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A constraint on the thickness-weighted average equation of motion deduced from energetics

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ABSTRACT

This study reviews the system governed by the thickness-weighted average (TWA) equation of motion, considering energetics. It is known that the TWA equation of motion based on the primitive equation describes the fluid motion with the residual mean velocity defined as the TWA velocity and is written in the same form as the nondissipative primitive equation, except that the eddy momentum fluxes (the interfacial form stress and Reynolds flux associated with eddy motion) are embedded in this equation. Also, incompressibility and density (buoyancy) conservation in the adiabatic condition hold in this system. In this study, considering that the TWA system satisfies a time mean energy conservation of the primitive equation system, we obtain an energy equation showing that the rate of change of eddy energies (the sum of the kinetic and potential energies of the eddies) along pathlines with the residual mean velocity is caused by the work done by the eddy momentum fluxes. This relation is analogous to the relation between internal energy and the dissipation function in a viscous fluid. This study also reconsiders the TWA system in terms of Hamiltonian dynamics. Regarding the eddy energies and the eddy momentum fluxes as analogous to the internal energy and the viscous momentum fluxes, respectively, the methodology of the variational principle for a viscous fluid can be applied to the TWA system. The Lagrangian density in this system is defined as the mean kinetic energy minus the mean potential energy and the eddy energies. Minimizing this Lagrangian density integrated over space and time under the constraints of the incompressibility equation, the buoyancy equation, and the equation of the eddy energies yields the TWA equation of motion. If we neglect the eddy energies in the Lagrangian density and the constraint of the equation of the eddy energies, the resulting equation in the variational calculus is merely the nondissipative primitive equation. This suggests that considering these is essential for describing the motion in the TWA system. Moreover, we inferred from the equation of the eddy energy that the TWA equation of motion can be expressed in a different form in which the isotropic component of the eddy momentum fluxes is included as a part of the pressure. Applying this modified equation to the issue of downstream decaying mechanism of the western boundary current extension jets, it can be interpreted that the deceleration of the jet is caused by the pressure induced by the eddies.

Keywords. thickness-weighted average, energetics, variational principle, mesoscale eddy

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1. Introduction

In recent years, there has been a movement for formulating a macroscopic equation system to describe ocean circulations under the circumstance that mesoscale eddies occur. Although the mesoscale eddies with horizontal and temporal scales of 100 km and 100 days, respectively (e.g., Chelton et al. 2007), are unable to be fully resolved from observations, they have an important role in transporting density along with passive scalars (Jayne and Marotzke 2002; Meijers, Bindoff, and Roberts 2007; Sumata et al. 2010; Tsujino et al. 2010; Aoki et al. 2013). The eddy transport of the density causes a mass transport, which is referred to as an eddy-induced velocity or a bolus velocity (Rhines 1982; Gent and McWilliams 1990; Gent et al. 1995; McDougall and McIntosh 1996, 2001; Treguier, Held, and Larichev 1997; Greatbatch 1998; Griffies 1998; Plumb and Ferrari 2005). Thus, in the mean state, such as a low-pass-filtered oceanic field, the velocity causing the net mass transport is given by the sum of the mean and the bolus velocities. This net velocity is called a residual mean velocity or a Lagrangian mean velocity. Accordingly, to estimate the residual mean velocity, physical oceanographers have addressed parameterization of the bolus velocity (Gent and McWilliams 1990; Killworth 1997; Visbeck et al. 1997; Aiki, Jacobson, and Yamagata 2004; Cessi 2008; Ferrari et al. 2010). On the other hand, instead of parameterizing the bolus velocity, one can formulate a equation system by setting the residual mean velocity as a prognostic variable, which gives a macroscopic fluid motion (de Szoeke and Bennet 1993; Ferreira and Marshall 2006; Young 2012). In this formulation, the effects of eddies are embedded as eddy momentum fluxes in the macroscopic equation of motion.

A thickness-weighted average (TWA) formulation of the primitive equations has potential for providing a macroscopic system describing low-pass-filtered ocean circulation. The TWA system premises a set of the primitive equations: the equation of motion with the Boussinesq and hydrostatic approximations, the incompressibility equation, and the buoyancy equation. The TWA equation of motion, originally proposed by de Szoeke and Bennet (1993) and further argued in terms of the tensor analysis by Young (2012), is given by averaging the momentum equation in buoyancy coordinates weighted by the isopycnal thickness. The TWA equation of motion expressed in the height coordinates has the same form as the primitive equation, where the prognostic variable is the residual mean velocity and eddy momentum fluxes composed of an interfacial form stress and Reynolds fluxes emerge. The residual mean velocity satisfies the incompressibility condition. Also, the buoyancy equation is written in the same form as that in the primitive equation, where the advection is caused by the residual mean velocity and the buoyancy diffusion is defined as its TWA. These equations of motion give the set of fundamental equations in the TWA system (de Szoeke and Bennet 1993; Young 2012).

However, the energetics of the TWA system has not been considered. As we will see in Section 3a, the total energy, given by the sum of the kinetic and potential energies ($\mathcal{K}_M$ and $\mathcal{P}_M$, respectively; for convenience, we refer to these as the mean kinetic and potential energies) for this system, generally is not conserved in a volume, V, surrounded by rigid boundaries due to the work done by the eddy momentum fluxes ($\mathcal{W}_{\text{Flux}}$):

$$\frac{d}{dt}\int_V d^3x\ (\mathcal{K}_M + \mathcal{P}_M) = \int_V d^3x\ \mathcal{W}_{\text{Flux}}. \tag{1}$$

where $d^3x \equiv dxdydz$. However, we note that the TWA system must satisfy a time mean energy conservation of the primitive equation in the period of the time average supposed in TWA because no additional assumption is imposed on the primitive equations. This implies that the total energy for the TWA system is altered, and the corresponding energy equation can be written in a conservative form.

The TWA formulation of the energetics of the primitive equation system may give a hint (Bleck 1985; Aiki and Yamagata 2006; Aiki and Richards 2008). Under the adiabatic and nondissipative conditions, this energetics shows the conservation of the sum of the kinetic and potential energies for the mean field and those for the eddy field (K_E and P_E, respectively) in a volume of interest:

$$\frac{d}{dt}\int_V d^3x\ (\mathcal{K}_M + \mathcal{P}_M + \mathcal{K}_E + \mathcal{P}_E) = 0. \tag{2}$$

Comparing equations (1) with (2) leads to the following relation between the eddy energies and the eddy momentum fluxes:

$$\frac{d}{dt}\int_V d^3x\ (\mathcal{K}_E + \mathcal{P}_E) = -\int_V d^3x\ \mathcal{W}_{\text{Flux}}. \tag{3}$$

From equations (1) and (3), we see that the mean energies are interactively converted to the eddy energies through the work done by the eddy momentum fluxes, indicating that the mean variables in the TWA system are closely related to the eddy energies. However, because momentum and energy of the primitive equation system in the TWA framework have been individually explored in previous studies, the influence of the eddy energy on the fluid motion in the TWA system has not been argued.

Moreover, considering the eddy energy is important for the parameterization problem of the eddy fluxes. For instance, in the quasi-geostrophic (QG) system, the norm of the tensor of the eddy momentum fluxes is bounded by the total eddy energy, and each component of the eddy momentum flux tensor can be written using the eddy energy and the geometric parameters of eddy shape such as the orientation of principle axis and anisotropy of eddies (Marshall, Maddison, and Berloff 2012; Maddison and Marshall 2013). Thus, by parameterizing the geometric features, the prognostic equation of the eddy energy can be solved, which leads to developing a closed system. Given that the equations governing low-frequency motion in the QG system are expressed in similar forms to those in the TWA system (e..g., Marshall, Maddison, and Berloff 2012), a similar argument can be extended to the TWA system. To do this, however, we need to clarify how the prognostic equation of the eddy energy is described in this system.

The purpose of this study is to comprehensively understand the TWA system by taking account of energetics. In this study, to simplify the problem, we suppose an adiabatic nondissipative fluid in the absence of any external force such as wind force because the mesoscale eddies, associated with instability of geostrophic sheared flows, are basically adiabatic unforced phenomena. As a corollary to the energetics, we will derive the equation of the eddy energy in a local form. Moreover, we propose a variational principle for the TWA system. In terms of this principle, in general, the equation of motion is identical to minimizing the Lagrangian density corresponding to the energy to be conserved over a time interval (e.g., Landau and Lifshitz 1976). Although a rich literature of the variational principle for the fluid dynamics now exists (Salmon 1988, 2013; Holm, Marsden, and Ratiu 1998, 2002; Kambe 2004; Bennet 2006; Fukagawa and Fujitani 2010, 2012), that for the TWA system has not been addressed. The problem in this regard is that we cannot derive the energy field to be conserved from the hitherto known set of equations in the TWA system as shown in equation (1). However, we will see that additional consideration of the eddy energies and its governing equation enables us to apply the variational principle for the TWA system, and they are essential for describing the motion in this system.

This article is organized as follows. In Section 2, we derive the hitherto known set of equations in the TWA system. Section 3 consists of two parts. First, we show the detail of the aforementioned problem on the energy equation for the hitherto known TWA system. Second, revisiting the time mean energetics for the primitive equation system in the TWA framework, we formulate the local energy equation of the eddy energies. In Section 4, we validate the importance of the equation of the eddy energy in terms of the variational principle. In Section 5, we discuss an arbitrary nature of the definition of the eddy momentum fluxes and show that its isotropic component can be included in the pressure in the TWA equation of motion. Section 6 contains the summary and conclusion.

2. TWA equation of motion

The TWA system is developed based on the primitive equations. In the primitive equation system assuming a Boussinesq fluid, the density of seawater is approximated as $\rho(x, y, z, t) = \rho_0 + \delta\rho(x, y, z, t) \simeq \rho_0$, where ρ_0 is a constant reference density. Let $\mathbf{v} \equiv (u, v, w)$ and $\mathbf{u} \equiv (u, v)$ be the three- and two-dimensional velocity vectors, respectively; p be the pressure; and b be the buoyancy defined as $b = -g\delta\rho/\rho_0$, where g is the gravitational acceleration. In the adiabatic nondissipative conditions, the primitive equations (i.e., the momentum equation, buoyancy equation and incompressibility equation) are given by

$$D_t\mathbf{u} + f\mathbf{k} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla_H p, \tag{4}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b, \tag{5}$$

$$D_t b = 0, \tag{6}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{7}$$

where $\nabla \equiv (\partial_x, \partial_y, \partial_z)$ and $\nabla_H \equiv (\partial_x, \partial_y)$ are the three- and two-dimensional spatial gradient operators, respectively; $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$ is a Lagrangian time derivative; and f is the Coriolis parameter. As established in previous studies (e.g., Young 2012), the boundary condition in this system is assumed to be $\mathbf{v} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is a unit vector normal to the spatial boundaries.

Supposing the buoyancy b is monotonically increasing with height, we can transform the coordinates from (x, y, z, t) to $(x, y, b(x, y, z, t), t)$. The former and the latter are referred to as z-coordinates and b-coordinates, respectively. Following the transition rules between the coordinate systems (e.g., Vallis 2006), derivatives with respect to z and b are related by

$$\frac{\partial}{\partial b} = \frac{\partial z}{\partial b}\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial b}{\partial z}\frac{\partial}{\partial b}, \tag{8}$$

and temporal and horizontal derivatives in these coordinate systems are related by

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tilde{x}_i} - \frac{\partial z}{\partial \tilde{x}_i}\frac{\partial}{\partial z}, \tag{9}$$

where the partial derivatives with respect to $(x_1,\ x_2,\ x_3) \equiv (x,\ y,\ t)$ and $(\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3) \equiv (\tilde{x},\ \tilde{y},\ \tilde{t})$ denote those holding z and b fixed, respectively, following the notation employed in Young (2012). In particular, for the buoyancy, this relation reduces to

$$\frac{\partial b}{\partial x_i} = -\frac{\partial z}{\partial \tilde{x}_i}\frac{\partial b}{\partial z}. \tag{10}$$

Applying this relation to the buoyancy equation (6), we have

$$\left(\frac{\partial}{\partial \tilde{t}} + \mathbf{u} \cdot \tilde{\nabla}\right) z = w, \tag{11}$$

where $\tilde{\nabla} \equiv (\partial_{\tilde{x}},\ \partial_{\tilde{y}})$. From this equation and the equation of incompressibility mapped into b-coordinates, we obtain the mass conservation equation[2] expressed in b-coordinates:

$$\frac{\partial z_b}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\mathbf{u} z_b) = 0. \tag{12}$$

Similarly, the horizontal component of the equation of motion in b-coordinates is written as

$$\left(\frac{\partial}{\partial \tilde{t}} + \mathbf{u} \cdot \tilde{\nabla}\right) \mathbf{u} + f\mathbf{k} \times \mathbf{u} = -\frac{1}{\rho_0}\tilde{\nabla}\pi, \tag{13}$$

where $\pi(\tilde{x}, \tilde{y}, b, \tilde{t}) \equiv p(x, y, z(\tilde{x}, \tilde{y}, b, \tilde{t}), t) - \rho_0 b z(\tilde{x}, \tilde{y}, b, \tilde{t})$ is the Montgomery potential, by which the hydrostatic equation is expressed as $\pi_b = -\rho_0 z$.

2. Equation (12) gives the mass conservation because the density is constant for b-coordinates.

We shall derive the TWA system. Taking the temporal average of equation (12) yields

$$\frac{\partial \bar{z}_b}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\hat{\mathbf{u}} \bar{z}_b) = 0, \tag{14}$$

where the overbar denotes temporal average and $\hat{\mathbf{u}}$ is the residual mean velocity defined as TWA of the velocity vector: $\hat{\mathbf{u}} \equiv \overline{z_b \mathbf{u}}/\bar{z}_b$ (Andrews 1983; Bleck 1985; de Szoeke and Bennet 1993). Hereafter, we write $\hat{A} \equiv \overline{z_b A}/\bar{z}_b$ for any variable A. The residual mean velocity can be decomposed into the isopycnal mean velocity and the bolus velocity, $\hat{\mathbf{u}} = \bar{\mathbf{u}} + \overline{\mathbf{u}' z'_\rho}/\bar{z}_\rho$, where we adopted the Reynolds decomposition, $A = \bar{A} + A'$. The bolus velocity represents the mass transport caused by eddies. From equation (14), we find that the residual mean velocity satisfies the time mean mass conservation; this velocity gives the net motion of the fluid. Applying the TWA to equation (13) with the aid of equations (12) and (14), we have the TWA equation of motion expressed in b-coordinates:

$$\left(\frac{\partial}{\partial \tilde{t}} + \hat{\mathbf{u}} \cdot \tilde{\nabla} \right) \hat{\mathbf{u}} + f \mathbf{k} \times \hat{\mathbf{u}} = -\frac{1}{\rho_0} \tilde{\nabla} \bar{\pi} + \mathbf{F}, \tag{15}$$

where

$$\mathbf{F} \equiv -\frac{1}{\bar{z}_b} \left\{ \tilde{\nabla} \left(\frac{\overline{z'^2}}{2} \right) + \tilde{\nabla} \cdot \left(\bar{z}_b \widehat{\mathbf{u}'' \mathbf{u}''} \right) + \frac{\partial}{\partial \rho} \left(\frac{\overline{z' \tilde{\nabla} \pi'}}{\rho_0} \right) \right\} \tag{16}$$

is an eddy momentum flux vector (see Young [2012] for detailed derivation of this vector). When deriving this equation, we adopted the decomposition, $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}''$, where $\mathbf{u}''$ satisfies $\overline{z_b \mathbf{u}''} = 0$ (de Szoeke and Bennett 1993). The latter yields the relation $\overline{z_b \mathbf{u}\mathbf{u}} = \bar{z}_b \hat{\mathbf{u}} \hat{\mathbf{u}} + \bar{z}_b \widehat{\mathbf{u}'' \mathbf{u}''}$. Although the double primed variable is peculiar to the TWA framework, this can be approximated as $A'' \simeq A'$ in the QG limit because we have $A'' = A' - \overline{A' z'_\rho}/\bar{z}_\rho$ by definition, and the second term can be neglected due to the fact that $z'/\bar{z}_\rho$ is on the order of the Rossby number (e.g., Vallis 2006). The second and third terms of the eddy momentum fluxes are called the isopycnal Reynolds flux and the interfacial form stress associated with fluctuation of buoyancy surface, respectively (see Young 2012).

To complete the TWA system, we define the vertical velocity as

$$w^{\#} \equiv \left(\frac{\partial}{\partial \tilde{t}} + \hat{\mathbf{u}} \cdot \tilde{\nabla} \right) \bar{z}. \tag{17}$$

Notice that this vertical velocity does not mean the TWA of equation (11) but is defined so as to satisfy the incompressibility condition (de Szoeke and Bennet 1993), that is,

$$\nabla \cdot \mathbf{v}^{\#} = 0, \tag{18}$$

where $\mathbf{v}^{\#} \equiv (\hat{u}, \hat{v}, w^{\#})$ is referred to as the residual mean velocity vector. The boundary condition in this system is $\mathbf{v}^{\#} \cdot \mathbf{n} = 0$, which can be deduced from the no-normal-flow boundary condition for the instantaneous field (Aiki and Richards 2008). In the TWA system, we can regard the variables $\hat{A}$, $A^{\#}$, and $\bar{A}$ as macrovariables and the variables A' and A'' as microvariables associated with eddies. The TWA system can, thus, be interpreted as the system describing the macromotion of the fluid forced by the eddy momentum fluxes associated with the micromotion.

The TWA equation of motion can be expressed in z-coordinates. Consider the transformation between the coordinates $(x, y, \bar{z}, t)$ and $(x, y, b^{\#}(x, y, \bar{z}, t), t)$, where $b^{\#} \equiv b$. In this article, we use this notation only for the hydrostatic equation and the density equation shown later. The aforementioned transition rules can be employed by replacing z with $\bar{z}$, but we will not distinguish these variables in the rest of this article if we express the derivative and integral operators. Applying the transition rules, after some manipulation, we have

$$D_t^{\#}\hat{\mathbf{u}} + f\mathbf{k} \times \hat{\mathbf{u}} = -\frac{1}{\rho_0}\nabla_H p^{\#} + \nabla \cdot \gamma_{ij}, \tag{19}$$

where we defined a Lagrangian time derivative,[3] $D_t^{\#} \equiv \partial_t + \mathbf{v}^{\#} \cdot \nabla$; the eddy momentum flux tensor, γ_{ij} (Appendix 1); and the pressure, $p^{\#}(x, y, \bar{z}, t) \equiv \bar{\pi}(\tilde{x}, \tilde{y}, b^{\#}(x, y, \bar{z}, t), t) + \rho_0 b^{\#}(x, y, \bar{z}, t)\bar{z}$, satisfying the hydrostatic relation

$$0 = -\frac{1}{\rho_0}\frac{\partial p^{\#}}{\partial z} + b^{\#}. \tag{20}$$

Also, equation (17) is equivalent to the conservation of the buoyancy:

$$D_t^{\#} b^{\#} = 0. \tag{21}$$

Equations (18) to (21) give the hitherto known set of equations in the TWA system. We may, in passing, note that the TWA equation of motion can hold for nonrotating stably stratified turbulent flows because the TWA system only assumes b as a monotonically increasing function of z (e.g., Young 2012).

As previous studies have pointed out (e.g., Young 2012), apart from the eddy momentum fluxes and dropping symbols such as #, the TWA system is identical to the primitive equations. This implies that the eddy momentum fluxes distinguish between these systems. In other words, the existence of the eddy momentum fluxes guarantees that the “fluid particle" moves with the residual mean velocity. In Section 4, we will see, in terms of the variational principle, that this fact is closely related to the existence of the eddy energies and its governing equation derived in the next section.

3. The Lagrangian time derivative in the TWA system in z-coordinates can be expressed in advective and flux form in b-coordinates (Appendix 2).

3. Energetics

a. Problem on the hitherto known TWA system

We derive the energy equation from the hitherto known set of equations in the TWA system. A vector form of the TWA equation of motion can be written as

$$D_t^{\#}\hat{\mathbf{u}} + f\mathbf{k} \times \hat{\mathbf{u}} = -\frac{1}{\rho_0}\nabla p^{\#} + b\mathbf{k} + \nabla \cdot \gamma_{ij}. \tag{22}$$

Multiplying this by $\rho_0 \mathbf{v}^{\#}$ yields

$$D_t^{\#}\left(\rho_0 \frac{|\hat{\mathbf{u}}|^2}{2}\right) = -\nabla \cdot (\mathbf{v}^{\#} p^{\#}) + \rho_0 w^{\#} b + \rho_0 \mathbf{v}^{\#} \cdot \nabla \cdot \gamma_{ij}, \tag{23}$$

where we applied the incompressibility condition for the pressure term. Using equation (17) and taking into account that b is constant in b-coordinates, we have

$$\rho_0 w^{\#} b = \left(\frac{\partial}{\partial \tilde{t}} + \hat{\mathbf{u}} \cdot \tilde{\nabla}\right)(\rho_0 b \bar{z}) = D_t^{\#}(\rho_0 b \bar{z}). \tag{24}$$

Second equality is derived from using the relation (A13); see Appendix 2. Substituting this into equation (23) yields the equation of energy for the sum of the mean kinetic and potential energies,

$$D_t^{\#}\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} = \rho_0 \mathbf{v}^{\#} \cdot \nabla \cdot \gamma_{ij} - \nabla \cdot (\mathbf{v}^{\#} p^{\#}), \tag{25}$$

or using the incompressibility condition,

$$\frac{\partial}{\partial t}\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} = \rho_0 \mathbf{v}^{\#} \cdot \nabla \cdot \gamma_{ij} - \nabla \cdot \left[\mathbf{v}^{\#} p^{\#} + \mathbf{v}^{\#} \rho_0 \left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right]. \tag{26}$$

Integrating over the volume under the no-normal-flow boundary condition for $\mathbf{v}^{\#}$ gives

$$\frac{d}{dt}\int_V d^3x \left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} = \int_V d^3x \ \rho_0 \mathbf{v}^{\#} \cdot \nabla \cdot \gamma_{ij}. \tag{27}$$

Integrands in this equation correspond to $\mathcal{K}_M$, $\mathcal{P}_M$, and $\mathcal{W}_{\text{Flux}}$ in equation (1), respectively.

As mentioned in Section 1, the total energy consisting of the sum of the kinetic and potential energies for the mean state is not conserved due to the work done by the eddy momentum fluxes. Although either self-vanishing of the work done by the eddy momentum fluxes in the volume integral or additional inclusion of external forces may lead to the conservation of the total energy, for the former there is no physical basis, and the latter is inconsistent with the fact that the eddy momentum fluxes can intrinsically occur in the ocean. Accordingly, for the energy in the TWA system to be conserved, the work done by the eddy momentum fluxes is required to be written as a time evolution of a state function, which will be shown in the next subsection.

b. Equation of the eddy energy and its physical interpretation

Because the eddy energy in the TWA framework is defined as the difference of the mean energy from the total energy in b-coordinates (e.g., Aiki and Richards 2008), we start with reformulating the equation of the mean energy in b-coordinates. The TWA equation of motion (equation 15) multiplied by $\rho_0 \hat{\mathbf{u}} \bar{z}_b$ and the equation for the vertical velocity (equation 17) multiplied by $-\rho_0 b \bar{z}_b$, with the aid of equation (14), give the equation for the mean kinetic and potential energies, respectively,

$$\frac{\partial}{\partial \tilde{t}}\left(\bar{z}_b \rho_0 \frac{|\hat{\mathbf{u}}|^2}{2}\right) + \tilde{\nabla} \cdot \left(\hat{\mathbf{u}} \bar{z}_b \rho_0 \frac{|\hat{\mathbf{u}}|^2}{2}\right) = -\hat{\mathbf{u}} \cdot \bar{z}_b \nabla_H p^\# + \rho_0 \bar{z}_b \hat{\mathbf{u}} \cdot \mathbf{F}, \tag{28}$$

$$-\frac{\partial}{\partial \tilde{t}}(\rho_0 b \bar{z} \bar{z}_b) - \tilde{\nabla} \cdot (\hat{\mathbf{u}} \rho_0 b \bar{z} \bar{z}_b) = -\rho_0 b w^\# \bar{z}_b. \tag{29}$$

Although the previous studies have performed volume integrals of these equations to develop the energy diagram for the mean and eddy kinetic and potential energies in a volume budget, we abandon it in favor of seeking local equations of the sum of the mean energies and that of the eddy energies.

Now, we notice that using the hydrostatic equation and the incompressibility equation, the term of $\rho_0 b w^\#$ in the right-hand side of equation (29) can be written as

$$\begin{aligned} \rho_0 b w^\# &= p_z^\# w^\# \\ &= p_z^\# w^\# + p^\# \nabla \cdot \mathbf{v}^\# \\ &= \nabla \cdot (p^\# \mathbf{v}^\#) - \hat{\mathbf{u}} \cdot \nabla_H p^\#. \end{aligned}$$

We see that the second term in the bottom line in this relation is identical to the first term in the right-hand side of equation (28). This means that the mean kinetic and potential energies are converted each other through the terms of $\hat{\mathbf{u}} \cdot \nabla_H p^\#$ and $\rho_0 b w^\#$ in equations (28) and (29), respectively. Thus, using this relation, the equation for the sum of the mean kinetic and potential energies becomes

$$\frac{\partial}{\partial \tilde{t}}\left\{\bar{z}_b \rho_0 \left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} + \tilde{\nabla} \cdot \left\{\bar{z}_b \hat{\mathbf{u}} \rho_0 \left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} = \rho_0 \bar{z}_b \hat{\mathbf{u}} \cdot \mathbf{F} - \bar{z}_b \nabla \cdot (\mathbf{v}^\# p^\#), \tag{30}$$

which results in the same energy equation as equation (25) by mapping the terms in the left-hand side to z-coordinates (Appendix 2) and replacing $\mathbf{F}$ with $\nabla \cdot \gamma_{ij}$.

The temporal average of the equation of motion (equation 13) and that of the equation of vertical velocity (equation 11) multiplied by $\rho_0 \mathbf{u} z_b$ and $-\rho_0 b z_b$, respectively, with the aid of equation (12), give the total kinetic and potential energies, respectively,

$$\frac{\partial}{\partial \tilde{t}}\left(\bar{z}_b \rho_0 \frac{\widehat{|\mathbf{u}|^2}}{2}\right) + \tilde{\nabla} \cdot \left(\hat{\mathbf{u}} \bar{z}_b \rho_0 \frac{\widehat{|\mathbf{u}|^2}}{2}\right) + \tilde{\nabla} \cdot \mathbf{B}_K = -\overline{z_b \mathbf{u} \cdot \tilde{\nabla} \pi}, \tag{31}$$

$$-\frac{\partial}{\partial \tilde{t}}(\rho_0 b \hat{z} \bar{z}_b) - \tilde{\nabla} \cdot (\hat{\mathbf{u}} \rho_0 b \hat{z} \bar{z}_b) + \tilde{\nabla} \cdot \mathbf{B}_P = -\rho_0 b \overline{w z_b}, \tag{32}$$

where $\mathbf{B}_K \equiv \bar{z}_b \rho_0 \widehat{\mathbf{u}''|\mathbf{u}|^{2''}}/2$ and $\mathbf{B}_P \equiv -\bar{z}_b \rho_0 b \widehat{\mathbf{u}''z''}$, which are energy fluxes by eddies. Note that b is constant for the temporal average in b-coordinates. Following the same procedure just below equation (29), $\rho_0 bw$ can be rewritten as

$$\rho_0 bw = \nabla \cdot (p\mathbf{v}) - \mathbf{u} \cdot \nabla_H p,$$

whose TWA yields

$$\rho_0 b\overline{wz_b} = \overline{z_b \nabla \cdot (p\mathbf{v})} - \overline{z_b \mathbf{u} \cdot \tilde{\nabla}\pi},$$

where we used the fact that $\nabla_H p = \tilde{\nabla}\pi$. Similar to the case of the mean energies, this relation expresses the conversion between the total kinetic and potential energies.

Subtracting equations (28) and (29) from equations (31) and (32), we have the equations of the eddy kinetic and potential energies, respectively,

$$\frac{\partial}{\partial \tilde{t}}\left(\bar{z}_b \rho_0 \frac{\widehat{|\mathbf{u}''|^2}}{2}\right) + \tilde{\nabla} \cdot \left(\hat{\mathbf{u}}\bar{z}_b \rho_0 \frac{\widehat{|\mathbf{u}''|^2}}{2}\right) = -\overline{z_b \mathbf{u} \cdot \tilde{\nabla}\pi} + \bar{z}_b \hat{\mathbf{u}} \cdot \tilde{\nabla}_H p^{\#} - \rho_0 \bar{z}_b \hat{\mathbf{u}} \cdot \mathbf{F} - \tilde{\nabla} \cdot \mathbf{B}_K, \quad (33)$$

$$-\frac{\partial}{\partial \tilde{t}}\left(\bar{z}_b \rho_0 b \frac{\overline{z' z_b'}}{\bar{z}_b}\right) - \tilde{\nabla} \cdot \left(\hat{\mathbf{u}}\bar{z}_b \rho_0 b \frac{\overline{z' z_b'}}{\bar{z}_b}\right) = -\rho_0 b\overline{wz_b} + \rho_0 b w^{\#} \bar{z}_b - \tilde{\nabla} \cdot \mathbf{B}_P, \quad (34)$$

where we used $\widehat{|\mathbf{u}|^2} = \bar{z}_b |\hat{\mathbf{u}}|^2 + \bar{z}_b \widehat{|\mathbf{u}''|^2}$ and $\hat{z} = \bar{z} + \overline{z' z_b'}/\bar{z}_b$. Introducing a variable, $\mathcal{S}^{\#}$, denoting the sum of the eddy kinetic and potential energies,

$$\mathcal{S}^{\#} \equiv \frac{\widehat{|\mathbf{u}''|^2}}{2} - b\frac{\overline{z' z_b'}}{\bar{z}_b}, \quad (35)$$

the equation for the sum of the eddy kinetic and potential energies is given by

$$\frac{\partial}{\partial \tilde{t}}(\bar{z}_b \rho_0 \mathcal{S}^{\#}) + \tilde{\nabla} \cdot \left(\hat{\mathbf{u}}\bar{z}_b \rho_0 \mathcal{S}^{\#}\right) = -\rho_0 \bar{z}_b \hat{\mathbf{u}} \cdot \mathbf{F} - \overline{z_b \nabla \cdot (\mathbf{v}p)} + \bar{z}_b \nabla \cdot (\mathbf{v}^{\#} p^{\#}) - \tilde{\nabla} \cdot (\mathbf{B}_K + \mathbf{B}_P). \quad (36)$$

Mapping the terms on the left-hand side into z-coordinates (Appendix 2) and replacing $\mathbf{F}$ with $\nabla \cdot \gamma_{ij}$, we have the equation of the eddy energies in a local formulation:

$$\rho_0 D_t^{\#} \mathcal{S}^{\#} = -\rho_0 \mathbf{v}^{\#} \cdot \nabla \cdot \gamma_{ij} + G, \quad (37)$$

where

$$G \equiv -\left[\frac{1}{\bar{z}_b}\overline{z_b \nabla \cdot (\mathbf{v}p)} - \nabla \cdot (\mathbf{v}^{\#} p^{\#})\right] - \frac{1}{\bar{z}_b}\tilde{\nabla} \cdot (\mathbf{B}_K + \mathbf{B}_P). \quad (38)$$

The rate of change of the eddy energies is caused by the work done by the eddy momentum fluxes and the scalar function defined as G. The term for the eddy momentum fluxes also appeared in the energy equation of the mean state (equation 25) and, thus, causes a reversible conversion between the mean and eddy energies. Such a conversion is caused by the occurrence of the Reynolds flux and the interfacial form stress associated with barotropic and baroclinic instabilities (e.g., Aiki and Richards 2008). The function G contains the pressure fluxes and energy fluxes by eddies. The former is written as the difference of the total pressure flux in a time average from the pressure flux by the residual mean velocities and, thus, can be interpreted as the pressure flux associated with eddy motion. The function, G, however, has the property that it vanishes if it is integrated over the volume. The vanishing of the first[4] and second terms of G is due to the boundary conditions of $\mathbf{v}$ and $\mathbf{v}^{\#}$, respectively, and that of the third term is demonstrated in Appendix 3.

By definition, taking account of the eddy energies leads to the conservation of the total energy. The sum of equations (25) and (37) gives the local energy equation for the total energy,

$$D_t^{\#}\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z} + \mathcal{S}^{\#}\right)\right\} = -\nabla\cdot(\mathbf{v}^{\#}p^{\#}) + G, \tag{39}$$

or in flux form,

$$\frac{\partial}{\partial t}\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z} + \mathcal{S}^{\#}\right)\right\} = -\nabla\cdot\left[\mathbf{v}^{\#}\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z} + \mathcal{S}^{\#}\right)\right] - \nabla\cdot(\mathbf{v}^{\#}p^{\#}) + G. \tag{40}$$

This is the energy equation, that the TWA system must satisfy. In this energy equation, the total energy is the sum of the mean kinetic energy, the mean potential energy, and the eddy energies. The local temporal change of the total energy is caused by the flux of the total energy, the mean pressure flux, and the fluxes in G. The work done by the eddy momentum fluxes does not appear in the total energy budget because it is the term representing the conversion between the mean and eddy energies. Applying the boundary condition of $\mathbf{v}^{\#}$ and by the nature of G, the volume integral of this equation gives

$$\frac{d}{dt}\int_V d^3x\,\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z} + \mathcal{S}^{\#}\right)\right\} = 0. \tag{41}$$

Making the notational substitutions $\rho_0|\hat{\mathbf{u}}|^2/2 \longrightarrow \mathcal{K}_M$, $-\rho_0 b\bar{z} \longrightarrow \mathcal{P}_M$, and $\rho_0\mathcal{S}^{\#} \longrightarrow \mathcal{K}_E + \mathcal{P}_E$, we have the energy conservation shown in equation (2), which is the same as that in previous studies (Bleck 1985; Aiki and Yamagata 2006; Aiki and Richards 2008).

4. Notice that the vertical integral of the first term of G is equal to $\int dz\nabla\cdot\overline{(\mathbf{v}p)}$ by the "pile-up rule" termed in Aiki and Yamagata (2006).

Generally, for a perfect or a dissipative fluid with a conservative force, the total energy to be conserved consists of the kinetic energy, potential energy, and internal energy (e.g., Landau and Lifshitz 1987). Compared with this, the eddy energies, $\mathcal{S}^{\#}$, can be regarded as internal energy inherent in the TWA system. The internal energy in the general sense follows the thermodynamic relation, in which its variation depends on work done by pressure and the variation of entropy. For a viscous fluid, the rate of change of entropy along the path line is caused by the dissipation function (e.g., Landau and Lifshitz 1987). Regarding the eddy momentum fluxes as analogous to viscous momentum fluxes, the relation (37) tells us that $\mathcal{S}^{\#}$ works like entropy in this system. To be exact, different from the general meaning of entropy, $\mathcal{S}^{\#}$ can take a negative value depending on the sign of the work done by the eddy momentum fluxes. Although, if we assume, however, that the eddy momentum fluxes are written as a form of Fickian diffusion, $\mathcal{S}^{\#}$ may be qualified as entropy in the general sense, and discussion of this point may be beyond the scope of this article.

In this system, there seems to be no contribution of the work done by the pressure to the variation of the internal energy. This is attributed to the assumption of the Boussinesq fluid. Consider the thermodynamic relation without entropy variation:

$$d\epsilon(\rho) = \frac{p}{\rho^2} d\rho, \tag{42}$$

where ϵ denotes internal energy. Replacing the exterior derivative, d, with the Lagrangian time derivative, D_t, and using the mass conservation, $D_t\rho + \rho\nabla \cdot \mathbf{v} = 0$ (e.g., Landau and Lifshitz 1987), this relation can be rewritten as

$$D_t\epsilon = -\frac{p}{\rho}\nabla \cdot \mathbf{v}. \tag{43}$$

For the Boussinesq fluid, the variation of internal energy vanishes due to incompressibility of the fluid. Thus, in the TWA system premising the Boussinesq fluid, the work done by the pressure never occurs.

4. The TWA equation of motion in terms of variational principle

Here, we argue the necessity of the equation of the eddy energies for deriving the TWA equation of motion by the variational principle. In Section 3, premising the fluid motion following the TWA momentum equation, it was shown that the eddy energy and its governing equation are necessary to satisfy the time mean energy conservation of the adiabatic nondissipative primitive equations. In terms of the variational principle, in turn, they are necessary to describe the fluid motion itself in the TWA system.

In the variational principle, generally, the Lagrangian density for a three-dimensional perfect fluid with a conservative force, ϕ, is given by $\mathcal{L} = \rho[\mathbf{v}^2 - \epsilon(\rho, s) - \phi]$ (e.g., Salmon 1988). Minimizing this Lagrangian density integrated over space and time under the constraints of the conservation of mass and entropy yields the Euler equation (e.g., Salmon 1988;

Holm, Marsden, and Ratiu 1998; Kambe 2004; Bennet 2006). For an adiabatic nondissipative Boussinesq fluid including the primitive equation system, internal energy is omitted from the Lagrangian, and consequently, constraints are replaced with the incompressibility and buoyancy conservation (e.g., Holm, Marsden, and Ratiu 1998, 2002). Fukagawa and Fujitani (2012) extended the variational principle for a dissipative fluid based on an argument of a nonholonomic system (e.g., Landau and Lifshitz 1976). In this section, regarding the eddy momentum fluxes as analogous to viscous momentum fluxes, we apply the variational principle proposed in their study for the TWA system. Also, for simplicity, we omit notation characterizing the variables in the TWA system such as # and do not treat the Coriolis term, because it is nonessential for this system as mentioned in Section 2.

a. Methodology and the case excluding eddy energies

The variational principle can be formulated in the Eulerian or Lagrangian description, and the former is adopted in this study. In this case, the velocity of the fluid is a variable of the Lagrangian density instead of the position of the fluid particle. Thus, to specify the position of each particle, we need to introduce the Lagrangian coordinates. Let $\mathbf{x}(\mathbf{A}, t)$ be the position at time t for a fluid particle labeled by $\mathbf{A} \equiv (A_1, A_2, A_3)$, which gives the Lagrangian coordinates. The relation between the Lagrangian coordinates and the velocity field is given by

$$\frac{\partial A_i}{\partial t} + \mathbf{v} \cdot \nabla A_i = 0, \tag{44}$$

which may be written as

$$v_j = -\frac{\partial x_j}{\partial A_i}\frac{\partial A_i}{\partial t} \tag{45}$$

(e.g., Fukagawa and Fujitani 2012). In the latter form, we used Einstein's summation convention for the index variable "i." Hereafter, we use similar notation for summation.

Using the Lagrangian coordinates, we rewrite the incompressibility equation and the buoyancy equation in different forms. Assigning the Lagrangian coordinates so as to satisfy $dV_A \equiv \rho dV_X$, where dV_A and dV_X are volume elements made by $\mathbf{A}$ and $\mathbf{x}$ (i.e., $dV_A \equiv dA_1 \wedge dA_2 \wedge dA_3$ and $dV_X \equiv dx \wedge dy \wedge dz$, respectively), we have the relation

$$J^{-1} \equiv \frac{\partial(A_1, A_2, A_3)}{\partial(x, y, z)} = \rho \tag{46}$$

(e.g., Salmon 1998). For the TWA system based on the Boussinesq fluid, this relation is approximated as

$$J^{-1} \simeq \rho_0, \tag{47}$$

which is another expression of the incompressibility condition (equation 18) (e.g., Salmon 1998). By the temporal integration, the conservation of buoyancy (equation 21) can be expressed as

$$b(\mathbf{x}(\mathbf{A}, t), t) - b_{\text{init}}(\mathbf{A}) = 0, \tag{48}$$

where b_{init} denotes the buoyancy field at initial time. Note that the Lagrangian coordinates can be regarded as the initial position of the fluid particle because they are fixed on the moving fluid.

Based on the energy conservation for the TWA system (equation 41), the Lagrangian density in the Eulerian description to be considered here should be given by

$$\mathcal{L} \equiv J^{-1}\left(\frac{\mathbf{u}^2}{2} + bz - \mathcal{S}\right), \tag{49}$$

where we replaced ρ_0 with J^{-1} following the relation (47). Although $\mathcal{S} = 0$ in this case, we leave $\mathcal{S}$ to the Lagrangian density with an eye to the case including the eddy energies. The action is given by the integral of the Lagrangian density over space and time with respect to the constraints of incompressibility, buoyancy conservation, and the conservation of the Lagrangian coordinates,[5] that is,

$$I_{NE}[J^{-1}, \mathbf{v}, b, \mathbf{A}, K, \Lambda, \boldsymbol{\beta}] \equiv \int_{t_1}^{t_2} dt \int_V d^3x \left\{\mathcal{L} + K(\rho_0 - J^{-1}) + \Lambda(b - b_{\text{init}}) + \beta_i \left(\frac{\partial A_i}{\partial t} + \mathbf{v}\cdot\nabla A_i\right)\right\}, \tag{50}$$

where K, λ, and β_i are undetermined Lagrange multipliers. The end-point condition is defined as $\delta\mathbf{A}(\mathbf{x}, t_1) = \delta\mathbf{A}(\mathbf{x}, t_2) = 0$. Using the method of Lagrange multiplier, the stationary conditions that minimize the action (i.e., $\delta I_{NE} = 0$, with respect to δK, $\delta\Lambda$, $\delta\beta_i$, $\delta\mathbf{v}$, δJ^{-1}, δb, and $\delta\mathbf{A}$, respectively), give (47), (48), (44), and

$$\delta\mathbf{v} : (u, v, 0) = -\frac{1}{\rho_0}\beta_i \nabla A_i, \tag{51}$$

$$\delta J^{-1} : K = \frac{\mathbf{u}^2}{2} + bz - \mathcal{S}, \tag{52}$$

$$\delta b : \Lambda = -\rho_0 z, \tag{53}$$

$$\delta\mathbf{A} : D_t\beta_i = -\Lambda\frac{\partial b_{\text{init}}}{\partial A_i}. \tag{54}$$

5. The conservation of the Lagrangian coordinates is referred to as Lin's constraint, which enables us to give a rotation of the velocity field derived by the Eulerian variational calculus (Lin 1963).

For deriving equation (54), we applied the end-point condition and the no-nomal-flow boundary condition. Furthermore, the variation with respect to δJ^{-1} can be rewritten as

$$\begin{aligned}\left(\frac{\mathbf{u}^2}{2}+bz-\mathcal{S}-K\right)\delta J^{-1} &= \left(\frac{\mathbf{u}^2}{2}+bz-\mathcal{S}-K\right)\frac{\partial J^{-1}}{\partial(\partial A_i/\partial x_j)}\delta\frac{\partial A_i}{\partial x_j}\\ &= \frac{\partial}{\partial x_j}\left[\left(\frac{\mathbf{u}^2}{2}+bz-\mathcal{S}-K\right)J^{-1}\frac{\partial x_j}{\partial A_i}\delta A_i\right]\\ &\quad - J^{-1}\frac{\partial x_j}{\partial A_i}\frac{\partial}{\partial x_j}\left[\left(\frac{\mathbf{u}^2}{2}+bz-\mathcal{S}-K\right)\right]\delta A_i, \end{aligned} \tag{55}$$

where we used the integration by parts and the facts that

$$\frac{\partial J^{-1}}{\partial(\partial A_i/\partial x_j)} = J^{-1}\frac{\partial x_j}{\partial A_i}, \tag{56}$$

and

$$\frac{\partial}{\partial x_j}\frac{\partial J^{-1}}{\partial(\partial A_i/\partial x_j)} = 0. \tag{57}$$

The first term of equation (55) vanishes for the volume integral because $\mathbf{v}\cdot\mathbf{n} = -(\partial x_j/\partial A_i)(\delta A_i/\delta t)n_j = 0$ at the boundary. Thus, the stationary condition with respect to $\delta\mathbf{A}$ can be replaced with

$$D_t\beta_i = \left[J^{-1}\frac{\partial}{\partial x_j}\left(-\frac{\mathbf{u}^2}{2}-bz+\mathcal{S}+K\right)-\Lambda\frac{\partial b}{\partial x_j}\right]\frac{\partial x_j}{\partial A_i}, \tag{58}$$

where we used equation (48).

We define the Lagrangian time derivative as $\partial_t + L_v$, where L_v denotes the Lie derivative, which is given by $\mathbf{v}\cdot\nabla$ for a scalar and $\nabla(\mathbf{v}\cdot) - \mathbf{v}\times\nabla\times$ for a cotangent vector and is commutative with the gradient and rotational operators (Schutz 1980; Holm, Marsden, and Ratiu 1998; Fukagawa and Fujitani 2010). The Lagrangian time derivative of equation (51) becomes

$$\left(\frac{\partial}{\partial t}+L_v\right)\mathbf{u} = -\frac{1}{\rho_0}(D_t\beta_i)\nabla A_i. \tag{59}$$

Substituting equations (53) and (58) into this equation yields

$$D_t\mathbf{u} = -\frac{1}{\rho_0}\nabla(\rho_0 K) + b\mathbf{k}, \tag{60}$$

where we used $J^{-1} = \rho_0$ and $\mathcal{S} = 0$. If we regard the Lagrange multiplier, K, as the pressure, that is, $K \equiv p/\rho_0$, we have the TWA equation of motion excluding the eddy momentum fluxes. Note that this equation is the same as the adiabatic nondissipative primitive equation.

b. Case including eddy energies

Fukagawa and Fujitani (2012) proposed the variational principle for a dissipative fluid by using a nonholonomic constraint. Because the internal energy is generally a function of density and entropy, the fluid motion is influenced by the states of these variables. For the adiabatic perfect fluid, the conservations of mass and entropy give these constraints. For a dissipative fluid, however, the entropy, follows the relation such that

$$\rho T D_t s = \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \nabla \cdot \boldsymbol{\theta}, \tag{61}$$

where T is temperature, s is entropy, σ_{ij} is the viscous stress tensor, and $\boldsymbol{\theta}$ is the heat flux vector (e.g., Landau and Lifshitz 1987). The first term on the right-hand side of this equation gives the so-called dissipation function. Supposing an adiabatic wall at the boundaries and the no-normal-flow boundary condition, the volume integral of this equation yields

$$\int_V d^3x \; \rho T \frac{\partial s}{\partial t} = \int_V d^3x \; \left\{ -\mathbf{v} \cdot \left(\nabla \cdot \sigma_{ij} + \rho T \nabla s \right) \right\}. \tag{62}$$

Expressing $\mathbf{v}$ as equation (45), where $\mathbf{v}$ means instantaneous velocity only in this case, and replacing ∂_t with δ, we have

$$\int_V d^3x \; \rho T \delta s = \int_V d^3x \; \left\{ \frac{\partial x_j}{\partial A_i} \cdot \left(\nabla \cdot \sigma_{ij} + \rho T \nabla s \right) \right\} \delta A_i, \tag{63}$$

which is the nonholonomic constraint on the dissipative fluid.

Regarding the eddy energies and the eddy momentum fluxes as analogous to an entropy and a viscosity, respectively, we apply their argument to the TWA system. Integrating the equation of the eddy energies (equation 37) over the volume and using equation (45) yields

$$\int_V d^3x \; \rho_0 \delta \mathcal{S} = \int_V d^3x \; \frac{\partial x_j}{\partial A_i} \cdot \left(\rho_0 \nabla \cdot \gamma_{ij} + \rho_0 \nabla \mathcal{S} \right) \delta A_i, \tag{64}$$

which is the nonholonomic constraint for the TWA system. By this relation, the variation with respect to the eddy energies $\delta \mathcal{S}$ can be replaced with that to the Lagrangian coordinates $\delta \mathbf{A}$. The action to be considered here is defined as

$$I_E[J^{-1}, \mathbf{v}, b, \mathbf{A}, K, \Lambda, \boldsymbol{\beta}, \mathcal{S}] \equiv \int_{t_1}^{t_2} dt \int_V d^3x \left\{ \mathcal{L} + K(\rho_0 - J^{-1}) + \Lambda(b - b_{\text{init}}) + \beta_i \left(\frac{\partial A_i}{\partial t} + \mathbf{v} \cdot \nabla A_i \right) \right\}, \tag{65}$$

where we notice that the action is also a function of the eddy energies $\mathcal{S}$. Although the stationary conditions with respect to $\delta \mathbf{v}$, δJ^{-1}, and δb are the same as equations (51), (52), and (53), respectively, that to $\delta\mathbf{A}$ is given by equation (58) and the integrands of equation (64), that is,

$$\delta\mathbf{A}\colon\ D_t\beta_i = \left[J^{-1}\frac{\partial}{\partial x_j}\left(-\frac{\mathbf{u}^2}{2} - bz + S + K\right) - \Lambda\frac{\partial b}{\partial x_j} - \rho_0\nabla\cdot\gamma_{ij} - \rho_0\nabla\mathcal{S}\right]\frac{\partial x_j}{\partial A_i}. \tag{66}$$

Employing the same procedure to yield equation (60), we have the TWA equation of motion including the eddy momentum fluxes:

$$D_t\mathbf{u} = -\frac{1}{\rho_0}\nabla p + b\mathbf{k} + \nabla\cdot\gamma_{ij}. \tag{67}$$

By comparison with the case of noneddy energies, we find that the equation of the eddy energies is necessary to derive the TWA equation of motion. This equation can also be expressed as a Hamiltonian formulation (Appendix 4).

5. Discussion of the pressure induced by eddies

As mentioned in Section 3, the equation of the eddy energies (equation 37) includes the term G whose volume integral becomes zero. If the tensor of the eddy momentum fluxes includes a component, σ_{ij}, satisfying $\int_V d^3x\ \mathbf{v}^\# \cdot \nabla \cdot \sigma_{ij} = 0$, then this component can be absorbed into G, and we can define a modified tensor, $\gamma_{ij} - \sigma_{ij}$. An isotropic tensor gives an example, say, $\sigma_{ij} = \mu$, where μ is a scalar function. We can easily find that with the aid of the boundary condition of $\mathbf{v}^\#$, the volume integral of $\mathbf{v}^\# \cdot \nabla\mu$ vanishes because this can be written in a flux form due to the incompressibility. Denoting $\Gamma_{ij} \equiv \gamma_{ij} - \mu\delta_{ij}$, where δ_{ij} denotes the Kronecker delta, and $G^\dagger \equiv G - \mathbf{v}^\# \cdot \nabla\mu$, the equation of the eddy energy is rewritten as

$$\rho_0 D_t^\# \mathcal{S}^\# = -\rho_0\mathbf{v}^\# \cdot \nabla\cdot\Gamma_{ij} + G^\dagger. \tag{68}$$

Likewise, replacing γ_{ij} with Γ_{ij} in the energy equation for the mean state (equation 25) yields

$$D_t^\#\left\{\rho_0\left(\frac{|\hat{\mathbf{u}}|^2}{2} - b\bar{z}\right)\right\} = \rho_0\mathbf{v}^\#\cdot\nabla\cdot\Gamma_{ij} - \nabla\cdot(\mathbf{v}^\# P), \tag{69}$$

where

$$P \equiv p^\# - \rho_0\mu. \tag{70}$$

These transformations enable us to express the TWA equation of motion differently, that is, in a vector form,

$$D_t^\#\hat{\mathbf{u}} + f\mathbf{k}\times\hat{\mathbf{u}} = -\frac{1}{\rho_0}\nabla P + b\mathbf{k} + \nabla\cdot\Gamma_{ij}. \tag{71}$$

Notice that this equation of motion is the one in which we just changed the notation for $p^{\#}$ and γ_{ij} in equation (22). That is, the TWA system is invariant for those transformations. The transformation of pressure (equation 70) indicates that the eddy momentum fluxes have the role of inducing pressure. Taking into account that an isotropic component of a stress tensor is generally given by the sum of the diagonal components of the tensor, that of the eddy momentum fluxes can be defined as $\mu \equiv \gamma_{11} + \gamma_{22} + \gamma_{33} = -\overline{z'^2 b_z^{\#}}/\rho_0 - \widehat{u''u''} - \widehat{v''v''}$ (Appendix 1).

We consider the problem of the zonal extent of the eastward jet of the western boundary current extension such as the Kuroshio Extension or the Gulf Stream as an example of the application of the eddy-induced pressure. Analyzing the solution of a two-layer QG potential vorticity (PV) equation with rigid surface and flat bottom, Waterman and Jayne (2011) showed that eddy PV flux associated with the instability of the unstable idealized eastward jet occurs equatorward in the upstream and poleward in the downstream. They suggested that the former acts to decay the jet consistent with barotropic instability and the latter leads to the formation of the jet's flanking recirculations in the upstream through the westward propagation of Rossby waves. They also reported that the spatial structure of the eddy PV flux is almost explained by barotropic model. Such a complex picture of eddy-mean flow interaction has been confirmed in several studies based on numerical experiment and observation (Jayne, Hogg, and Malanotte-Rizzoli 1996; Jayne and Hogg 1999; Mizuta 2009; Waterman, Hogg, and Jayne 2011; Waterman and Hoskins 2013).

Waterman and Hoskins (2013) analyzed the solution of the barotropic QG PV equation employed in Waterman and Jayne (2011), focusing on the Reynolds flux by decomposition into

$$\begin{pmatrix} \overline{u'^2} & \overline{v'u'} \\ \overline{v'u'} & \overline{v'^2} \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} + \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \tag{72}$$

where $K \equiv \overline{u'^2 + v'^2}/2$, $M \equiv \overline{u'^2 - v'^2}/2$, and $N \equiv \overline{u'v'}$. From these expressions, divergence of the eddy PV flux giving the forcing of the mean PV field can be written as $\nabla \cdot \overline{\mathbf{u}'\zeta'} = -2M_{xy} + N_{xx} - N_{yy}$, where $\zeta' \equiv v'_x - u'_y$ denotes relative vorticity by eddy. According to their study, the divergence of the eddy PV flux is accounted for by the terms N_{yy} and M_{xy}. The contribution of the N_{yy} term concentrates along the flank of the jet in the upstream and has a role in stabilizing the jet consistent with barotropic instability. The contribution of the M_{xy} term is significant throughout the domain and, in particular, solely accounts for the aforementioned poleward eddy PV flux in the downstream, which drives the recirculations. The distribution of the M_{xy} term in the downstream represents that the direction of the elongation of eddies alters from east-west to north-south toward the downstream along the jet, in association with the deceleration of the jet. From these results, considering the mechanism that causes the deceleration of the jet may be a key to understanding the eddy-mean flow interaction of the eastward jet.

We reconsider the result by Waterman and Hoskins (2013) in terms of the momentum equation. Although the eddy kinetic energy, K, does not appear in the forcing term of the PV equation because it is a rotational component of the eddy PV flux, it becomes the forcing in the momentum equation, which potentially accelerates or decelerates the motion. It is interesting to note that K is dominant in the tensor of the Reynolds flux (equation 72) and occurs to the east of the site where the jet begins to weaken (see their Fig. 3 and Fig. 4a), suggesting that the occurrence of the eddy kinetic energy is likely to decelerate the jet in addition to the effect of N. Now, we notice that the TWA system includes the barotropic QG system because it is also included in the primitive equation system that the TWA system premises. Thus, the eddy momentum fluxes formulated in the TWA system are consistent with the tensor of the Reynolds flux in (equation 72) if we consider the barotropic QG fluid.[6] Accordingly, in terms of the modified version of the TWA equation of motion (equation 71), the eddy kinetic energy can be interpreted as the pressure induced by eddies in the barotropic QG system.

6. Summary and conclusion

This study has dealt with the TWA system in terms of energetics. The total energy (the kinetic and potential energies for the mean state) derived from the hitherto known set of equations in this system generally is not conserved due to the work done by the eddy momentum fluxes. Supposing that the TWA system satisfies a time mean energy conservation of the primitive equations, however, yields an equation showing that the eddy momentum fluxes influence the variation of the eddy energies (the sum of the kinetic and potential energies for the eddies) along the path lines with the residual mean velocity. Regarding the eddy momentum fluxes as analogous to viscous momentum fluxes, the eddy energies can be interpreted as internal energy in the TWA system by analogy to a dissipative fluid. Although the work done by pressure in the equilibrium state generally can cause the variation of the internal energy, this is not the case in the TWA system due to the incompressibility of the fluid.

The process of identifying the eddy energies as the internal energy can be viewed as coarse graining of microphenomena. In the TWA system, in which the residual mean velocity, $\hat{\mathbf{u}}$, and accompanying state of buoyancy surfaces, $\bar{z}$, are regarded as macrovariables, deviations from them associated with the eddy motion are implicitly defined as microvariables. Instead of describing the individual motions associated with the eddies, we define a characteristic of these motions by a statistical manipulation (temporal average in this case). This is analogous to the fact that one defines a mean energy of molecular motions as a state function in statistical mechanics.

6. The TWA momentum equation for the barotropic fluid in a rigid surface and flat bottom can be derived by just performing the temporal average of the primitive equation because the layer thickness of the fluid is constant. The Reynolds decomposition in the time mean barotropic momentum equation yields the Reynolds fluxes. If we apply the QG approximation of this fluid, the Reynolds flux tensor is identical to that used in Waterman and Hoskins (2013).

Table 1. A fundamental set of equations for the thickness-weighted average system.

Momentum (horizontal)	$D_t^{\#}\hat{\mathbf{u}} + f\mathbf{k}\times\hat{\mathbf{u}} = -\frac{1}{\rho_0}\nabla_H p^{\#} + \nabla\cdot\gamma_{ij}$
Momentum (vertical)	$0 = -\frac{1}{\rho_0}\frac{\partial p^{\#}}{\partial z} + b^{\#}$
Incompressibility	$\nabla\cdot\mathbf{v}^{\#} = 0$
Buoyancy	$D_t^{\#}b^{\#} = 0$
Internal energy	$D_t^{\#}\mathcal{S}^{\#} = \mathbf{v}^{\#}\cdot\nabla\cdot\gamma_{ij} + \rho_0^{-1}G$

In this study, we showed that taking account of the equation of the eddy energies successfully derives the TWA equation of motion by the variational principle. If we otherwise neglect the eddy energies, the motion of the fluid is the same as that expected in the nondissipative primitive equation, even if we claimed that the prognostic variable is the residual mean velocity. This fact suggests that the equation of the eddy energies is important for determining the motion of the macrofluid in the TWA system. In other words, the term discerning the TWA system from the primitive equation system is not the residual mean velocity, but rather the eddy momentum fluxes embedded in the equation of motion.

From these results, this study proposes a fundamental set of equations in the TWA system, adding the equation of the eddy energies (Table 1). As mentioned in Section 5, this system is invariant for the alteration that we include the isotropic component of the eddy momentum fluxes as a part of the pressure; these terms cannot be distinguished from each other. Based on this concept, we can give an interpretation that the downstream decaying of the western boundary current extension is caused by the pressure induced by eddies. Also, we expressed the TWA system in the Lagrangian and Hamiltonian formulations, which will be helpful to describe this system in terms of the Hamiltonian dynamics in the future.

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APPENDIX 1

Tensor representation of the eddy momentum fluxes

We express the eddy momentum flux vector $\mathbf{F}$ using a tensor, γ_{ij}. Defining $\Phi \equiv \overline{z'^2}/2$; $V_{ij} \equiv \bar{z}_b\widehat{u_i''u_j''}$, where $(u_1, u_2) \equiv (u, v)$; and $\mathbf{C} \equiv \overline{z'\tilde{\nabla}\pi'}/\rho_0$, the eddy momentum flux vector $\mathbf{F}$ becomes

$$\mathbf{F} \equiv -\frac{1}{\bar{z}_b}\tilde{\nabla}\Phi - \frac{1}{\bar{z}_b}\tilde{\nabla} \cdot V_{ij} - \frac{\partial \mathbf{C}}{\partial z}. \tag{A1}$$

Using the transition rule (equation 9), after some manipulation, the first term of this formula is expressed in Cartesian coordinates as

$$\frac{1}{\bar{z}_b}\tilde{\nabla}\Phi = \nabla_H \left(b_z^{\#}\Phi\right) + \frac{\partial}{\partial z}(-\Phi\nabla_H b^{\#}). \tag{A2}$$

This can be written in a tensor form as

$$\frac{1}{\bar{z}_b}\tilde{\nabla}\Phi = \nabla \cdot \xi_{ij}, \tag{A3}$$

where

$$\xi_{ij} \equiv \Phi \begin{pmatrix} b_z^{\#} & 0 & 0 \\ 0 & b_z^{\#} & 0 \\ -b_x^{\#} & -b_y^{\#} & 0 \end{pmatrix} \tag{A4}$$

and $\nabla \cdot \xi_{ij} \equiv (\partial/\partial x_i)\xi_{ij}$. Because a similar relation to (A2) holds for the tensor, V_{ij}, the second term of (A1) becomes

$$\frac{1}{\bar{z}_b}\tilde{\nabla} \cdot V_{ij} = \nabla \cdot \eta_{ij}, \tag{A5}$$

where

$$\eta_{ij} \equiv \begin{pmatrix} \widehat{u''u''} & \widehat{u''v''} & 0 \\ \widehat{v''u''} & \widehat{v''v''} & 0 \\ -\frac{\nabla_H b^{\#}}{b_z^{\#}} \cdot \widehat{\mathbf{u}''u''} & -\frac{\nabla_H b^{\#}}{b_z^{\#}} \cdot \widehat{\mathbf{u}''v''} & 0 \end{pmatrix}. \tag{A6}$$

Also, denoting $\mathbf{C} \equiv (C_1, C_2)$, the third term of (A1) can be written as

$$\frac{\partial \mathbf{C}}{\partial z} = \nabla \cdot \zeta_{ij}, \tag{A7}$$

where

$$\zeta_{ij} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & 0 \end{pmatrix}. \tag{A8}$$

Because there is a relation between the coordinates such that $\tilde{\nabla}\pi' = \nabla_H \pi' - \pi'_z(\nabla_H b^{\#})/b_z^{\#}$, the vector $\mathbf{C}$ can be expressed as a horizontal vector in Cartesian coordinates. The eddy momentum flux tensor, γ_{ij}, is defined as the sum of the following tensors:

$$\gamma_{ij} \equiv -(\xi_{ij} + \eta_{ij} + \zeta_{ij}). \tag{A9}$$

Note that $\gamma_{i3} = 0$, meaning that this tensor does not influence the vertical component of the residual velocity, and, therefore, the work done by this tensor on the residual vertical velocity is zero: $w^{\sharp}\partial\gamma_{i3}/\partial x_i = 0$. Also, a generalization of the eddy momentum flux tensor is shown in Maddison and Marshall (2013).

APPENDIX 2

On expression of the Lagrangian time derivative in the TWA system

This appendix compiles some expressions of the Lagrangian time derivatives in the TWA system. Although they have been shown in previous studies, showing them again here may be helpful to the readers.

Consider a tracer equation for A forced by Q in the TWA system such that

$$\frac{\partial A}{\partial t} + \mathbf{v}^{\#} \cdot \nabla A = \frac{\partial A}{\partial t} + \nabla \cdot (\mathbf{v}^{\#} A) = Q. \tag{A10}$$

The first equality comes from the incompressibility. With the aid of equation (17), the transformation of this equation into b-coordinates yields

$$\frac{\partial A}{\partial \tilde{t}} + \hat{\mathbf{u}} \cdot \tilde{\nabla} A = Q. \tag{A11}$$

Multiplying it by the thickness, $\bar{z}_b$, and using the time mean mass conservation in b-coordinates (equation 14) gives

$$\frac{\partial}{\partial \tilde{t}}(\bar{z}_b A) + \tilde{\nabla} \cdot (\hat{\mathbf{u}} \bar{z}_b A) = \bar{z}_b Q. \tag{A12}$$

Comparing these three equations, we have relations such that

$$\frac{\partial A}{\partial t} + \mathbf{v}^{\#} \cdot \nabla A = \frac{\partial A}{\partial t} + \nabla \cdot (\mathbf{v}^{\#} A) = \frac{\partial A}{\partial \tilde{t}} + \hat{\mathbf{u}} \cdot \tilde{\nabla} A = \frac{1}{\bar{z}_b}\left[\frac{\partial(\bar{z}_b A)}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\hat{\mathbf{u}} \bar{z}_b A)\right]. \tag{A13}$$

A similar argument can also be made for the primitive equations because the equation of incompressibility, the equation of the vertical velocity, and the equation of the mass conservation in b-coordinates in the primitive equations are written in the same form in the TWA system.

APPENDIX 3

Integral constraint

We show here that the volume integral of the terms $\bar{z}_b^{-1}\tilde{\nabla} \cdot \mathbf{B}_K$ and $\bar{z}_b^{-1}\tilde{\nabla} \cdot \mathbf{B}_P$ in Section 3 vanishes. Let Ω be the domain consisting of (x, y, b). Owing to the relations $dz = z_b db$ and $d\bar{z} = \bar{z}_b db$, this domain corresponds to the volumes in the x-y-z and x-y-$\bar{z}$ spaces, respectively. These volumes are the same because all of the buoyancy surfaces in a temporal mean, $\bar{z}$, are stacked in the volume for the x-y-z space.

Consider temporal and spatial changes of an arbitrary quantity, A, in a flux form such that

$$z_b\psi \equiv (z_b A)_{\tilde{t}} + \tilde{\nabla} \cdot (z_b \mathbf{u} A) = z_b[A_t + \nabla \cdot (\mathbf{v}A)], \tag{A14}$$

where for the second equality we used equation (A13) for the primitive equations. The integral of ψ over the domain, Ω, with the no-normal-flow boundary condition of $\mathbf{v}$ becomes

$$\int_\Omega d^2xdb\ z_b\psi = \frac{d}{dt}\int_\Omega d^2xdb\ z_b A, \tag{A15}$$

whose temporal average gives

$$\int_\Omega d^2xdb\ \bar{z}_b\hat{\psi} = \frac{d}{dt}\int_\Omega d^2xdb\ \bar{z}_b\hat{A} = \frac{d}{dt}\int_V d^3x\ \hat{A}. \tag{A16}$$

Likewise, consider temporal and spatial changes of $\hat{A}$ such that

$$\bar{z}_b\phi \equiv (\bar{z}_b\hat{A})_{\tilde{t}} + \tilde{\nabla} \cdot (\bar{z}_b\hat{\mathbf{u}}\hat{A}) = \bar{z}_b[\hat{A}_t + \nabla \cdot (\mathbf{v}^{\#}\hat{A})], \tag{A17}$$

where we used equation (A13) again for the second equality. As done for deriving equation (A15), the integral of ϕ over the domain with the no-normal-flow boundary condition for $\mathbf{v}^{\#}$ yields

$$\int_\Omega d^2xdb\ \bar{z}_b\phi = \frac{d}{dt}\int_V d^3x\ \hat{A}, \tag{A18}$$

which equals equation (A16). The detailed derivations of these volume integrals are found in (equation A4) and (equation A12) in Aiki and Richards (2008).

Now, the temporal average of equation (A14) yields

$$\bar{z}_b\hat{\psi} \equiv (\bar{z}_b\hat{A})_{\tilde{t}} + \tilde{\nabla} \cdot (\bar{z}_b\hat{\mathbf{u}}\hat{A}) + \tilde{\nabla} \cdot (\bar{z}_b\widehat{\mathbf{u}''A''}). \tag{A19}$$

The third term on the right-hand side of this equation corresponds to $\nabla \cdot \mathbf{B}_K$ or $\nabla \cdot \mathbf{B}_P$. Taking into account that the sum of the first and the second terms equals $\bar{z}_b\phi$, and the equality between equations (A16) and (A18), the volume integral of $\hat{\psi}$ over the domain leads to

$$\int_\Omega d^2xdb\ \tilde{\nabla} \cdot (\bar{z}_b\widehat{\mathbf{u}''A''}) = \int_V d^3x\ \frac{1}{\bar{z}_b}\tilde{\nabla} \cdot (\bar{z}_b\widehat{\mathbf{u}''A''}) = 0. \tag{A20}$$

Thus, the volume integral of $\bar{z}_b^{-1}\tilde{\nabla} \cdot \mathbf{B}_K$ and $\bar{z}_b^{-1}\tilde{\nabla} \cdot \mathbf{B}_K$ vanishes.

APPENDIX 4

Hamiltonian

Fukagawa and Fujitani (2010, 2012) successfully obtained a Hamiltonian formulation for perfect and dissipative fluids, employing Pontryagin's minimum principle (Pontryagin et al. 1962). In this principle, the state, $\mathbf{q}$, controlled by the input, $\mathbf{v}$, is defined as

$$\frac{d}{dt}\mathbf{q} \equiv \mathbf{F}(\mathbf{q}(t), \mathbf{v}(t)), \tag{A21}$$

and we seek the optimal input, $\mathbf{v}^*$, minimizing the factional

$$\int_{t_1}^{t_2} dt\ L(\mathbf{q}(t), \mathbf{v}(t)) \tag{A22}$$

on the end-point condition that

$$\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0. \tag{A23}$$

The optimal input is obtained by minimizing the action

$$\begin{aligned} I[\mathbf{q}, \mathbf{p}, \mathbf{v}] &= \int_{t_1}^{t_2} dt\ \left[L(\mathbf{q}, \mathbf{v}) + \mathbf{p} \cdot \left(\frac{d}{dt}\mathbf{q} - \mathbf{F}(\mathbf{q}, \mathbf{v})\right)\right] \\ &= \int_{t_1}^{t_2} dt\ \left[-H(\mathbf{q}, \mathbf{p}, \mathbf{v}) + \mathbf{p} \cdot \frac{d}{dt}\mathbf{q}\right], \end{aligned} \tag{A24}$$

where $\mathbf{p}$ is the undetermined multiplier called costate and

$$H(\mathbf{q}, \mathbf{p}, \mathbf{v}) \equiv -L(\mathbf{q}, \mathbf{v}) + \mathbf{p} \cdot \mathbf{F} \tag{A25}$$

is the Hamiltonian for this system. Let $\mathbf{v}^*(\mathbf{q}, \mathbf{p})$ be the optimal input, which satisfies

$$\left.\frac{\partial H(\mathbf{q}, \mathbf{p}, \mathbf{v}^*)}{\partial \mathbf{v}^*}\right|_{\mathbf{q},\ \mathbf{p}} = 0. \tag{A26}$$

Denoting $H^*(\mathbf{q}, \mathbf{p}) \equiv H(\mathbf{q}, \mathbf{p}, \mathbf{v}^*)$, the preoptimized action is defined as

$$I^*[\mathbf{q}, \mathbf{p}, \mathbf{v}] = \int_{t_1}^{t_2} dt\ \left[-H^*(\mathbf{q}, \mathbf{p}) + \mathbf{p} \cdot \frac{d}{dt}\mathbf{q}\right]. \tag{A27}$$

Solving the stationary conditions of this action for $\mathbf{q}$ and $\mathbf{p}$ yields the canonical equation given by

$$\frac{dq_i}{dt} = \frac{\partial H^*(\mathbf{q}, \mathbf{p})}{\partial p_i}, \tag{A28}$$

$$\frac{dp_i}{dt} = -\frac{\partial H^*(\mathbf{q}, \mathbf{p})}{\partial q_i}. \tag{A29}$$

For applying this principle to the TWA system, we let the state and the costate be $\mathbf{q} \equiv A_i$ and $\mathbf{p} \equiv \beta_i$, respectively, and $\mathbf{F} \equiv -\mathbf{v} \cdot \nabla A_i$. Consider minimizing the functional given by

$$\int_V d^3x\, \mathcal{L}(\mathbf{q}, \mathbf{v}, \mathcal{S}) \tag{A30}$$

under the constraint in a form of equation (A21). Different from the basic theory of this principle, this Lagrangian is also a function of the eddy energies related to the nonholonomic constraint (equation 64). We define the Lagrangian in the form of

$$\mathcal{L}(\mathbf{q}, \mathbf{v}, S) \equiv J^{-1}\left(\frac{\mathbf{v}^2}{2} + bz - \mathcal{S}\right) + K(\rho_0 - J^{-1}), \tag{A31}$$

where $\mathbf{v} \equiv (u, v, \alpha w)$ and α is an aspect ratio; w is redefined as αw. In the limit of $\alpha \to 0$, we have the same Lagrangian as equation (49). The definition of the Lagrangian using the three-dimensional velocity vector with the aspect ratio is an analogy from an inclusive definition of the Lagrangian for a nonhydrostatic and hydrostatic system (e.g., Holm, Marsden, and Ratiu 2002). By definition, the Hamiltonian corresponding to equation (A25) can be written as

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, \mathbf{v}, \mathcal{S}) = -\mathcal{L}(\mathbf{q}, \mathbf{v}, \mathcal{S}) - \beta_i \mathbf{v} \cdot \nabla A_i, \tag{A32}$$

from which the optimal input is deduced as $\mathbf{v}^* = -\beta_i \nabla A_i$ and the corresponding Hamiltonian is found to be

$$\mathcal{H}^*(\mathbf{q}, \mathbf{p}, \mathcal{S}) \equiv \mathcal{H}(\mathbf{q}, \mathbf{p}, \mathbf{v}^*, \mathcal{S}) = J^{-1}\left(\frac{\mathbf{v}^{*2}}{2} - bz + \mathcal{S}\right) - K(\rho_0 - J^{-1}). \tag{A33}$$

Notice that for $\alpha \to 0$, this preoptimized Hamiltonian is equivalent to the total energy for the TWA system. Using this Hamiltonian, the preoptimized action is given by

$$I^*[\mathbf{q}, \mathbf{p}, \mathcal{S}] = \int_{t_1}^{t_2} dt \int_V d^3x \left[-\mathcal{H}^*(\mathbf{q}, \mathbf{p}, \mathcal{S}) + \mathbf{p} \cdot \frac{d}{dt}\mathbf{q}\right]. \tag{A34}$$

With the aid of equations (48) and (37), the stationary conditions with respect to $\delta\mathbf{p}$ and $\delta\mathbf{q}$ (i.e., $\delta\beta_i$ and δA_i) yield Hamilton's equations as follows:

$$\frac{\partial A_i}{\partial t} = \frac{\partial \mathcal{H}^*}{\partial \beta_i} - \frac{\partial}{\partial x_j}\frac{\partial \mathcal{H}^*}{\partial(\partial \beta_i/\partial x_j)}, \tag{A35}$$

$$\frac{\partial \beta_i}{\partial t} = -\frac{\partial \mathcal{H}^*}{\partial A_i} + \frac{\partial}{\partial x_j}\frac{\partial \mathcal{H}^*}{\partial(\partial A_i/\partial x_j)} - \rho_0 \frac{\partial x_j}{\partial A_i} \cdot \nabla \cdot \gamma_{ij}. \tag{A36}$$

These equations are equivalent to equations (44) and (66), respectively. Also, omitting the term of the eddy momentum fluxes, these equations become the so-called canonical equation, which corresponds to the nondissipative primitive equation.

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