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Axioms for Factor Augmenting Technical Progress

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In much of economic growth theory it is postulated either that technical progress is purely labor augmenting or that progress is purely capital augmenting. There is increasing evidence, however, that if progress is factor augmenting at all, it is both labor and capital augmenting. Mrs. Robinson [3] and Uzawa [4] have given a necessary and sufficient condition (on the relationship over time between the average and marginal products of capital) for progress to be purely labor augmenting. There exists an analogous condition for progress to be purely capital augmenting. But the conditions on which progress is factor augmenting, meaning that progress is labor augmenting, capital augmenting or both, has apparently not been explored. The purpose of this essay is to axiomatize the general case of factor augmenting technical progress. The axiomatization produces interpretations of the rates of capital and labor augmentation.

Consider a production function which makes output, Q, a function of homogeneous capital, K, labor, L (both measured in physical units) and time, t:

$$Q = F(K, L; t) \tag{1}$$

We suppose that there are constant returns to scale and that the function satisfies the usual neoclassical conditions of being twice differentiable with positive marginal products and diminishing marginal rate of substitution everywhere. Then

(2) $$q = f(k, t)$$

(3) $$f_k(k, t) > 0$$

(4) $$f_{kk}(k, t) < 0$$

where

(5) $$q = Q/L$$

(6) $$k = K/L$$

(7) $$f(k, t) = F(k, 1; t)$$

and

$$f_k = \partial f/\partial k \; , \quad f_{kk} = \partial^2 f/\partial k^2 \; .$$

There is technical progress when $f_t > 0$.

Technical progress is said to be _Harrod neutral_ at a particular capital-output ratio or its reciprocal, the average product of capital, if and only if, when the average product of capital is constant, the marginal product of capital (and hence capital's share) is likewise constant. Uzawa [4] has shown, on the above assumptions, that if and only if progress is Harrod neutral for _all_ capital-output ratios, the production function can be written in the form

(8) $$F(K, L; t) = G[K, A(t)L] \; , \qquad A(t) > 0 \; .$$

where $A(t)$ is a function only of time. Equation (8) defines _purely labor augmenting progress_. So far as output and distributive shares are concerned, it is as if progress "augments" the labor input. The quantity $\dot{A}(t)/A(t)$ is said to be the (proportionate) rate of labor augmentation.

Similarly, Fei and Ranis [2] have studied the counterpart for labor of Harrod neutrality. (They call it U-neutrality because they believe it to be of relevance for the analysis of underdeveloped economies.) We shall say that technical progress is Fei-Ranis neutral at a given labor-output ratio or its reciprocal, the average product of labor, if and only if, when the average product of labor is constant, the marginal product of labor (and hence distributive shares) is constant. By a proof similar to Uzawa's of the previous theorem, it can be shown that if and only if progress is Fei-Ranis neutral for all labor-output ratios, the production function can be written in the form

$$F(K, L; t) = G[B(t)K, L], \quad B(t) > 0, \tag{9}$$

where $B(t)$ is a function only of time. This defines purely capital augmenting progress. $\dot{B}(t)/B(t)$ is the rate of capital augmentation. This special case of factor augmenting progress, like the previous special case, is of importance.[1]

Progress is said to be Hicks neutral at a given capital-labor ratio if and only if, when the capital-labor ratio is constant, the average and marginal products of capital increase proportionally (so that distributive shares are constant). As is well-known, if and only if progress is Hicks neutral for

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[1] In a one-good vintage model with ex post and ex ante substitutability alike, an "aggregate production function" representing aggregate output as a function of aggregate labor and "effective capital" exists if and only if all capital-embodied progress can be described as purely capital augmenting.

all capital-labor ratios, the production function can be written

$$(10) \qquad F(K, L;t) = G[A(t)K, A(t)L] = A(t)\, G(K, L) , \quad A(t) > 0 .$$

The last equality follows from constant returns to scale. This might be called iso-factor augmenting progress.

It may be remarked that these three neutralities coexist at a particular time and capital-labor ratio if and only if the elasticity of substitution is unitary at that $\underline{k}$ and $\underline{t}$. (See [2].) Hence, technical progress is simultaneously everywhere Harrod neutral (purely labor augmenting), everywhere Fei-Ranis neutral (purely capital augmenting) and everywhere Hicks neutral (iso-factor augmenting) if and only if $F(K, L; t)$ is Cobb-Douglas -- where "everywhere" means "for all capital-labor ratios."

Technical progress is said to be factor augmenting if and only if the production function can be put into the form

$$(11) \qquad F(K, L; t) = G[B(t)K, A(t)L] , \quad B(t) > 0 , \; A(t) > 0 ,$$

where $B(t)$ and $A(t)$ are functions only of time. (If technical progress is to be non-negative for all capital-labor ratios and all production functions satisfying (2), (3) and (4), then both rates of factor augmentation must be non-negative: $\dot{B}/B \geq 0$, $\dot{A}/A \geq 0$.)

We wish now to axiomatize (11) as previous writers have done for (8), (9) and (10). Consider first the following proposition: Given any initial capital-output ratio, there necessarily exists some positive function, say $P(t)$, such that when the average product of capital changes over time proportionally to $P(t)$, the marginal product of capital likewise changes proportionally to $P(t)$, so that distributive shares are constant. This says merely that there must exist some path of the capital-labor ratio (or equivalently some path of the capital-output ratio) which will preserve distributive shares in the face of technical progress. Without restrictions on the character of technical progress, the function $P(t)$ may differ for different initial capital-output ratios. (Of course, if $P(t)$ is constant for some initial capital-output ratio, we have Harrod neutrality for that ratio.)

Consider now the following restriction on the character of technical progress:

> Condition: For all initial capital-output ratios, there exists a positive function $B(t)$, where $B(t)$ is a function only of time, such that when the average product of capital increases proportionally to $B(t)$, the marginal product likewise increases proportionally to $B(t)$, so that distributive shares are constant.

This condition generalizes the concept of Harrod neutrality everywhere. It is clear that if $B(t)$ is constant, this is the condition for Harrod neutrality everywhere and hence for purely labor augmenting progress. If $\dot{B} > 0$, and if this condition happens to be satisfied when the average product of labor is constant, then we have Fei-Ranis neutrality everywhere and hence pure capital augmentation. If $\dot{B} > 0$ and the condition is satisfied by a constant capital-labor ratio, the progress is everywhere Hicks neutral and iso-factor augmenting. But these are merely special cases.

The following theorem will now be proved along the lines of Uzawa's proof [4] of the aforementioned theorem on labor augmenting progress:

> Theorem: Technical progress can be represented as factor augmenting, i.e., the relation (11) is satisfied, if and only if the above Condition is met.

To express the theorem differently, if at time t the rate of change of the capital-output ratio necessary to keep shares constant is independent of the capital-output ratio, then (and only then) technical progress is factor augmenting at time t.

Proof: Sufficiency is proved first. By (3) and (4), the production relation (2) may be transformed into one making output per head, q, a function of the capital-output ratio, denoted by x:

(12) $$q = \varphi(x, t)$$

or, without loss of generality,

(13) $$q = \Phi[B(t)x, t] ,$$

where $B(t)$ is a function of t only and where

(14) $$k = xq .$$

Differentiating (13) and (14) with respect to x, q and k, we have

(15) $$dq = \Phi_1 B(t)dx ,$$

(16) $$dk = x\,dq + qdx$$

where Φ_1 denotes the partial derivative of Φ with respect to the first argument, $B(t)x$.

Solving (15) and (16) with respect to dq and dk , we obtain

(17) $$\frac{\partial q}{\partial k} = \frac{\Phi_1 B(t)}{\Phi + B(t)x\Phi_1} .$$

Now if the previously stated Condition is satisfied, then there exists some function $B(t) > 0$, such that, for all $\underline{x}$, $\frac{\partial q}{\partial k} \frac{1}{B(t)}$ is constant when $B(t)x$ is constant. Then the right-hand side of (18) below is independent of the second argument, $\underline{t}$, being a function only of $B(t)x$:

(18) $$\frac{\partial q}{\partial k} \frac{1}{B(t)} = \frac{\Phi_1[B(t)x, t]}{\Phi[B(t)x, t] + B(t)x \, \Phi_1[B(t)x, t]}$$

That is,

(19) $$\frac{\Phi_1[B(t), x]}{\Phi[B(t)x, t] + B(t)x\Phi_1[B(t)x, t]} = h[B(t)x] ,$$

where $h[B(t)x]$ is a function of $B(t)x$ only.

From (19) we have

(20) $$\frac{\Phi_1}{\Phi} = \frac{h[B(t)x]}{1 - B(t)x \, h[B(t)x]} .$$

The relation (20) indicates that Φ_1/Φ is independent of the argument $\underline{t}$; hence the function $\Phi[B(t)x, t]$ is decomposable:

(21) $$\Phi[B(t)x, t] = A(t) \, \psi \, [B(t)x] .$$

From (13) and (21) we have

(22) $$B(t)x = \psi^{-1} \left[\frac{q}{A(t)}\right] ,$$

where ψ^{-1} is the inverse function of ψ .

The relation (22) together with (14) imply

(23) $$\frac{B(t)k}{A(t)} = \frac{q}{A(t)} \quad \psi^{-1} \left[\frac{q}{A(t)}\right] .$$

Hence $q/A(t) = Q/A(t)L$ is some function of $B(t)k/A(t) = B(t)K/A(t)L$,

(24) $$\frac{q}{A(t)} = g \left[\frac{B(t)k}{A(t)} \right]$$

or

(25) $$Q = A(t)L \, g \left[\frac{B(t)K}{A(t)L} \right] .$$

Therefore

(26) $$Q = G[B(t)K, A(t)L]$$

where $G(K, L) = g(k)L$.

To prove necessity, let the relation (11) be satisfied. Then output per augmented labor is some function of the ratio of augmented capital to augmented labor, by virtue of constant returns to scale:

(27) $$q = A(t) \; g \left[\frac{B(t)k}{A(t)} \right] \text{, say.}$$

By virtue of (3) and (4), the augmented capital-output ratio is a monotonically increasing function of the augmented capital-augmented labor ratio, so that we may write

(28) $$q = A(t) \; \psi \; [B(t)x]$$

for some $\psi[B(t)x]$. Hence

$$(29) \qquad \frac{\Phi_1}{\Phi + B(t)x\,\Phi_1} = \frac{\psi'[B(t)x]}{\psi[B(t)x] + B(t)x\,\psi'[B(t)x]} ,$$

which is independent of $\underline{t}$, being a function only of $B(t)x$. Hence, from (18), $\frac{\partial q}{\partial k}\,\frac{1}{B(t)}$ is independent of $\underline{t}$ for constant $B(t)x$, i.e., for average product of capital increasing proportionally with $B(t)$. Hence, when $F(K, L; t)$ is of the form (11), our Condition is satisfied.

Q.E.D.

It should be remarked that our Condition could have been expressed in terms of the average and marginal products of labor. The condition would then be that, for every initial labor-output ratio, there exists a positive function $A(t)$ such that when the average product of labor increases proportionally with $A(t)$, the marginal product also increases in proportion to $A(t)$, so that distributive shares are constant. It can be proved analogously to the above proof that technical progress is describable as factor augmenting if and only if this latter condition is met. Therefore these two conditions are equivalent.

The "necessity" part of the theorem shows that the rate of capital augmentation can be interpreted as the rate of increase of the output-capital ratio when factor shares are constant; similarly, the rate of labor augmentation can be interpreted as the rate of increase of the output-labor ratio when shares are constant.

It is of interest to relate the <u>rate</u> of technical progress and various measures of the <u>bias</u> of technical progress to the (proportionate) rates of labor and capital augmentation, $\hat{A}$ and $\hat{B}$ respectively. The rate of technical progress, R, is defined by

$$(30) \qquad R(k, t) = F_t/F$$

The measure of Hicksian bias, $\tilde{B}$, used by Fei and Ranis [2], Diamond [1] and others, is the proportionate rate of increase of F_K/F_L for fixed capital-labor ratio. Hence

$$(31) \qquad \tilde{B}(k, t) = (F_{Kt}/F_K) - (F_{Lt}/F_L)$$

Both R and $\tilde{B}$ may be functions of $\underline{k}$ and $\underline{t}$. At a given $\underline{k}$, capital's competitive share will be increasing (decreasing) if $\tilde{B} > 0$ (< 0). Thus progress is, in the Hicksian sense, labor saving, neutral or capital saving according as $\tilde{B} > 0$, $\tilde{B} = 0$ or $\tilde{B} < 0$.

Further, we have

$$(32) \qquad a(k, t) = F_K K/F = G_1 B(t)K/G$$

$$(33) \qquad \sigma(k, t) = F_K F_L/F F_{KL} = G_1 G_2/G G_{12}$$

where $a(k, t)$ denotes the capital elasticity of output (capital's competitive share), $\sigma(k, t)$ the elasticity of substitution, and subscripts partial derivatives of the functions $F(K, L; t)$ and $G[B(t)K, A(t)L]$.

Differentiating $Q = G[B(t)K, A(t)L]$ partially with respect to time yields

$$(34) \qquad R = a\hat{B} + (1-a)\hat{A}$$

where $\hat{X}$ denotes the proportionate rate of change of any variable $X(t)$ over time, i.e., $(dX/dt)/X$.

Using (32), (33) and linear homogeneity of G , one can derive the following:

$$(35) \qquad F_{Kt}/F_K = \hat{B} - \frac{1-a}{\sigma}(\hat{B} - \hat{A})$$

$$(36) \qquad F_{Lt}/F_L = \hat{A} + \frac{a}{\sigma}(\hat{B} - \hat{A})$$

Hence

$$(37) \qquad \tilde{B} = \frac{1-\sigma}{\sigma}(\hat{A} - \hat{B})$$

Thus progress which is predominantly labor augmenting $(\hat{A} - \hat{B} > 0)$ will be Hicks-labor saving $(\tilde{B} > 0)$, neutral or capital saving according as the elasticity of substitution is less than, equal or greater than one. Factor augmenting progress is always Hicks neutral when $\sigma = 1$.

I shall define the Harrodian bias of technical progress, H , as the (proportionate) rate of growth of the marginal product of capital (equivalently, of capital's share) when the capital-output ratio is fixed. Then, as can be seen from Fei and Ranis, [1] H is related to R and $\tilde{B}$ as follows:

$$(38) \qquad H(k, t) = \left(\frac{\sigma-1}{\sigma}\right)R + (1-a)\tilde{B}$$

Technical progress is Harrod neutral at given $\underline{k}$ and $\underline{t}$, if $H = 0$. If $H > 0$,

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[1] Fei and Ranis use D as a measure of Harrodian bias, where D is defined as the rate of increase of the capital-output ratio at a given marginal product of capital. From their equation (2.3), it can be seen that $D = \sigma H$, so that the two measures will have the same algebraic sign.

the marginal product of capital is rising for given capital-output ratio, so that capital's share is rising; such technical change will be called labor saving in Harrod's sense. Similarly, $H < 0$ indicates capital saving progress in Harrod's sense.

Finally, I define a measure of bias in the sense of Fei and Ranis. Let Z be the (proportionate) rate of increase of the marginal product of labor (equivalently, of labor's share) when the labor-output ratio is fixed. Then, as can be seen from Fei and Ranis,[1]

$$Z(k, t) = \frac{\sigma-1}{\sigma} R - a\tilde{B} \tag{39}$$

Progress is Fei-Ranis neutral if $Z = 0$. If $Z > 0$, labor's share is rising and so progress is Fei-Ranis labor using (or capital saving). If $Z < 0$, progress is Fei-Ranis labor saving.

Using the expressions for R and $\tilde{B}$ in terms of $\hat{A}$ and $\hat{B}$ that appear in (34) and (37), we find that

$$H = \frac{\sigma-1}{\sigma} \hat{B} \tag{40}$$

$$Z = \frac{\sigma-1}{\sigma} \hat{A} \tag{41}$$

Hence, if technical progress is factor-augmenting and if $\sigma \neq 1$, then Harrod neutrality requires zero capital augmentation while Fei-Ranis neutrality requires zero labor augmentation. Suppose that, as is widely believed, $\sigma < 1$ in the relevant range, and suppose further that both rates of factor augmentation

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[1] Fei and Ranis use U , the rate of increase of the labor-output ratio for a fixed marginal product of labor. From their (4.3) we have $U = \sigma Z$.

are positive (for which there is some evidence in studies which postulate factor augmentation). Then technical progress is capital saving in the sense of Harrod and it is labor saving in the sense of Fei and Ranis.

Let us adopt an aggregative model -- with purely disembodied, factor augmenting progress -- of growth in the United States business sector over the past five or six decades. The special character of that growth allows us to compute the average rates of factor augmentation without knowledge of the elasticity of substitution. The previous theorem permits us to identify $\hat{B}$ as the rate of increase of the output-capital ratio, and $\hat{A}$ as the rate of increase of the output-labor ratio, when factor shares are constant. Since factor shares have shown little trend in the U. S. while the capital-output ratio has declined over the past fifty years at roughly .5 percent per annum and output per unit of labor has increased at roughly 2.5 percent per annum, we may conclude that, on average, $\hat{B} = .005$ and $\hat{A} = .025$. If these estimates are correct, progress has been predominantly labor augmenting, but not purely labor augmenting.

Unfortunately, it is not possible to read from the data the elasticity of substitution. [1] We require statistical technique (using time series exhibiting variability)-- involving assumption of constant substitution elasticity or of some relationship between the elasticity and the ratio of augmented capital to augmented labor.

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[1] We have only two independent equations -- one for the growth of output and the other for the growth of one factor price (or share) -- containing three unknowns, A , B and σ .

When such statistical analysis of production models has advanced considerably, we will be able to make some appraisal of the empirical convenience of the factor augmentation hypothesis. But we will never be able to refute it for it will always be possible to give a factor augmentation interpretation of any past growth path, even if the path of the substitution elasticity is prespecified. That is perhaps the charm of the hypothesis. But it may not be -- undoubtedly, it is not -- a correct hypothesis. The theorist is wise to use it sparingly, only for the sake of more general theorems which its use may suggest.

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