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### Two Papers Commerating Domar's “ ‘Burden of the Debt’ and the National Income

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TWO PAPERS COMMEMORATING DOMAR'S

"'BURDEN OF THE DEBT' AND THE NATIONAL INCOME"

On The Feasibility of Targeted Growth Through Taxation  
and the Social Virtue of Private Thrift

E. S. Phelps

Alternative Fiscal Policies for Targeted Growth

T. N. Srinivasan

September 12, 1963

ON THE FEASIBILITY OF TARGETED GROWTH THROUGH TAXATION  
AND THE SOCIAL VIRTUE OF PRIVATE THRIFT

Edmund S. Phelps \*

We celebrate here the twentieth anniversary (almost) of Professor Evsey Domar's classic paper, "The 'Burden of the Debt' and the National Income." <sup>1/</sup> That paper dispelled the fears of many that an economy which supported aggregate demand by means of a budgetary deficit would eventually sink under the weight of the taxes necessary to service the rapidly rising public debt. Domar showed that if the deficit required for full employment is proportional to the national income and the full-employment income grows at a constant relative rate then the debt as a ratio to national income will approach a finite limit; and if, further, the interest rate is constant then so too will the debt service as a ratio to income. Moreover, if interest on the public debt is a part (but not the whole) of "taxable income" then the debt service as a ratio to taxable income will approach a limit which is less than 100%.

However, the behavior of the tax rate, as distinct from the debt service-income ratio, was neglected by Domar. This note repairs that omission and proceeds to study the feasibility and determinants of the income tax rate

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\* I am grateful to T. N. Srinivasan for contributions to this paper.

<sup>1/</sup> American Economic Review, 34 (1944), 798-827. Reprinted in Readings in Fiscal Policy (Irwin, 1955), 479-502, and with a new foreword, in E. D. Domar, Essays in the Theory of Economic Growth (Oxford, 1957), 35-69.

an proportionate algebraic deficit required to maintain the economy on the targeted full-employment path. Our chief findings are described at the conclusion of the paper.

The model.

In the spirit of the Domar paper we postulate the following:

1. There is a full-employment path of money national income,  $\bar{Y}_t$ , (possibly the only full employment path or one among many) having the property of a constant rate of growth,  $\lambda$  :

$$(1) \quad \bar{Y}_t = \bar{Y}_0 e^{\lambda t}$$

To insure that actual income,  $Y_t$ , takes this path is the objective of fiscal and monetary policies.

2. To realize this income path, investment expenditures must be some constant proportion,  $k$ , of targeted income:

$$(2) \quad \bar{I}_t = k \bar{Y}_t$$

3. Associated with this path is a constant rate of interest,  $i$ , which the government pays on the public debt. This is the interest rate that the monetary authorities find necessary to bring about the required path of investment demand.
4. Desired government expenditures,  $\bar{G}_t$ , are a constant proportion,  $\gamma$ , of targeted national income, and desired non-interest transfer payments by

government,  $B_t$ , ("benefits") are a constant proportion,  $\beta$ , of targeted income:

$$(3) \quad \bar{G}_t = \gamma \bar{Y}_t$$

$$(4) \quad \bar{B}_t = \beta \bar{Y}_t$$

We assume that  $\gamma + k < 1$  so that resources are left for private consumption.

5. Private consumption expenditure demand is a constant proportion,  $\pi$ , of disposable income. Letting  $D_t$  denote the debt,  $T_t$  taxes, and  $Y_t$  actual money income:

$$(5) \quad C_t = \pi(Y_t + \iota D_t + B_t - T_t) \quad , \quad 0 < \pi \leq 1 .$$

6. Taxes are proportional to taxable income,  $Y_t + \iota D_t$ ,  $\tau_t$  being the proportional tax rate:

$$(6) \quad T_t = \tau_t(Y_t + \iota D_t)$$

7. Any shortfall between taxes and government outlays is financed by the issue of interest-bearing debt. Hence the rate of change of the debt,  $\dot{D}_t = \frac{dD_t}{dt}$ , equals the "deficit":

$$(7) \quad \dot{D}_t = (\gamma + \beta) Y_t + \iota D_t - \tau_t(Y_t + \iota D_t)$$

If the target national income is to be realized then, by virtue of the income identity  $C_t = Y_t - I_t - G_t$ , the required consumption expenditure at time  $t$ ,  $\bar{C}_t$ , is

$$(8) \quad \bar{C}_t = \bar{Y}_t(1 - k - \gamma)$$

Equations (5) and (8) imply what disposable income must be in order that consumption demand meet the requirement. Using the budget identity,  $T_t + a_t Y_t = (\gamma + \beta)Y_t + tD_t$  where  $a_t$  denotes the deficit as a proportion of national income, we can express disposable income in terms of  $a_t$ ,  $Y_t$  and  $\gamma$ :

$$(9) \quad Y_t + tD_t + B_t - T_t = Y_t(1 + a_t - \gamma)$$

whence, by (5),

$$(10) \quad C_t = \pi(1 + a_t - \gamma)Y_t$$

Therefore if  $\bar{Y}_t$  is to be realized, meaning  $Y_t = \bar{Y}_t$  and  $C_t = \bar{C}_t$ , then  $a_t$  must satisfy the equation, derived from (8) and (10),

$$\pi(1 + a_t - \gamma) = 1 - \gamma - k$$

or

$$(11) \quad a = \frac{(1-\pi)(1-\gamma) - k}{\pi} .$$

As Domar knew, the required proportionate deficit is constant over time. It will be positive if  $\pi$ ,  $\gamma$  and  $k$  are small. As  $\pi$  approaches unity  $\underline{a}$  approaches  $-k$ ; as  $\pi$  approaches zero  $\underline{a}$  increases without limit. However for every  $\pi$ ,  $0 < \pi \leq 1$  there is an  $\underline{a}$ ,  $-k < \underline{a} < \infty$ , which will satisfy (11).

Recognizing that the proportionate deficit is constant over time we can write

$$(12) \quad \dot{D}_t = a\bar{Y}_t, \quad \underline{a} = \text{constant.}$$

Finally, we can derive from the budget equation (7) the required tax rate at time  $t$ , as a function of  $\bar{Y}_t$ ,  $D_t$  and the required  $\underline{a}$ :

$$(13) \quad \tau_t = \frac{\iota D_t}{\bar{Y}_t + \iota D_t} + \frac{(\gamma + \beta - a)\bar{Y}_t}{\bar{Y}_t + \iota D_t}$$

The tax rate will have to be revised continuously if  $D_t/\bar{Y}_t$  changes over time.

#### Domar's results

Domar showed that if  $a > 0$  a kind of steady state is approached (if the initial state is different from it) in which  $D_t/\bar{Y}_t$  and  $\iota D_t/(\bar{Y}_t + \iota D_t)$  are constant. To prove this he substituted (1) in (12) to obtain the differential equation (writing  $Y_t$  in place of  $\bar{Y}_t$  for neatness).

$$(14) \quad \dot{D}_t = aY_0 e^{\lambda t},$$

the solution of which is

$$(15) \quad D_t = D_0 + \frac{a}{\lambda} Y_0 (e^{\lambda t} - 1)$$

where  $D_0$  is the (initial) amount of debt at  $t = 0$ . It is immediately clear that the debt tends eventually to grow at the rate  $\lambda$ , no faster than output.

From (15) we obtain

$$(16) \quad \frac{D_t}{Y_t} = \frac{D_0}{Y_0} e^{-\lambda t} + \frac{a}{\lambda} - \frac{a}{\lambda} e^{-\lambda t}$$

Hence

$$(17) \quad \lim_{t \rightarrow \infty} \frac{D_t}{Y_t} = \frac{a}{\lambda}$$

As a matter of notation, let the limiting value of any variable,  $x_t$ , be denoted  $x_\infty$ . Hence  $x_\infty = \lim_{t \rightarrow \infty} x_t$ .

Continuing, we also obtain from (17)

$$(18) \quad \frac{tD_\infty}{Y_\infty} = a \frac{t}{\lambda}$$

and

$$(19) \quad \frac{tD_\infty}{Y_\infty + tD_\infty} = \frac{tD_\infty/Y_\infty}{1 + tD_\infty/Y_\infty} = a \left( \frac{t}{\lambda + at} \right)$$



Domar points out that, provided  $\lambda > 0$ , this last "debt service" ratio -- which he identifies as a measure of the debt's burdensomeness -- is smaller than 100%.

It is curious however that Domar repeatedly terms this ratio the "tax rate." If it is the tax rate that Domar was concerned about -- will deficits today necessitate incentive-crushing tax rates in the future? -- he should have examined the behavior of the tax rate,  $\tau_t$ . Let us do this.

The behavior of the tax rate.

From (13) we have

$$\tau_{\infty} = \frac{tD_{\infty}}{Y_{\infty} + tD_{\infty}} + (\gamma + \beta - a) \left( \frac{Y_{\infty}}{Y_{\infty} + tD_{\infty}} \right)$$

which, upon substitution of (19), yields

$$(20) \quad \tau_{\infty} = \frac{a(t - \lambda) + \lambda(\gamma + \beta)}{\lambda + a t}$$

Now if the tax rate should equal or exceed 100% there would be no incentive to work in which case the desired growth path would become infeasible. Even if the limiting tax rate should be less than 100% but very large it would be desirable to reconstruct the model to make the targeted growth path a function of work incentives which may in turn be affected by a steep rise of income tax rates. We hope the reader, in the spirit of the Domar model, will settle for an analysis of the following

question: On what conditions will the limiting tax rate,  $\tau_{\infty}$ , be smaller than 100%? That is, assuming work incentives are unimpaired by tax rates up to (but not including) 100%, on what conditions will the targeted growth path be feasible?

Simple arithmetic shows, assuming  $\lambda > 0$ , that when  $at + \lambda > 0$  then  $\tau_{\infty} < 1$  if and only if  $a > \gamma + \beta - 1$ . This states that if the proportionate algebraic deficit,  $\underline{a}$ , is greater than  $-\frac{\lambda}{t}$  then the limiting tax rate will be under 100% provided that the algebraic deficit is also greater than  $\gamma + \beta - 1$  (which may or may not already be implied by the former inequality). When  $at + \lambda < 0$  then  $\tau_{\infty} < 1$  if and only if  $a < \gamma + \beta - 1$ . This states that if the proportionate surplus,  $-\underline{a}$ , is greater than  $\frac{\lambda}{t}$  then the tax rate will be under 100% provided that the surplus is greater than  $1 - \gamma - \beta$  (which may or may not be implied by the former inequality). Equivalently, there are two "safe" (feasible) zones in which the limiting tax rate is less than 100%: one zone in which the algebraic deficit is larger than  $\max\left[\gamma + \beta - 1, -\frac{\lambda}{t}\right]$  (which will be a little negative) and another zone in which the algebraic deficit is smaller than  $\min\left[\gamma + \beta - 1, -\frac{\lambda}{t}\right]$  -- that is, there is a large surplus. The zone between these two is infeasible. A diagram would probably be illuminating but before we can draw it we need to know in what direction  $\tau_{\infty}$  changes with changes in  $\underline{a}$ .

Suppose that  $\pi$  (the propensity to consume) were to decrease so that a larger  $\underline{a}$  was required to fulfill the consumption demand requirement of the targeted income path. While this would certainly call for an initial reduction of the tax rate (to produce the larger deficit) would it also permit a lower limiting tax rate (and hence a permanently smaller tax rate) despite the eventually higher debt service? Taking the derivative of  $\tau_{\infty}$  in (20) with respect to  $\underline{a}$  we find that, if it exists (implying  $\lambda + at \neq 0$ ), its algebraic sign is the sign of

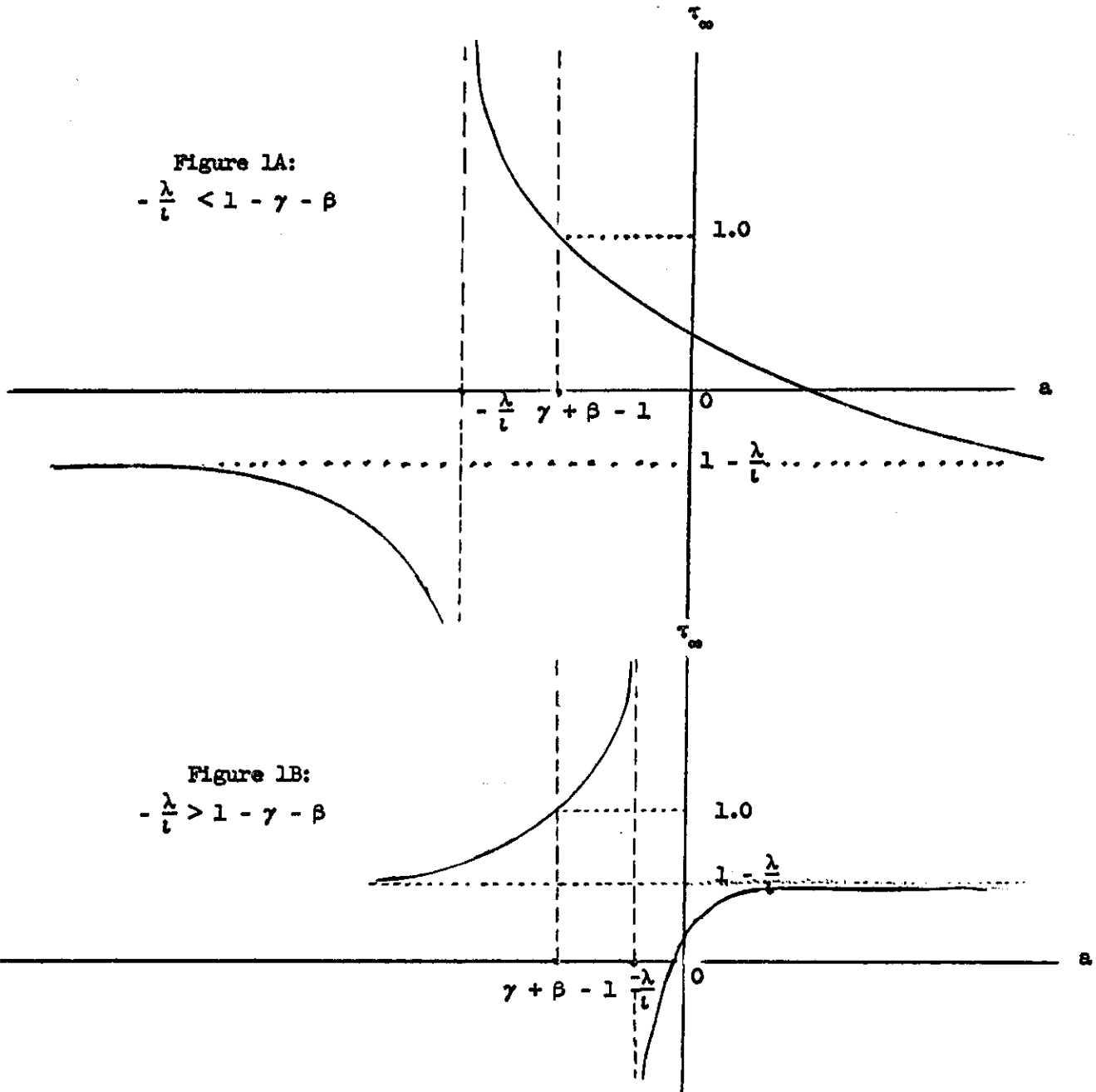
$$t - \lambda - t(\gamma + \beta) .$$

This expression is negative, and hence an increase of the required  $\underline{a}$  will reduce both the initial and eventual tax rate, if and only if

$$(21) \quad \frac{t}{\lambda}(1 - \gamma - \beta) < 1 \quad \text{or} \quad -\frac{\lambda}{t} < \gamma + \beta - 1 .$$

Both those readers who believed that an increase of thrift spells an eventual rise of tax rates (since it leads to a higher debt-income ratio) and those readers who believed that it spells a fall of tax rates (since it permits a reduced algebraic surplus and therefore, for a given debt ratio, lower taxes) will be surprised to find that neither result is necessary. Evidently the rise of the debt service ratio will be insufficient to maintain by itself to generate the larger deficit -- so that a permanently smaller tax rate will be required (though not as small as in the early years of the larger deficit)-- provided the interest rate is not too large.

And now the promised diagram: Figure 1A illustrates our results when  $-\frac{\lambda}{t} < \gamma + \beta - 1$  and Figure 1B illustrates the case in which  $-\frac{\lambda}{t} > \gamma + \beta - 1$ . The only feature of these diagrams which has not yet been explained can be deduced from (20): As  $a \rightarrow \pm \infty$ ,  $\tau_{\infty} \rightarrow 1 - \frac{\lambda}{t}$ , so that  $\tau_{\infty}$  is asymptotic to the line  $\tau_{\infty} = 1 - \frac{\lambda}{t}$ . 2/



The diagrams describe our results better than words can but let us attempt a brief summary: First, concerning feasibility, should a constant proportionate deficit be required to achieve the targeted full employment income path the ratio of debt service to taxable income will reach some upper limit, less than 100%, rather than increase without limit, as Domar showed. We have demonstrated that the same is true of the income tax rate. But should a surplus that is between  $1 - \gamma - \beta$  and  $\frac{\lambda}{t}$  (as a proportion to national income) be required the tax rate will eventually reach 100% at which point, presumably, incentives to work will break down and the targeted full employment growth pattern will cease to be feasible.

Second, concerning the effect of private thrift, should a proportionate deficit be required, an increase of private thrift, by increasing the proportionate deficit required, will raise the limiting ratio of debt service to income, as Domar showed. But despite that rise, the increased deficit will permit a permanently smaller tax rate if the foregoing condition on  $t$ ,  $\lambda$ ,  $\gamma$  and  $\beta$  expressed in (21) is satisfied. Should a surplus be required, the results are less certain: If the foregoing condition is

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2/ (From preceding page 10)

In discussing (11) we showed that  $\underline{a}$  need never be smaller algebraically than  $-k$ . But this lower limit on the required  $\underline{a}$  does not avert the possibility that the required  $\underline{a}$  may fall in the infeasible zone. We assume only that  $1 - \gamma - k > 0$ , hence  $K < 1 - \gamma$  or  $-k > \gamma - 1$ . Even if  $\beta = 0$  this means that the required  $\underline{a}$  could be smaller algebraically than  $-\frac{\lambda}{t}$ . And since  $\beta > 0$  is possible,  $\underline{a}$  can also be smaller algebraically than  $\gamma + \beta - 1$ . Therefore nothing precludes the required  $\underline{a}$  from falling in the intermediate infeasible zone of either Figure 1A or 1B.

satisfied and the initially required surplus was sufficiently small that the economy is on the upper curve of Figure 1A, then a permanent reduction of tax rates will be possible if both the old surplus (hence also the new) is feasible; if the old surplus is not feasible, the increase of thrift may make the (new and smaller) required surplus feasible, and hence prevent the tax rate from reaching 100%; but if the new surplus is also infeasible, tax rates can be lower only temporarily as the inevitable approach to a 100% tax rate is merely postponed a little. On the other hand, if the old required surplus was so large as to place the economy on the lower curve of Figure 1A, then the increase of thrift, by reducing the surplus to a level in the infeasible zone, will eventually cause the tax rate to be positive rather than negative and even to reach 100%; but if the reduction of the required surplus is large enough to avoid this zone, the tax rate will escape the latter calamity and merely rise from its initial (possibly negative value eventually to some higher (possibly positive) limiting value. (A shift from a negative to a positive tax rate is due to the government's move from a creditor position so strong that it must levy negative taxes to a creditor position sufficiently weak that  $\tau_{\infty}$ , the limiting net tax rate, is positive and possibly very large due to the requirement of a surplus over outlays.)

There are very many cases here so that the following empirical observations may help the reader in deciding which case is the most interesting. In the United States, very roughly:

- $t = .04$  (nominal interest rate on long-term government bonds)
- $\lambda = .05$  (nominal growth rate, i.e., growth rate of money GNP)
- $\gamma = .20$  (government expenditures as a ratio to GNP)
- $\beta = .05$  (government noninterest transfers as a ratio to GNP)

so that  $\frac{t}{\lambda}(1 - \gamma - \beta) = .60 < 1$ .

The U. S. is therefore "in" Figure 1A. Indeed, even the most "capital starved" (high  $t$ ) and "least publicly needy" (low  $\gamma + \beta$ ) of real-life capitalistic economies surely satisfy this condition. Further since even the most aggressive U. S. growth targets (high  $\lambda$ ) are unlikely to require a surplus in excess of .75 as a proportion of national income, it seems certain that we are on the upper curve of Figure 1A and safely in the feasible zone!

### Conclusions

We have found that in the Domar model an increase of the propensity to save, by increasing the proportionate deficit required to maintain the economy on its target full employment path, will permit a permanently smaller tax rate, despite the concomitant rise of the debt service-income ratio, if the rate of interest on government debt is smaller than the rate of growth of money national income. The necessary and sufficient condition is weaker if the government makes non-interest outlays.

If originally a very large surplus was required to meet the growth target, the increase of thrift, by reducing the required surplus, may so

reduce the government's creditor position (hence government interest receipts) that the tax rate will eventually have to be raised above what it would otherwise have been. Indeed, if the new required surplus lies in a critical intermediate zone, the tax rate may reach 100% at which point (if not before) the full-employment "growth target" -- by which we include the mix between public and private expenditures -- will have to be scrapped. On the other hand, the reduction of the required surplus dictated by the increase of thrift may remove the economy from this zone and thereby make possible the targeted growth through lower tax rate less than 100%.

Moral: Private thrift is surely virtuous if it is necessary to the feasibility of the targeted growth path. But if it is not, thrift may yet have a virtue: Where the above condition on the rate of interest is satisfied, thrift may be virtuous not because it releases the resources for targeted capital formation -- for in this case taxation can do that -- but because it permits the growth target to be achieved by lighter taxation than would otherwise be required.



ALTERNATIVE FISCAL POLICIES FOR TARGETED GROWTH

T. N. Srinivasan \* \*\*

Phelps considers an economy whose targeted full employment output  $\bar{Y}(t)$  grows exponentially over time. Constant proportion  $k$  of  $\bar{Y}(t)$  needs to be invested at each  $t$  if the growth path  $Y(t)$  is to be sustained. Postulating that (a) government expenditures and transfer payments form a constant proportion of income and (b) consumers save a constant proportion of their disposable income, Phelps derives the required budget deficit to close the gap between ex ante saving and required investment along the path  $\bar{Y}(t)$ .

We shall drop the assumption that government expenditures form a constant proportion of income and derive the set of alternative combinations of budget deficit and government expenditure that will maintain the equality along the path  $\bar{Y}(t)$  of savings and required investment. A subset of this set which will include Phelps' case will be examined.

Using Phelps' notation we have:

$$(1) \quad C_t = \pi \left( \bar{Y}_t + \mu D + B_t - T_t \right)$$

$$(2) \quad \dot{D}_t = G_t + B_t + \mu D_t - T_t$$

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\* For a somewhat similar analysis on the basis of a different model see John Cornwell, "Three Paths to Full Employment Growth," Quarterly Journal of Economics, February 1963.

\*\* This paper reports on research carried out under a grant from the National Science Foundation."

Now private saving equals  $(1 - \pi) (\bar{Y}_t + \iota D_t + B_t - T_t)$  and public saving equals  $-\dot{D}_t$ . Total saving  $S_t$  will therefore be given by

$$(3) \quad S_t = (1 - \pi) (\bar{Y}_t + \iota D_t + B_t - T_t) - \dot{D}_t$$

Substituting for  $T_t$  from (2) in (3) we have,

$$(4) \quad S_t = (1 - \pi) (\bar{Y}_t - G_t) - \pi \dot{D}_t$$

Now the required investment along the path  $\bar{Y}_t$  is  $k \bar{Y}_t$ . Equating  $S_t$  with required investment we have

$$(1 - \pi) (\bar{Y}_t - G_t) - \pi \dot{D}_t = k \bar{Y}_t \quad \text{or}$$

$$(5) \quad \pi \dot{D}_t + (1 - \pi) G_t = (1 - \pi - k) \bar{Y}_t$$

Equation (5) will yield the set of "feasible" combinations of  $\dot{D}_t$  and  $G_t$  which will sustain the full employment path of income, provided the following nonnegativity conditions are met:<sup>1/</sup>

$$(6) \quad C_t \geq 0$$

and

$$(7) \quad G_t \geq 0$$

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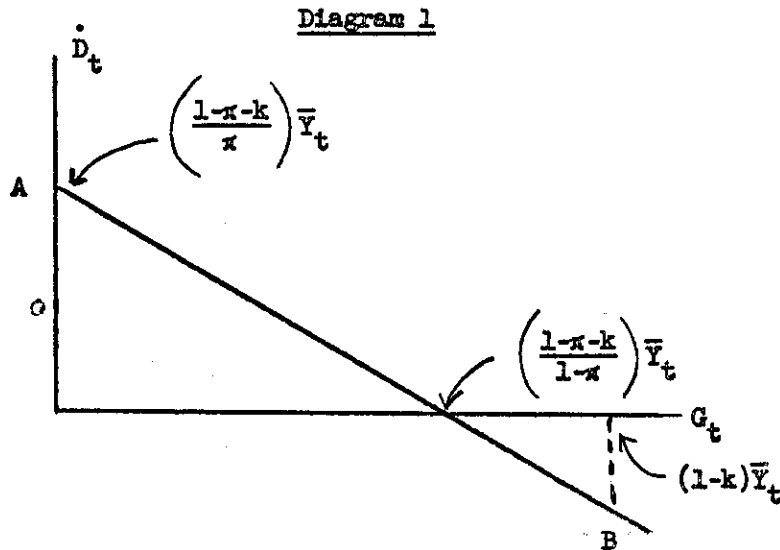
<sup>1/</sup> For feasibility one might also require that the ratio of taxes to taxable income is bounded above. We have not imposed this requirement in what follows.

Since  $C_t$  along the full employment path is given by  $(1 - k)\bar{Y}_t - G_t$

we can combine (6) and (7) into

$$(8) \quad 0 \leq G_t \leq (1-k)\bar{Y}_t$$

Thus (5) together with (8) yields the set of feasible combinations of  $\dot{D}_t$  and  $G_t$ . The following diagram illustrates this set:



Any combination on the segment  $AB$  is a feasible combination.

We first note that along  $AB$ , a unit increase in government expenditure is equivalent to a decrease of  $\left(\frac{1-k}{x}\right)$  units in deficit. This is as it should be for a unit increase in government expenditure can be accommodated along the path  $\bar{Y}_t$  only by a unit decrease in consumption. If the deficit remained unchanged, taxes will go up by one unit yielding only  $x$  units of

reduction in consumption. Hence taxes will have to go up by  $\frac{1}{\pi}$  units thus reducing the deficit by  $\left(\frac{1}{\pi} - 1\right) = \frac{1-\pi}{\pi}$  units.

Let us consider some special cases:

Case I: Balanced budget situation i.e.  $\dot{D}_t = 0$  for all  $t$ .

It can be seen from equation (5) that the required government expenditure

$$(9) \quad G_t = \left(\frac{1-\pi-k}{1-\pi}\right) \bar{Y}_t$$

Using (2) and the assumption that  $B_t = \beta \bar{Y}_t$  we get

$$(10) \quad T_t = \left[ \frac{(1-\pi)(1+\beta) - k}{1-\pi} \right] \bar{Y}_t + tD_0$$

From (9) and (10) it is obvious that both  $G_t$  and  $T_t$  are decreasing functions of  $\pi$ . Hence, an increase in private thrift (i.e., a decrease in  $\pi$ ) will necessitate higher taxes and government expenditures if budget balance is to be maintained along the full employment path. It is also seen that as time goes to infinity, the limiting debt service ratio is zero and the limiting tax rate  $\tau_\infty = \lim_{\pi \rightarrow \infty} \frac{T_t}{Y_t + tD_t} = \frac{(1-\pi)(1+\beta) - k}{1-\pi}$ .

Case II: Taxes form a constant proportion  $\tau$  of taxable income, i.e.,

$$(11) \quad T_t = \tau \left[ \bar{Y}_t + \iota D_t \right]$$

Substituting (11) in (2) we get

$$(12) \quad \begin{aligned} G_t &= -\beta \bar{Y}_t - \iota D_t + \tau(\bar{Y}_t + \iota D_t) + \dot{D}_t \\ &= -(\beta - \tau)\bar{Y}_t - \iota(1 - \tau)D_t + \dot{D}_t \end{aligned}$$

Let us assume for a moment that  $\tau$  is such that the combination  $(G_t, \dot{D}_t)$  obtained by solving (12) and (5) simultaneously is feasible in the sense that it does not violate (8). Now substituting for  $G_t$  from (12) in (5) we get:

$$(13) \quad \dot{D}_t - \iota(1 - \pi)(1 - \tau)D_t = \left\{ (1 - \pi)(1 + \beta - \tau) - k \right\} \bar{Y}_t$$

Since  $\bar{Y}_t = \bar{Y}_0 e^{\lambda t}$  we can solve (13) for  $D_t$  and get

$$(14) \quad D_t = \hat{d} \bar{Y}_t + (d_0 - \hat{d})\bar{Y}_0 e^{\theta t}$$

where

$$(15) \quad \hat{d} = \frac{(1 - \pi)(1 + \beta - \tau) - k}{\lambda},$$

$$(16) \quad d_0 = \frac{D_0}{\bar{Y}_0}$$

and

$$(17) \quad \theta = \tau (1-\pi)(1-\tau) .$$

Hence the combination  $(G_t, \dot{D}_t)$  which will simultaneously satisfy (12) and (5) is given by:

$$(18) \quad \dot{D}_t = \lambda \hat{d} \bar{Y}_t + \theta (d_0 - \hat{d}) \bar{Y}_0 e^{\theta t}$$

$$(19) \quad G_t = \frac{1}{1-\pi} \left[ (1-\pi-k-\pi\lambda\hat{d}) \bar{Y}_t - \pi\theta (d_0 - \hat{d}) \bar{Y}_0 e^{\theta t} \right]$$

We can now examine the conditions under which  $G_t$  given by (19) will

satisfy the feasibility conditions (8). First we note that since

$\bar{Y}_t = \bar{Y}_0 e^{\lambda t}$ , if  $\theta > \lambda$  then  $G_t$  given by (19) will violate either the lower or the upper limit given by (8) for large enough  $t$  provided  $d_0 \neq \hat{d}$ .

If the initial conditions do result in  $d_0$  being equal to  $\hat{d}$  then the

feasibility conditions take the form  $(1-\pi)(1-k) \geq 1-\pi-k-\pi\lambda\hat{d} \geq 0$ .

after some simplification we can write this as:

$$(20) \quad (1+\beta) \geq \tau \geq (1+\beta) - \frac{(1-k)}{\pi}$$

The debt/income ratio is a constant  $\hat{d}$  for all  $t$ . The deficit/income ratio is a constant  $\lambda\hat{d}$ . The government expenditure/income ratio is a constant

$\frac{1-\pi-k-\pi\lambda\hat{d}}{1-\pi} = 1-k-\pi(1+\beta-\tau)$ . It is also obvious that all the three ratios

increase as private thrift increases (i.e.,  $\pi$  decreases). Thus

an increase in private thrift would necessitate a larger deficit, debt and government expenditures relative to income if full employment is to be maintained indefinitely. <sup>1/</sup>

Let us now drop the assumption that  $d_0 = \hat{d}$ . Then a necessary condition <sup>2/</sup> for feasibility of  $G_t$  given by (19) is  $\theta < \lambda$  or  $\tau > 1 - \frac{\lambda}{\epsilon(1-\pi)}$ .

This is only a necessary condition. The set of necessary and sufficient conditions can be obtained as follows: Let us assume that  $\theta(d_0 - \hat{d}) < 0$ .

Then  $G_t$  will lie in  $[0, (1-k)\bar{Y}_t]$  for all  $t$  if and only if:

$$(20) \quad (1-\pi-k-\pi\lambda\hat{d}) - \pi\theta(d_0 - \hat{d}) \leq (1-\pi)(1-k)$$

and

$$(21) \quad 1-\pi-k-\pi\lambda\hat{d} \geq 0$$

If  $\theta(d_0 - \hat{d}) > 0$ , then the conditions become

$$(22) \quad (1-\pi-k-\pi\lambda\hat{d}) - \pi\theta(d_0 - \hat{d}) \geq 0$$

and

$$(23) \quad 1-\pi-k-\pi\lambda\hat{d} \leq (1-\pi)(1-k)$$

<sup>1/</sup> It is worth remembering that we are maintaining the assumption  $d_0 = \hat{d}$ . Since  $\hat{d}$  depends on  $\pi$ , the above discussion is valid only when  $d_0$  is so changed as to maintain equality with  $\hat{d}$ . The same qualification applies to the other cases also.

<sup>2/</sup> The case  $\theta = \lambda$  will require special treatment since the differential equation (13) will then have a solution different from (14). This is left for the reader to work out.

In either case the three ratios (debt/income, deficit/income and government expenditure/income) converge to  $\hat{d}$ ,  $\lambda\hat{d}$  and  $1-k-\pi(1+\beta-\tau)$  respectively. The behavior of these asymptotic ratios with respect to changes in  $\pi$  has been discussed already.

Case III: Government expenditures are a constant proportion  $\gamma$  of income. This has been discussed by Phelps.

Case IV: Advocates of constant government expenditures over time may work out the case  $G_t = G_0$  for all  $t$ . We shall state the asymptotic results:

$$\begin{aligned} \lim_{t \rightarrow \infty} \left( \frac{\dot{D}_t}{\bar{Y}_t} \right) &= \left( \frac{1-\pi-k}{\pi} \right) \\ \lim_{t \rightarrow \infty} \left( \frac{D_t}{\bar{Y}_t} \right) &= \left( \frac{1-\pi-k}{\lambda\pi} \right) \\ \lim_{t \rightarrow \infty} \left( \frac{T_t}{\bar{Y}_t + \iota D_t} \right) &= \frac{\beta + \left( \frac{1+\pi-k}{\lambda\pi} \right) (\iota - \lambda)}{1 + \left( \frac{1-\pi-k}{\lambda\pi} \right) \iota} \end{aligned}$$