption of capital stock at some future time when more individuals can participate in the enjoyment of that process. But with population growth continuing forever, the moment when that maneuver pays off best, by the criterion adopted, never arrives!

(C) The model of Inagaki [1966] differs from model (B) only in assuming exogenous exponential technological progress of the product-augmenting type as shown in entry (C,III). Inagaki finds that it is now sufficient for the existence of an optimal path if ρ' is positive and also exceeds a quantity dependent on the following parameters: the rate of technological progress α , and the asymptotic values of the elasticities of marginal utility (β) and of the production function (γ) . The details are shown in the entries (C, I, III, IV) of Table 2. The elasticities β , γ enter because per capita consumption and capital per worker can now grow without bound, a circumstance that produces new mathematical complications. Thus, for instance, a sufficiently strong decrease in marginal utility as consumption becomes large is needed for an optimal path to exist.

Inagaki gives some computations for a special utility function with a constant elasticity β of marginal utility, where $0 < \beta < 1$.

(D) Mirrlees [1965] differs from Inagaki mainly in assuming that the exogenous exponential technological progress is of the labor-augmenting type, as expressed in (D,III). In addition, he adopts the interpretation of the optimality criterion as an integral over a sum of individual utilities.

Mirrlees then finds the condition (D,IV)

$$\rho - \lambda \gtrless -\alpha(\beta-1)$$

to be necessary and sufficient for the existence of an optimal path by the overtaking criterion. This condition fits in naturally with those obtained in models (B) and (C). He also finds that consumption and capital stock, both taken "per augmented worker," approach finite asymptotic levels $\hat{z}$, $\hat{x}$, respectively, defined in (D,V).

Mirrlees obtains more explicit results by choosing a special form for the utility function which complements that of Inagaki. The elasticity of the marginal utility $u'(c)$ again has a constant value β throughout, but this time $\beta \gtrless 1$. This part of the paper also contains computations of numerical properties of optimal paths for two specific choices of the production function.

(E) In any of the optimality criteria considered, the discount rate, whether zero or positive, is always a constant. A criterion defined recursively, and in which the discount <u>factor</u> $\varphi(C)$

itself depends on the prospective consumption level C, was developed by Koopmans [1960][6] in a model using a discrete time variable. Beals and Koopmans [1962] experimented with the maximization of this objective function in a constant technology with constant returns to capital alone. It was found that an optimal path approaching finite and positive asymptotic levels of consumption and capital can exist only if the discount rate $\frac{1-\varphi(C)}{\varphi(C)}$ either increases with increasing consumption $(\varphi'(C) < 0)$, or, if constant, just happens to equal the constant rate of return on capital. Many economists feel, however, that if the discount rate is to be at all variable, it is more plausible to have it decrease when consumption levels increase.

3. Some discussion of results obtained.

What have we learned from the logical exercises carried out by our authors?

We may say first of all that the sequence of exercises has not yet run its course. While each model discussed corrects some basic lack of realism present in its predecessor, we shall need to examine below what further aspects of reality will have to be incorporated before a convergence of results can possibly be hoped for.

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6 See also Koopmans, Diamond, and Williamson [1964].

At the same time, the very difficulties encountered, and the ways around them found in some cases, have been highly instructive about the nature of the problem, and about possible directions of further research. I will therefore arrange my comments under the headings of a "list of troubles."

Trouble 1 is the paradox of the indefinitely postponed splurge. We have found that, in models (B), (C), (D), and (E), the existence of an optimal path depends on inequalities in terms of the parameters. Mathematically, nonexistence of a solution can occur because the set of programs is not compact. The interesting finding is in what circumstances nonexistence does arise. The common trait of these cases is that, if the discount factor falls below a critical value (e.g., if $\rho' < 0$ in (B), or $\rho < \lambda - \alpha(\beta-1)$ in (D)), a situation arises where further postponement of some ultimate consummation is always rated as an improvement of a path.[7]

The moral is, in my opinion, that one cannot adopt ethical principles without reference to the anticipated population growth and to the technological possibilities. Any proposed optimality criterion needs to go through a mathematical screening, to determine whether it does indeed bear on the problem at hand, under the

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[7] A similar situation arises even at $\rho' = 0$ in Ramsey's case of constant population if both marginal utility and marginal productivity of capital surpass some positive number at all levels of consumption and of capital intensity.

circumstances assumed. More specifically, too much weight given to generations far into the future turns out to be self-defeating. It does nobody any good. How much weight is too much has to be determined in each case.

Trouble 2 arises from asymptotic distortion of reality in the assumptions. If we think of an economy that remains limited to the planet earth, it is obvious that population cannot go on increasing exponentially forever. It can also be questioned whether technological progress can continue indefinitely at a rate exceeding some positive constant -- even though at present we are obviously very far from exhaustion of further possibilities.

With regard to these difficulties it should be kept in mind that many of the answers given by speculative models like those listed in Table 2, especially as they bear on an intermediate future, may not depend too strongly on assumptions about a more distant future. Whether this is the case can be examined by sensitivity analysis. For instance, Samuelson [1965] and Cass [1964] have independently shown that, if in model (B) one adopts a long but finite horizon T, then the optimal path depends noticeably on the prescribed terminal capital stock per worker k_t only near the end of the horizon. For the remainder of the period, the path follows very closely the course that would have been optimal for an infinite horizon.[8]

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[8] It might be thought that this type of sensitivity analysis would also be the way out of Trouble 1. But that is not so. It is true that even if $\rho' < 0$ in model (B) "optimal" paths with different prescribed

values of k_T are close together for most of the time if T is large. But the course they hug, as already explained, builds up capital per worker to such a high level as to depress consumption below the highest indefinitely maintainable level. Moreover, even if k_T is put at the golden rule level, say, the timing of the final splurge for which all this capital has been built up is completely determined by the arbitrary horizon T. I interpret this outcome as confirmation of our discarding of the case $\rho' < 0$, by a sensitivity analysis employing a finite horizon.

Trouble 3 is the unverifiability of crucial assumptions, that is, of assumptions recognized as important by sensitivity analysis. The contrast between the solutions of models (C) and (D) suggests that the form of the production function $F(L,K,t)$ for some given $t = t_o$, and (for some forms) the type of technological change over time, have an important influence on the optimal path. In one particular year t_o one can at best observe directly L, K, and $F(L,K,t_o)$. If one is willing to assume perfectly competitive markets -- or its equivalent in perfect planning and associated perfect valuation -- then one can also observe the derivatives F_L and F_K indirectly. But even a time series of values (L,K,F,F_L,F_K) in which K/L changes only slowly gives little information about the form of $F(L,K,t)$ for ratios K/L somewhat different from those observed. In fact, the economy does not produce the information from which the shape of the production function for values of K/L substantially different from the observed ratio can be determined, even if one considers only presently known technological principles. The application of these principles to different K/L ratios will not have been worked out in the normal pursuit of efficient and/or profitable operation, under the price-ratios F_L/F_K experienced. This difficulty would exist even if the type of technological progress -- whether labor-, or capital-, or product-augmenting, or a combination of these -- were to be revealed to us by some providential intelligence. It is compounded if the same meager data are, in addition, to be our principal source of information about the type of technological progress to be expected.

Trouble 4. It should be said on the other side that perhaps the main service to be hoped for from optimal growth models at any point in time is some help with decisions, or some evaluation of anticipated developments, for five or at most ten years ahead, say. There is in the modern economy a good deal of information in the hands of scientists, engineers, managers and officials that bears on impending technological changes. However, (trouble 4) <u>the models used cannot absorb such pertinent information as we have</u> regarding technological change to come. Presumably, the information referred to is slow in affecting evaluations such as F_L , F_K . It certainly has not yet influenced the data L , K , F to which we habitually fit aggregate production functions. The information is, rather, of a disaggregate kind. It concerns expected changes in the input-to-output ratios of best processes for producing specific commodities.

These thoughts point to the need for disaggregation in our models. So does

Trouble 5, the neglect of resources other than labor. Important progress toward disaggregative models of optimal growth, after Malinvaud's fundamental paper of 1953, has been made recently in four papers by Gale [1965], McFadden [1966], and Radner [1964, 1966] respectively. Gale's paper uses a constant technology of the von Neumann type, in which presumably information about process changes can also be introduced. It has a bundle of labor services as an exogenously and exponentially growing composite resource. Radner's papers introduce resources other than labor as well, with Cobb-Douglas production functions that limit the growth rate of the economy to a weighted average of the individual growth rates of the various resource availabilities. An important contribution of McFadden's paper is a distinction between two ways in which resources can enter into growth models. Resources may have a role so important that the curve of their availability over time limits the growth rate of production. All models in Table 2, Gale's model, and those of Radner's models in which resources occur at all, are of this type. McFadden's paper is concerned with models of the second type. In these models the technology is such that resources are not indispensable for capital accumulation. That is, in the extreme case where all output is allocated to capital formation, output targets for a distant future would be limited only by technology and by the initial capital stock. Resources enter on a par with consumption in that the availability of more resources will permit given output targets for a distant future to be attained with less tightening of the belt in the meantime.

At this point my list of troubles shades over into a list of questions. These questions arise in large part from a feeling of uneasiness about the entire framework in which the portrayal of preference, technology, and population growth has been approached so far. The formulae by which we have been trying to capture these phenomena bear the marks of their intellectual parentage in the classical immutable laws of the physical sciences. They have no provision for the continual adjustment of preference, knowledge, practice and custom to new experience and observation. In brief, they lack the flexibility that is an essential trait of all human response to a changing environment.

Any one generation determines only the size and composition of the capital stock it hands on to the next generation.[9] It cannot

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[9] It is convenient for the exposition here to shift over to expressions that are really more appropriate to models with a discrete time variable, where one generation occupies one time unit.

prescribe the similar decision to be taken by the next generation, nor the optimality criterion, if any, applied to that decision. Its influence on subsequent generations' choices is limited to what example and persuasion may achieve. The significance of the capital stock that is handed on is, therefore, that it determines, insofar as is possible at the time, the range of choices among alternative consumption paths that will be open to the later generations. Each generation's choice thus involves a weighing of a little more consumption for itself against a little wider range of choice (including higher consumption paths) for its descendants. By representing the stake of future generations in the decision as a set of paths, one avoids prejudging the criterion for further choices within that set. This leads to

Question 6: Is it possible and useful to develop and apply the concept of a preference ordering over sets of consumption paths within which further choices will be required as time goes on? I have attempted a very preliminary exploration of this question elsewhere [Koopmans, 1964].

A specific aspect of flexibility concerns the relative valuation placed on a high level of consumption as against a rising level of consumption. The studies we have reviewed value only the level of consumption. One should not prejudge future generations' response to

Question 7: <u>Does an increase in consumption levels over time have a value apart from that of the level attained?</u>

The need for flexibility in the representation of preferences arises from the fact that values do and should change as circumstances change. There is a corresponding need for flexibility in the representation of production possibilities and of changes therein. We must recognize the fact that knowledge of the extent of production possibilities, and of the means and pace of their enlargement, is gained only through experience in their use and extension. Optimization and exploration thus have to be engaged in simultaneously, with the latter serving to guide and strengthen the former. The problem takes on some of the aspects of the ascent of a mountain wrapped in fog. Rather than searching for a largely invisible optimal path, one may have to look for a good rule for choosing the next stretch of the path with the help of all information available at the time. Simulation studies in various hypothetical unknown technological landscapes may help in the evaluation of alternative rules.

Shifting labels once more, I would put my main conclusion from these considerations in the form of

Recommendation 8: <u>It is desirable that models of optimal growth be designed so as to require, and make use of, only information actually or potentially available at the time of decisions affecting growth</u>.

I have left population policy for the last, because it seems to me hardest, both conceptually and practically. It is hard from a practical point of view because the formation of public policy has been hesitant in many countries, and its effect very limited. The problem is equally hard conceptually -- the aspect that concerns us here -- and I shall venture only a few comments.

While the idea of flexibility is also very important here, it may be useful first to face one problem that arises already if population policy is considered in the more rigid framework of the studies reviewed above. Suppose, then, that one allows the population growth rate to become a variable, and suppose one wishes to include an optimal population policy in the concept of an optimal growth path. We note that, before that question was raised, the utility function $u(c)$ in any of the models (A) through (E) could have been subjected to any linear increasing transformation (scale change) without thereby changing the ordering of paths. As soon as population becomes a policy variable, this is no longer so. Take, for instance, Mirrlees' criterion (D,I) as that one in Table 2 most suited for introducing population policy. Then the choice of the individual consumption level c_o for which $u(c_o) = 0$ is in effect a choice of that anticipated consumption level, below which the creation of additional human life is not regarded as justified. For any given per capita consumption level above c_o, the criterion takes the view of the Dutch proverb, "the more souls the more joy." But if, under the constraints of technology and/or resources, more souls means less per capita consumption, the criterion will strike a balance between these two.

Returning now to the idea of flexibility in relation to population policy, it should be clear that each generation will want to form its own valuations in such a matter -- to the extent that human reproduction is at all the result of rational evaluation.

There is an important difference between reproduction decisions and other more purely allocative decisions, such as those determining the amount and composition of consumption, production, investment, research and development. With regard to the latter decisions there is in the design of economic systems a certain leeway as to where to place the power to make each type of decision. The tendency of economists has been, for a long time in the individual enterprise economies, and more recently also in the socialist economies, to recommend on grounds of efficiency that each type of decision be located where the most pertinent knowledge and information is found, and then to try and see to it that the proper incentives are brought to bear on that decision maker. With regard to human procreation there is really no such leeway. The locus of decision is determined by the nature of the process, and only the most draconic measures could possibly shift it. Short of that, the principal remaining problem is one of supplying both information and incentives to the extent both needed and possible.

Prominent among pertinent incentives already naturally present are both the burdens and the joys of raising children, and, in countries without social security, the desire for a source of support in old age. The main respect in which incentives have been weaker under some conditions is precisely the point already raised. This is the choice of c_o , or, in more general terms, the strength of the concern with an acceptable economic opportunity for the child if it is to be brought forth. Where this concern is weak, an approach through incentives can improve matters only by enlightenment and persuasion, and by provision for support of the aged. Since these processes act slowly, the idea of an optimal population policy seems premature in many situations. The problem is more a matter of finding out in which direction to seek to change population growth in which circumstances, and how hard to seek to change it.

Tables and References of the
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TABLE 1. NOTATIONS

	Absolute	Per Unit of Labor
Consumption flow	C_t	c_t
Capital stock	K_t	k_t
Labor force	L_t	
Production function	$F(L, K)$	$f(k) = F(1,\frac{K}{L})$
Utility flow		$u(c)$

Population growth rate	λ
Discount rate	ρ
Rate of technological progress	α

THE OVERTAKING CRITERION
(Ramsey-von Weizsäcker)

Path C_t is better than path C_t^* if there exists $T > 0$ such that

$$\int_0^t u(C_\tau)d\tau > \int_0^t u(C_\tau^*)d\tau \quad \text{for all} \quad t \geqq T$$

TABLE 2. SOME OPTIMAL GROWTH MODELS WITH ONE COMMODITY, ONE RESOURCE (LABOR)

Identities: $F(L_t, K_t) = \dot{K}_t + C_t$, $c_t = C_t/L_t$, $k_t = K_t/L_t$

	(I) Optimality criterion ($u(c)$ strictly concave)	(II) Population = Labor Force = L_t =	(III) Production function (concave) $F(L,K)$ =	(IV) Optimal path — Existence	(V) Monotonic approach of	(VI) Computation for special cases
(A) Ramsey	$\int u(C_t)dt$	1	$f(K)=F(1,K)$	$\begin{cases}\text{if } u(C) \leqq u(\bar{C}) \text{ (bliss)} \\ \text{or } f(K) \leqq f(\bar{K}) \text{ (capital saturation)}\end{cases}$	(C_t, K_t) to $(\bar{C}, f^{-1}(\bar{C}))$ (C_t, K_t) to $(f(\bar{K}), \bar{K})$	
(B) Cass, Koopmans, Malinvaud	$\int e^{-\rho' t} u(c_t)dt$	$e^{\lambda t}$, $\lambda>0$	$Lf(K/L)$	if $\rho' \geqq 0$	(c_t, k_t) to $(f(\hat{k})-\lambda\hat{k}, \hat{k})$ $[f'(\hat{k}) = \lambda + \rho']$	graphically for $\rho' = 0$
(C) Inagaki	$\int e^{-\rho' t} u(c_t)dt$ $[\beta = -\lim_{c\to\infty} \frac{cu''(c)}{u'(c)}]$	ditto	$Le^{\alpha t} f(K/L)$ $[\gamma = \lim_{k\to\infty} \frac{kf'(k)}{f(k)}]$	if $\rho' > 0$ and $\rho' > \alpha \frac{1-\beta}{1-\gamma}$		$u(c)=c^{1-\beta}$, $0<\beta<1$, F Cobb-Douglas
(D) Mirrlees	$\int e^{-\rho t} L_t u(c_t)dt$	ditto	$Le^{\alpha t} f(K/Le^{\alpha t})$	if $[\rho' =]\ \rho-\lambda \geqq -\alpha(\beta-1)$	$(c_t e^{-\alpha t}, k_t e^{-\alpha t})$ to $(\hat{z}, \hat{x})$ $[f'(\hat{x}) = \beta\alpha + \rho]$ $[\hat{z} = f(\hat{x}) - (\alpha + \lambda)\hat{x}]$	$u(c) = \begin{cases}\log c, & [\beta=1] \\ -c^{1-\beta}, & 1<\beta\end{cases}$. F Cobb-Douglas or CES .
			Identity			
(E) Beals and Koopmans (discrete time)	$U(C_1, C_2, C_3, \ldots) = V(C_1, U(C_2, C_3, \ldots))$ $[0<\varphi(C)=\left(\frac{\partial V(C,U)}{\partial U}\right)_{U=U(C,C,\ldots)}<1]$	1	$K_{t+1} = (K_t - C_t)/\epsilon$ $0 < \epsilon < 1$	if $\phi'(C) < 0$ and $\phi(\hat{C}) = \epsilon$	(C_t, K_t) to $(\hat{C}, \frac{\hat{C}}{1-\epsilon})$	

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