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1. Introduction

In the recent literature on mathematical models of economic growth, attention has been devoted mainly on the existence and stability of competitive equilibria [1], [3], [9], [10]. An important feature of these models is a rather simple savings function such as (a) value of savings in terms of a numéraire at each point in time is a constant proportion of the value total output in terms of the same numéraire [3], (b) value of savings in terms of a numéraire is a linear function of wage income and consumer's net worth [1], (c) value of savings is equal to non-wage income [9], and (d) value of savings is a proportion of income, the constant of proportionality being a function of per capita income income and the instantaneous interest rate. [10].

The approach adopted in this paper is that the savings decision is a derived decision, i.e., it is an implication of the more basic behaviour of utility maximization over time. It is of interest, then, to investigate into the nature of that particular utility index (if any), the maximization of which will imply each of the savings functions (a) - (d) . However, this is a formidable task and we shall attempt to solve instead a somewhat different but related problem. We shall derive the savings implications of maximizing two specific utility indexes and compare them with (a) - (d) .

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In Section 2, we show that the savings function (c) is implied by the maximization of a rather simple intertemporal utility index viz., the indefinitely sustainable rate of consumption per worker. It may be pointed out that this result has been obtained for one sector models by Phelps [4] and for two sector models by Kurz [3] under more restrictive assumptions than those adopted in this paper.

Section 3 considers the Ramsey type utility index - viz., the utility associated with a growth path is the sum of the discounted future stream of consumption per worker, discount rate being positive.1/ Assuming that the consumer goods sector is more capital intensive than the capital goods sector, it is shown that, given the initial stock of capital, there exists a growth path which maximizes the utility index. The optimal paths corresponding to <u>different</u> initial stocks of capital but to the <u>same</u> discount rate converge to the <u>same</u> asymptotic balanced growth path. As the discount rate approaches zero from above, this asymptotic balanced growth path approaches the balanced growth path optimal with respect to the utility index of Section 2. The value of savings evaluated in terms of the efficiency prices associated with the optimal path, is shown to be, in general, <u>not</u> a constant proportion of income.

1/ There have been objections raised against this utility index for planning purposes because this indicates a form of impatience. Future consumption is discounted mainly because it is future consumption! See Koopmans [2] for the logical connection between impatience and the existence of continuous ordinal utility function over time.

However, the share of savings in income approaches a limit as time goes to infinity. This asymptotic share of savings is shown to exceed the share of non-wage income in total income, for all positive discount rates.

2. The Model

There are two sectors in the economy. Sector 1 produces a homogeneous consumer good and Sector 2 produces a homogeneous capital good. There are two homogeneous factors of production, labour and capital. The production function of Sector i $(i = 1, 2)$ is denoted by $F^i[K_i, L_i]$ where K_i is the capital input and L_i is the rate of labour input. The entire output of the consumer good at any point in time is instantaneously consumed and the output of the capital good is added to the capital stock. The existing stock of capital can be divided between the sectors in any desired fashion.[1]

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[1] This assumption is not essential to the argument in Section 2.

The labour force is assumed to grow exponentially at a positive rate θ.

The following are the assumptions about the production functions.[1]

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[1] Our assumptions are the same as in Uzawa [10].

(2.1) F^i is homogeneous of degree 1, twice continuously differentiable and concave.

Let $\delta_i = \frac{K_i}{L_i}$. Then $F^i(K_i, L_i) = L_i F^i(\delta_i, 1)$. Define $F^i(\delta_i, 1) = f^i(\delta_i)$. Let f^i_1 be the derivative of f^i with respect to δ_i and f^i_{11} be the derivative of f^i_1 with respect to δ_i

$$(2.2) \quad f^i(\delta_i) \left.\begin{matrix} = 0 & \delta_i = 0 \\ > 0 & \delta_i > 0 \\ = \infty & \delta_i \to \infty \end{matrix}\right\}, \quad f^i_1(\delta_i) \left.\begin{matrix} \infty & \delta_i = 0 \\ > 0 & \delta_i > 0 \\ = 0 & \delta_i \to \infty \end{matrix}\right\}$$

$$f^i(\delta_i) - \delta_i f^i_1(\delta_i) \left.\begin{matrix} = 0 & \delta_i = 0 \\ > 0 & \delta_i > 0 \\ = \infty & \delta_i > \infty \end{matrix}\right\} \qquad i = 1, 2$$

It can be seen that assumption (2.2) implies nonnegative marginal products for capital and labour in the production of either good. Concavity of F^i implies that $f^i_{11}(\delta_i) < 0$.

Let $\bar{K}(0)$ be the initial stock of capital and $K(t)$ be the stock of capital at time t. Let $C(t)$ be the rate of output of consumer goods at time t . Without loss of generality we can assume the initial labour force to be equal to 1 . Then the labour force at time t is $e^{\theta t}$, treating time as a continuous variable. Let $c(t) \equiv e^{-\Theta t} C(t)$ be the consumption per worker at time t. Let $L_i(t)$, $K_i(t)$ be the labour and capital inputs denoted to the production of the i^{th} good at time t $(i = 1, 2)$. Let $I(t)$ be the rate of output of capital good. Let $\dot{K}(t)$ be the rate of change of capital stock at time t. Then the following relations hold true at all $t \geqq 0$.

$$\dot{K}(t) \leqq I(t) \equiv F^1[K_1(t), L_1(t)] \tag{1}$$

$$K(t) = \overline{K(0)} + \int_0^t \dot{K}(u)\, du \tag{2}$$

$$c(t) \equiv e^{-\Theta t} C(t) \equiv e^{-\Theta t} F^2[K_2(t), L_2(t)] \tag{3}$$

$$L_1(t) + L_2(t) \leqq e^{\Theta t} \tag{4}$$

$$K_1(t) + K_2(t) \leqq K(t) \tag{5}$$

$$L_1(t), L_2(t), K_1(t), K_2(t), \dot{K}(t), \overline{K(0)} \geqq 0 \tag{6}$$

Inequalities (1) state that the rate of change of capital stock at any t cannot exceed the rate of output of capital goods. Equation [2] is a definitional relationship. Equation [3] states that consumption per worker is identically equal to the ratio of output of consumer goods to the labour force. Inequality [4] states that employment in the two sectors together cannot exceed the available labour at any t. Inequality [5] is a similar restraint on capital inputs. Inequalities [6] are nonnegativity restraints imposed to make economic sense.

Definition $\underline{c}$ is said to be a sustainable rate of consumption per worker if we can find $L_i(t)$, $K_i(t)$, $\overline{K(0)}$, $\dot{K}$ satisfying (1) - (6) such that $c(t) \geqq \underline{c}$ for all $t \geqq 0$. It may be noted that we are assuming that even the initial stock of capital is subject to choice.

We wish to maximize the sustainable rate of consumption per worker. Introducing the variables $\delta(t) \equiv e^{-\Theta t} K(t)$ $\overline{\delta(0)} \equiv \overline{K(0)}$, $\ell_1(t) \equiv e^{-\Theta t} L_1(t)$, $\ell_2(t) \equiv e^{-\Theta t} L_2(t)$ we can state this problem as:

Maximize $\underline{c}$ subject to

$$\underline{c} \leqq c(t) \equiv \ell_2(t) \; f^2 [\delta_2(t)] \tag{7}$$

$$\delta(t) \equiv \overline{\delta(0)} + \int_0^t \dot{\delta}(t) dt \tag{8}$$

$$\dot{\delta} + \Theta \delta \leqq \ell_1(t) \; f^1[\delta_1(t)] \tag{9}$$

$$\ell_1(t) \; \delta_1(t) + \ell_2(t) \; \delta_2(t) \leqq \delta(t) \tag{10}$$

$$\ell_1(t) + \ell_2(t) \leqq 1 \tag{11}$$

$$\ell_1(t), \ \ell_2(t), \ \delta_1(t), \ \delta_2(t), \ \delta, \ \dot{\delta} + \Theta\delta, \ \overline{\delta(0)}, \ \underline{c} \geqq 0 \tag{12}$$

It is shown in Appendix I that the maximand is bounded above. Since the initial value $\overline{\delta(0)}$ is also subject to choice, it is clear that we can confine our search for the solution to the above problem among the set of stationary solutions. Hence we can restate the problem, omitting the time argument and setting $\dot{\delta} = 0$, as follows:

$$\text{Maximize} \quad \underline{c} \tag{A}$$

$$\text{subject to} \quad \underline{c} \leqq \ell_2 f^2(\delta_2) \tag{13}$$

$$\Theta \delta \leqq \ell_1 f^1(\delta_1) \tag{14}$$

$$\ell_1 \delta_1 + \ell_2 \delta_2 \leqq \delta \tag{15}$$

$$\ell_1 + \ell_2 \leqq 1 \tag{16}$$

$$\ell_1, \ell_2, \delta_1, \delta_2, \delta, \underline{c} \geqq 0 \tag{17}$$

It is shown in Appendix I that there exists a unique solution to this problem. This solution is

$$\ell_1^* = \frac{\delta_2^* f_1^2(\delta_2^*)}{f^2(\delta_2^*)} \quad , \quad \ell_2^* = 1 - \ell_1^* \tag{18}$$

$$\delta^* = \ell_1^* \delta_1^* + \ell_2^* \delta_2^* , \quad \underline{c}^* = \ell_2^* f^2 (\delta_2^*) \tag{19}$$

where δ_1^* , δ_2^* are the unique solutions of the following two equations:

$$f_1^1[\delta_1] = \Theta \tag{20}$$

$$\frac{f^1[\delta_1]}{f_1^1[\delta_1]} - \delta_1 = \frac{f^2[\delta_2]}{f_1^2[\delta_2]} - \delta_2 \tag{21}$$

A straightforward interpretation of this solution is as follows. $f_1^1(\delta_1)$ is the marginal physical product of capital in the capital goods sector, when capital labour ratio δ_1 is used. Equation (20) states that the optimal capital labour ratio in the capital goods sector is one which makes the marginal physical product equal the rate of growth of the labour force. But the marginal physical product of capital is the own rate of interest on capital and the rate of growth of the labour force is also the rate of growth of output of capital goods and consumer goods. Thus equation (21) is another way of stating that the own rate of interest on capital equals the rate of growth of output. Equation (22) ensures that the ratio of marginal physical product of labour to that of capital is the same in the two sectors.

An alternative interpretation of the above solution is possible. Suppose one wished to sustain indefinitely one unit of consumption per worker, using capital labour ratios δ_1 and δ_2 for all $t \geqq 0$. Given that the labour force is growing at the rate Θ, the direct labour input needed to produce one unit of consumer good per worker at time t is $\eta_2 = e^{\Theta t} \cdot \dfrac{1}{f^2(\delta_2)}$.

For the capital labour ratios δ_1 and δ_2 to be maintained indefinitely, the rate of output of capital goods required at time $t = \Theta[\eta_1\delta_1 + \eta_2\delta_2]$ where η_1 is the employment in the capital goods sector. On the other hand, given η_1 and δ_1 the actual rate of output of new capital goods is $\eta_1 f^1(\delta_1)$.

Equating the required output to actual output we have

$\eta_1\delta_1 + \eta_2\delta_2 = \frac{1}{\Theta}\ \eta_1 f^1(\delta_1)$. Hence we can solve for η_1

and get $\eta_1 = \dfrac{\Theta\ \delta_2\ e^{\Theta t}}{f^2(\delta_2)[f^1(\delta_1)-\Theta\delta_1]}$. We can call η_1 the indirect

labour input needed at time t to sustain one unit of consumption per worker. Adding the direct and indirect labour inputs we have

$(\eta_1 + \eta_2) = \dfrac{e^{\Theta t}}{f^2(\delta_2)}\left[\dfrac{\Theta\ \delta_2}{(f^1(\delta_1)-\Theta\delta_1)} + 1\right]$. Since labour is

the only primary factor in this economy, the maximum sustainable rate of consumption per worker is achieved along a path which minimizes $\eta_1 + \eta_2$ or equivalently minimizes $e^{-\Theta t}[\eta_1 + \eta_2]$. Minimization $e^{-\Theta t}[\eta_1 + \eta_2]$ leads to equations (20) and (21) again, for the solution of optimal δ_1 and δ_2 .

In deriving the solution (18) - (21) by maximizing $\underline{c}$ subject to (13) - (17), we obtain a set of Lagrangean multipliers λ_1^* , λ_2^* , λ_3^* , λ_4^* associated respectively with restraints (13) - (16). It is shown in Appendix I that the values of these multipliers are:

$$\left.\begin{aligned} \lambda_1^* &= 1 & \qquad \lambda_3^* &= f_1^2(\delta_2^*) \\ \lambda_2^* &= \frac{1}{\Theta}\ f_1^2(\delta_2^*) & \qquad \lambda_4^* &= f^2(\delta_2^*) - \delta_2^*\ f_1^2(\delta_2^*) \end{aligned}\right\} \tag{22}$$

These multipliers can be given a natural price interpretation. First we observe that our solution is a stationary one in the ratio variables $\ell_1, \ell_2, \delta_1, \delta_2$ and δ. The absolute variables, $L_1(t)$, $L_2(t)$, $K_1(t)$, $K_2(t)$ and $K(t)$ grow exponentially at the same rate Θ. This means that except for the scale factor $e^{\Theta t}$, there is no change in the state of the economy over time. This means that the relative price interpretation that we give to the multipliers holds therefore for all t.

λ_1^* can be interpreted as the price per unit of consumer good. The fact $\lambda_1^* = 1$ means merely that the consumer good has been chosen as the numéraire.[1] λ_2^* can be interpreted as the price per unit of capital goods

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[1] In fact we could have made λ_1^* take any desired positive value p by maximizing $p\, \underline{c}$ instead of $\underline{c}$, subject to (13) - (17). This maximization will lead to the same solution as (18) - (21), the only difference being that all the λ's will be multiplied by p.

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in terms of the numéraire. λ_3^* is the rate of rental on capital stock again in terms of the numéraire. λ_4^* is the wage rate in terms of the numéraire.

Using the prices λ_i^*, let us examine the relationship between the value of savings and non-wage income. In view of the fact that there is only a change of scale over time, we can confine our attention to the initial values. Since the initial labour force is unity, the initial capital stock is equal to δ^*. Hence using (22),

$$\text{Non-wage income} \equiv \text{Rental on Capital Stock} = \lambda_3^* \delta^* = \delta^* f_1^2(\delta_2^*) .$$

Using (18) - (21) we can easily see that $\delta^{*} = \frac{1}{\Theta} \ell_1^{*} f^{1}(\delta_1^{*})$. Hence the non-wage income is equal to $\frac{1}{\Theta} \ell_1^{*} f^{1}(\delta_1^{*}) \cdot f_1^{2}(\delta_2^{*})$. Let us now compute the value of savings. The value of savings is equal to the value of investment which in turn equals the value of the output of capital goods. The rate of output of capital goods is $\ell_1^{*} f^{1}(\delta_1^{*})$ and the price per unit of capital good is $\lambda_2^{*} = \frac{1}{\Theta} f_1^{2}(\delta_2^{*})$. Hence the value of the output of capital goods is $\ell_1^{*} f^{1}(\delta_1^{*}) \cdot \frac{1}{\Theta} f_1^{2}(\delta_2^{*})$. This is precisely equal to the rental on capital stock. Hence, <u>along the balanced growth path which maximizes the sustainable rate of consumption per worker, value of savings evaluated at the efficiency prices associated with the path equals the non-wage income</u>.

It may be recalled that we assumed that even the initial stock of capital is subject to choice. If this is not so, the following question arises. Given (a) the initial stock of capital, (b) that prices are competitively determined, and (c) that the value of savings is equal to the non-wage income, what is the relationship between the resulting equilibrium path and our balanced growth path that maximizes the sustainable rate of consumption per worker? The answer is provided by Uzawa [9]. He shows that the competitive equilibrium path converges to our balanced growth path, whatever be the initial stock of capital, provided the consumer goods sector is more capital intensive than the capital goods sector. This proviso arises because, otherwise, our balanced growth path <u>may</u> be unstable as a steady state solution to his problem. It plays a similar role in our discussion in Section 3.

3. Growth paths that maximize the discounted future stream of consumption.

In this section we shall consider the Ramsey type utility function -- utility of a growth path is the sum of the discounted future stream of consumption per worker. The initial stock of capital will be taken as historically given. We shall be using the following capital intensity assumption [1/] in addition to assumptions (2.1) and (2.2).

(3.1) The consumer goods sector is more capital intensive than the capital goods sector. This means that if $\delta_1(\omega)$ and $\delta_2(\omega)$ are the unique solutions of $\frac{f^1[\delta_1]}{f^1_1[\delta_1]} - \delta_1 = \omega = \frac{f^2[\delta_2]}{f^2_1[\delta_2]} - \delta_2$ for any $\omega > 0$, then $\delta_1(\omega) < \delta_2(\omega)$.

The meaning of (3.1) becomes clear if we note that $\frac{f^i[\delta]}{f^i_1[\delta]} - \delta$ represents the ratio of marginal physical product of labour to that of capital in sector i if it uses the capital labour ratio δ . Hence, if we interpret ω as the ratio of wages to rents, then $\delta_i(\omega)$ is the capital labour ratio in Sector i which minimizes the unit cost of production. (3.1) states that for any wage-rent ratio the optimal capital labour ratio in the consumer goods sector exceeds that of the capital goods sector.

1/ We can dispense with this assumption if we are prepared to assume instead that the production functions in the two sectors are of the Cobb-Douglas type. See Appendix III and also Solow [6] .

Our problem is the following

$$\text{Maximize} \quad \int_0^\infty e^{-\rho t}\, c(t)\, dt \qquad \rho > 0 \tag{B}$$

subject to $$c(t) = \ell_2(t)\, f^2(\delta_2(t)) \tag{24}$$

$$0 \leqq \dot{\delta} + \Theta\delta \leqq \ell_1(t)\, f^1(\delta_1(t)) \tag{25}$$

$$\ell_1(t)\delta_1(t) + \ell_2(t)\delta_2(t) \leqq \delta(t) \equiv \overline{\delta(0)} + \int_0^t \dot{\delta}(u)du \tag{26}$$

$$\ell_1(t) + \ell_2(t) \leqq 1 \tag{27}$$

$$\delta,\ \ell_1,\ \ell_2,\ \delta_1,\ \delta_2 \geqq 0 \qquad \overline{\delta(0)} \text{ given} \tag{28}$$

It is shown in Appendix II that the solution to this problem falls under three cases depending on $\overline{\delta(0)}$. In order to distinguish these cases let us define the following two constants $\hat{\delta}_1$ and $\hat{\delta}_2$ by

$$f_1^1(\hat{\delta}_1) = [\rho + \Theta] = \epsilon \tag{29}$$

$$\frac{f^2(\hat{\delta}_2)}{f_1^2(\hat{\delta}_2)} - \hat{\delta}_2 = \frac{f^1(\hat{\delta}_1)}{f_1^1(\hat{\delta}_1)} - \hat{\delta}_1 \tag{30}$$

Given (2.1) and (2.2), there exist unique $\hat{\delta}_1, \hat{\delta}_2$ which satisfy (29) and (30). Given (3.1) it follows that $\hat{\delta}_2 > \hat{\delta}_1$. The meaning of (29) is that the capital labour ratio $\hat{\delta}_1$ makes the rate of interest on capital in the capital goods sector equal the sum of the discount rate and the rate of growth of population. Equation (30) ensures that given $\hat{\delta}_1$, the capital labour ratio in Sector 2 is chosen in such a way as to equate the ratios of the marginal physical product of labour to that of capital in the two sectors.

<u>Case I</u>. $\hat{\delta}_2 \geqq \overline{\delta(0)} \geqq \hat{\delta}_1$. The solution is

$$\left.\begin{aligned} \delta_1(t) &= \hat{\delta}_1 \qquad \delta_2(t) = \hat{\delta}_2 \\ \delta(t) &= \hat{\delta}(\infty) + [\overline{\delta(0)} - \hat{\delta}(\infty)]\, e^{-xt} \\ \ell_1(t) &= \frac{\hat{\delta}_2 - \delta(t)}{\hat{\delta}_2 - \hat{\delta}_1}, \quad \ell_2(t) = 1 - \ell_1(t) \end{aligned}\right\} t \geqq 0 \qquad (31)$$

where $x = \Theta + \dfrac{f^1(\hat{\delta}_1)}{\hat{\delta}_2 - \hat{\delta}_1}$ $\qquad \hat{\delta}(\infty) = \dfrac{1}{x}\left\{\dfrac{\hat{\delta}_2 f^1(\hat{\delta}_1)}{\hat{\delta}_2 - \hat{\delta}_1}\right\}$

Clearly $x > 0$, and $\hat{\delta}_2 > \hat{\delta}(\infty) > \hat{\delta}_1$.

<u>Case II</u>. $\overline{\delta(0)} > \hat{\delta}_2$. The solution is

$$\left.\begin{aligned} \ell_1(t) &= 0 \qquad \ell_2(t) = 1 \\ \delta(t) &= \overline{\delta(0)}\; e^{-\Theta t} = \delta_2(t) \end{aligned}\right\} 0 \leqq t \leqq \bar{t} \qquad (32)$$

$$\left.\begin{aligned} \delta_1(t) &= \hat{\delta}_1 \qquad \delta_2(t) = \hat{\delta}_2 \\ \delta(t) &= \hat{\delta}(\infty) + [\hat{\delta}_2 - \hat{\delta}(\infty)]\, e^{-x(t-\bar{t})} \\ \ell_1(t) &= \frac{\hat{\delta}_2 - \delta(t)}{\hat{\delta}_2 - \hat{\delta}_1}, \quad \ell_2(t) = 1 - \ell_1(t) \end{aligned}\right\} t > \bar{t} \qquad (33)$$

where $\overline{\delta(0)}\; e^{-\Theta\bar{t}} = \hat{\delta}_2$ (34)

Case III. $\hat{\delta}_1 > \overline{\delta(0)}$

Let $\tilde{\delta}(t)$ be the solution of the differential equation $\dot{\delta} + \Theta\,\delta = f^1(\delta)$ with the initial condition $\tilde{\delta}(0) = \overline{\delta(0)}$. Then the optimal solution is

$$\left.\begin{array}{ll} \ell_1(t) = 1 & \ell_2(t) = 0 \\ \delta(t) = \tilde{\delta}(t) = \delta_1(t) & \end{array}\right\} \; 0 \leq t \leq \underline{t} \tag{35}$$

$$\left.\begin{array}{ll} \delta_1(t) = \hat{\delta}_1 & \delta_2(t) = \hat{\delta}_2 \\ \delta(t) = \hat{\delta}(\infty) + [\hat{\delta}_1 - \hat{\delta}(\infty)]\, e^{-x(t-\underline{t})} & \\ \ell_1(t) = \dfrac{\hat{\delta}_2 - \delta(t)}{\hat{\delta}_2 - \hat{\delta}_1} & \ell_2(t) = 1 - \ell_1(t) \end{array}\right\} \; t > \underline{t} \tag{36}$$

where $\tilde{\delta}(\underline{t}) = \hat{\delta}_1$ (37)

Before discussing the properties specific to each of the above three cases, it is worthwhile to draw attention to the property common to all three. That is, as $t \to \infty$ the solution in all three cases approaches the <u>same</u> path. This path is given by the following:

$$\delta_1(t) = \hat{\delta}_1, \quad \delta_2(t) = \hat{\delta}_2 \qquad \delta(t) = \hat{\delta}(\infty) \tag{38}$$

$$\ell_1(t) = \frac{\hat{\delta}_2 - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1} \qquad \ell_2(t) = 1 - \hat{\ell}_1(t) \tag{39}$$

$$c(t) = \hat{c} = \ell_2(t)\, f^2(\hat{\delta}_2) = \left\{ \frac{\hat{\delta}(\infty) - \hat{\delta}_1}{\hat{\delta}_2 - \hat{\delta}_1} \right\} f^2(\hat{\delta}_2) \qquad (40)$$

This asymptotic path is a balanced growth path, since all the ratio variables δ_1, δ_2, δ, ℓ_1 and ℓ_2 remain constant over time. An economy moving along this path looks exactly the same over time except for a scale factor $e^{\beta t}$.

The behaviour of this asymptotic path as we let the discount rate ρ vary, is interesting. First, the asymptotic per capita consumption $\hat{c}$ given by (40) is a decreasing function of ρ. As $\rho \to 0$, $\hat{c}$ attains its maximum value. The limiting values $\hat{\delta}_1$, $\hat{\delta}_2$, etc., as $\rho \to 0$ are of interest in themselves. If we recall the definition of $\hat{\delta}_1$ and $\hat{\delta}_2$ as given by (29) and (30), we notice that as we let $\rho \to 0$, $\hat{\delta}_1 \to \delta_1^*$ and $\hat{\delta}_2 \to \delta_2^*$ where δ_1^* and δ_2^* satisfy

$$\left.\begin{aligned} f_1^1(\delta_1) &= \Theta \\ \frac{f^2(\delta_2)}{f_1^2(\delta_2)} - \delta_2 &= \frac{f^1(\delta_1)}{f_1^1(\delta_1)} - \delta_1 \end{aligned}\right\} \qquad (41)$$

Comparing equations (41) with (20) and (21) of Section 2 we find they are the same. This means that <u>the asymptotic path given by (38) - (40) approaches the path optimal with respect to the utility index of Section 2.</u>

Coming to the specific aspects of the three cases, we observe that in Case I, both capital goods and consumer goods are produced at all t. If $\overline{\delta(0)} > \hat{\delta}_2$ as in Case II, initially there is "too much" capital relative to labour and the economy optimally adjusts to this capital surplus by producing only consumer goods up to a certain point $\bar{t}$ in time. On the other hand if $\hat{\delta}_1 > \overline{\delta(0)}$ as in Case III, there is initially too much labour relative to capital and the economy optimally adjusts to this labour surplus by producing only capital up to a certain point $\underline{t}$ of time. Except for the differences discussed in this paragraph, the solutions for the three cases are essentially the same.

Income and Savings along the optimal path: Income and Savings are value concepts and we need a set of prices to convert physical magnitudes such as output of consumer goods and of capital goods into values. For this purpose we shall use the efficiency prices associated with our optimal solutions. These prices have the following interpretation: Given these prices and the behaviour assumption that producers choose inputs and outputs so as to maximize profits, the resulting growth paths will be the same as our optimal solutions. In other words, we can realize our optimal solutions through profit maximization, given these prices.

Let us use the following notation for prices (a price here means the value at time zero of a unit of a good or service becoming available at time t):

$p(t)$; price per unit of consumer good at t

$q(t)$: price per unit of capital good at t

$r(t)$: rate of rental per unit of capital stock at t

$w(t)$: wage rate at t .

In computing Savings and Income, we shall confine our attention only to Case I, since the other two cases reduce to this case after a finite interval of time. It is shown in Appendix II that the efficiency prices associated with the optimal solution for this case are

$$p(t) = e^{-\epsilon t} \qquad q(t) = \frac{1}{\epsilon} f_1^2(\hat{\delta}_2)\, e^{-\epsilon t} \tag{41}$$

$$r(t) = f_1^2(\hat{\delta}_2) e^{-\epsilon t} \qquad w(t) = [f^2(\hat{\delta}_2) - \hat{\delta}_2 f_1^2(\hat{\delta}_2)]\, e^{-\epsilon t} \tag{42}$$

where $\epsilon = \rho + \Theta$. Let $Y(t)$ be the income and $S(t)$ be the savings at t . It is clear that

$$Y(t) = w(t)e^{\Theta t} + r(t)\, K(t) = e^{\Theta t} \left\{ w(t) + r(t)\, \delta(t) \right\} \tag{43}$$

$$S(t) = q(t) \left\{ e^{\Theta t}\, \ell_1(t)\, f^1(\hat{\delta}_1) \right\} \tag{44}$$

Equation (43) states that total income equals wage income plus rental on capital stock. Equation (44) states that savings equal the value of the output of capital goods sector. Let us denote by $s(t)$, the ratio of savings to income. Using (43) and (44) we can state that the

$$\begin{aligned} \text{savings ratio} \equiv s(t) &= \frac{S(t)}{Y(t)} \\ &= \frac{q(t)\ell_1(t)f^1(\hat{\delta}_1)}{w(t) + r(t)\,\delta(t)} \\ &= \frac{\ell_1(t)f^1(\hat{\delta})}{\left[\left\{ \dfrac{f^2(\hat{\delta}_2)}{f_1^2(\hat{\delta}_2)} - \hat{\delta}_2 \right\} + \delta(t) \right] \epsilon} \end{aligned} \tag{45}$$

We know from (31) that $\ell_1(t)$ and $\delta(t)$ are not, in general, constant over time. Hence $s(t)$ is not, in general, constant over time. However, since $\ell_1(t)$ and $\delta(t)$ converge as $t \to \infty$, $s(t)$ also converges as $t \to \infty$. Let us define $s(\infty) = \lim_{t\to\infty} s(t)$. Then

$$s(\infty) = \frac{\left\{\lim\limits_{t\to\infty} \ell_1(t)\right\} f^1(\hat{\delta}_1)}{\left[\left\{\dfrac{f^2(\hat{\delta}_1)}{f_1^2(\hat{\delta}_1)} - \hat{\delta}_2\right\} + \hat{\delta}(\infty)\right] \epsilon}$$

$$= \frac{[\hat{\delta}_2 - \hat{\delta}(\infty)]\, f^1(\hat{\delta}_1)}{[\hat{\delta}_2 - \hat{\delta}_1]\left[\left\{\dfrac{f^2(\hat{\delta}_2)}{f_1^2(\hat{\delta}_2)} - \hat{\delta}_2\right\} + \hat{\delta}(\infty)\right] \epsilon}$$

Using (30) we can rewrite $s(\infty)$ as follows

$$s(\infty) = \left\{\frac{\hat{\delta}_2 - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1}\right\} \left\{\frac{f^1(\hat{\delta}_1)}{f^1(\hat{\delta}_1) + \epsilon\,(\hat{\delta}(\infty) - \hat{\delta}_1)}\right\} \tag{46}$$

It can be shown that $s(\infty)$ decreases as the discount rate ρ increases. This is as it should be since larger ρ implies that future consumption is worth relatively less compared to the present. Let us compare $s(\infty)$ with the asymptotic share $\Pi(\infty)$ of non-wage income to total income. Let $\Pi(t)$ be the share of capital in income at t. Then

$$\Pi(t) = \frac{r(t)\,\delta(t)}{w(t) + r(t)\,\delta(t)}$$

$$= \frac{\delta(t)}{\left[\frac{f^1(\hat{\delta}_1)}{\epsilon} - \hat{\delta}_1 + \delta(t)\right]}$$

$$\Pi(\infty) = \lim_{t\to\infty} \Pi(t) = \frac{\epsilon\,\hat{\delta}(\infty)}{\left[f^1(\hat{\delta}_1) + \epsilon(\hat{\delta}(\infty) - \hat{\delta}_1)\right]} \qquad (47)$$

Comparing (46) with (41) we find that $\Pi(\infty) - s(\infty)$ has the same sign as $\epsilon\,\hat{\delta}(\infty) - \left(\frac{\hat{\delta}_2 - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1}\right) f^1(\hat{\delta}_1)$. Using the definition of $\hat{\delta}(\infty)$ we can simplify this expression to the following

$$\epsilon\,\hat{\delta}(\infty) - \left(\frac{\hat{\delta}_2 - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1}\right) f^1(\hat{\delta}_1) = \frac{\hat{\delta}_2 f^1(\hat{\delta}_1)[\epsilon - \Theta]}{\Theta(\hat{\delta}_2 - \hat{\delta}_1) + f^1(\hat{\delta}_1)}$$

. But $\epsilon = \rho + \Theta$ and $\rho > 0$. Hence <u>for all positive discount rates the asymptotic share of non-wage income exceeds the asymptotic share of savings</u>.

4. Summary and Conclusions:

We examined the implications of the maximization of two specific indexes of utility over time in the context of a two sector model of growth. In Section 2, it was shown that the maximization of the sustainable rate of consumption per worker implied that the value of savings at each point in time (evaluated at the efficiency prices corresponding to the optimal path) was equal to the rents imputed to the stock of capital. It was shown that along the optimal path under this objective the direct and indirect labour needed to sustain indefinitely a unit consumption per worker was minimum.

In Section 3 the index of utility was the discounted future stream of consumption per worker, the discount rate being positive. With the additional assumption that the consumer goods sector was more capital intensive than the capital goods sector, it was shown that:

(a) Whatever be the given initial stock of capital, the optimal path for any positive discount rate, converged to the same balanced growth path.

(b) The asymptotic consumption per worker associated with the balanced growth path was a decreasing function of the discount rate, approaching its maximum as the discount rate approached zero.

(c) As the discount rate approached zero from above, the asymptotic balanced growth path approached that balanced growth path which maximized the sustainable rate of consumption per worker.

(d) Savings, evaluated at the efficiency prices associated with the optimal path corresponding to any given positive discount rate, was <u>not</u>, in general, a constant proportion of income. However, the share of savings in income converged to a constant as time approached infinity.

(e) The asymptotic savings rate was found to be a decreasing function of the discount rate.

(f) For all positive discount rates, the asymptotic share of non-wage income in total income exceeded the share of savings.

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APPENDIX I

The problem is to

maximize $\underline{c}$ subject to

$$\underline{c} \leqq \ell_2(t)\, f^2[\delta_2(t)] \tag{I.1}$$

$$0 \leqq \dot{\delta} + \theta\delta \leqq \ell_1(t)\, f^1[\delta_1(t)] \tag{I.2}$$

$$\ell_1(t)\,\delta_1(t) + \ell_2(t)\,\delta_2(t) \leqq \delta(t) = \overline{\delta(0)} + \int_0^t \dot{\delta}(u)du \tag{I.3}$$

$$\ell_1(t) + \ell_2(t) \leqq 1 \tag{I.4}$$

$$\overline{\delta(0)},\ \delta(t),\ \ell_1(t),\ \ell_2(t),\ \delta_1(t),\ \delta_2(t) \geqq 0 \quad \text{for all } t \geqq 0\ . \tag{I.5}$$

It is clear that the above problem is equivalent to the problem of maximizing $\text{Inf} \left\{\ell_2(t\ \ f^2[\delta_2(t)]\right\}$ subject to (I.2) - (I.5). Our first task will be to show that this maximand is bounded above. For showing this, it is enough if we show that, for large values of t , $\delta_2(t)$ is bounded above, <u>whatever</u> be the initial value $\overline{\delta(0)}$. This in turn is established if we prove that $\delta(t)$ is bounded above for large values of t , whatever the initial $\overline{\delta(0)}$.

We notice that, given $\overline{\delta(0)}$, $\delta(T)$ will be maximized for any T , if for all $t < T$, we set $\ell_2(t) = 0$ $\delta_2(t) = 0$ and $\dot{\delta} + \Theta\,\delta = f^1[\delta]$. Let $\tilde{\delta}(t)$ be the solution to this differential equation, with the initial condition $\tilde{\delta}(0) = \overline{\delta(0)}$. If $\delta(t)$ satisfies (I.2) - (I.4) , then clearly

$\delta(t) < \tilde{\delta}(t)$ for all $t > 0$. Hence if $\tilde{\delta}(t)$ is bounded above for large t, so is $\delta(t)$ and a fortiori $\delta_2(t)$ as well.

Let us consider the following differential equation:

$$\dot{\delta} + \theta\,\delta = f\,[\delta] \tag{I.6}$$

Given our assumptions (2.1) and (2.2) on $f^1[\delta]$, there exist unique δ^* and δ^{**} such that:

(a) $f^1[\delta] - \theta\,\delta$ attains its unique maximum in $[0, \infty]$ at $\delta = \delta^*$. Clearly $f_1^1[\delta^*] = \theta$

(b) $f^1[\delta] - \theta\,\delta > 0$ for $0 < \delta < \delta^{**}$

$= 0$ for $\delta = \delta^{**}$

< 0 for $\delta > \delta^{**}$

(c) $\delta^* < \delta^{**}$

Hence, whatever be the initial value $\overline{\delta(0)}$, the solution $\tilde{\delta}(t)$ of (I.6) converges to δ^{**}. This implies, given $\overline{\delta(0)}$, and any $\eta > 0$ we can find a $T \equiv T\left(\eta, \overline{\delta(0)}\right)$ such that for all $t > T$, $\tilde{\delta}(t) < \delta^{**} + \eta$. This completes the proof of the boundedness of $\tilde{\delta}(t)$ for large t.

Given that the initial value $\overline{\delta(0)}$ is subject to choice, it is clear that we can confine our attention to stationary solutions in solving the above maximization problem. In other words, we search for a solution which will maximize $\underline{c}$ subject to

$$\underline{c} \leqq \ell_2 f^2[\delta_2] \tag{I.7}$$

$$\Theta \delta \leqq \ell_1 f^1[\delta_1] \tag{I.8}$$

$$\ell_1\delta_1 + \ell_2\delta_2 \leqq \delta \tag{I.9}$$

$$\ell_1 + \ell_2 \leqq 1 \tag{I.10}$$

$$\underline{c} \quad \delta, \delta_1, \delta_2, \ell_1, \ell_2 \geqq 0 \tag{I.11}$$

The Lagrangean associated with this maximization problem is

$$\Phi = \underline{c} - \lambda_1 \left[\underline{c} - \ell_2 f^2[\delta_2]\right] - \lambda_2 \left[\Theta \delta - \ell_1 f^1[\delta_1]\right]$$
$$- \lambda_3 [\ell_1\delta_1 + \ell_2\delta_2 - \delta] - \lambda_4 [\ell_1 + \ell_2 - 1] \tag{I.12}$$

Maximizing Φ unconditionally, we have as necessary conditions,

$$\frac{\partial \Phi}{\partial \underline{c}} = 1 - \lambda_1 = 0 , \quad \frac{\partial \Phi}{\partial \delta} = - \lambda_2 \Theta + \lambda_3 = 0$$

$$\frac{\partial \Phi}{\partial \delta_1} = \lambda_2 \ell_1 f_1^1[\delta_1] - \lambda_3 \ell_1 = 0 , \quad \frac{\partial \Phi}{\partial \delta_2} = \lambda_1 \ell_2 f_1^2[\delta_2] - \lambda_3 \ell_2 = 0$$

$$\frac{\partial \Phi}{\partial \ell_1} = \lambda_2 f^1[\delta_1] - \lambda_3 \delta_1 = 0 , \qquad \frac{\partial \Phi}{\partial \ell_2} = \lambda_1 f^2[\delta_2] - \lambda_3 \delta_2 = 0 .$$

Given assumptions (2.1) and (2.2), these equations and (I.7)-(I.10) with inequalities replaced by equalities determine uniquely $\underline{c}^*$, δ^* , δ_1^* , δ_2^*, ℓ_1^* , ℓ_2^* , λ_1^* , λ_2^* . These are given by

$$\underline{c}^* = \ell_2^* f^2[\delta_2^*] \qquad \delta^* = \ell_1^* \delta_1^* + \ell_2^* \delta_2^*$$

$$\ell_1^* = \frac{\delta_2^* f_1^2[\delta_2^*]}{f^2[\delta_2^*]} \qquad \ell_2^* = 1 - \ell_1^*$$

$$\lambda_1^* = 1 \qquad \lambda_2^* = \frac{1}{\theta} f_1^2 [\delta_2^*]$$

$$\lambda_3^* = f_1^2[\delta_2^*] \qquad \lambda_4^* = f^2[\delta_2^*] - \delta_2^* f_1^2 [\delta_2^*]$$

where δ_1^* and δ_2^* are the unique solutions of

$$f_1^1[\delta_1] = \theta \quad \text{and} \quad \frac{f^1[\delta_1]}{f_1^1[\delta_1]} - \delta_1 = \frac{f^2[\delta_2]}{f_1^2[\delta_2]} - \delta_2 .$$

It may be remarked that the assumptions (2.1)-(2.2) imply that the restraint set of the above maximization problem is convex. We are maximizing a linear function $\underline{c}$ over a convex set. These considerations lead to the conclusion that the above solution is the unique globally maximal situation.

APPENDIX II

The problem is to

$$\text{maximize} \quad \int_0^\infty e^{-\rho t} \; c(t) \, dt \tag{II.1}$$

subject to

$$c(t) \equiv \ell_2(t) \; f^2[\delta_2(t)] \tag{II.2}$$

$$0 \leqq \dot{\delta}(t) + \theta \, \delta(t) \leqq \ell_1(t) \; f^1[\delta_1(t)] \tag{II.3}$$

$$\ell_1(t)\delta_1(t) + \ell_2(t)\delta_2(t) \leqq \delta(t) = \overline{\delta(0)} + \int_0^\infty \dot{\delta}(u) du \tag{II.4}$$

$$\ell_1(t) + \ell_2(t) \leqq 1 \tag{II.5}$$

$$c(t), \delta(t), \delta_1(t), \delta_2(t), \ell_1(t), \ell_2(t) \geqq 0 \qquad \overline{\delta(0)} \text{ (given)} \tag{II.6}$$

<u>Definition 1</u> Any set of time paths $c(t)$, $\delta(t)$, $\delta_1(t)$, $\delta_2(t)$, $\ell_1(t)$, and $\ell_2(t)$ which satisfy (II.2)-(II.6) is called a feasible solution.

<u>Definition 2</u> An optimal solution is a feasible solution which maximizes II.1 among all feasible solutions.

The following lemma is useful in proving the optimality of the solutions given in Section 3.

<u>Lemma</u> If a feasible solution $\hat{c}(t)$, $\hat{\delta}(t)$, $\hat{\delta}_1(t)$, $\hat{\delta}_2(t)$, $\hat{\ell}_1(t)$, $\hat{\ell}_2(t)$ and a set of auxiliary functions $q(t)$, $w(t)$, $r(t)$ exist such that for all $t \geqq 0$ <u>and</u> for <u>all</u> feasible solutions $c(t)$, $\delta(t)$, $\delta_1(t)$, $\delta_2(t)$, $\ell_1(t)$ $\ell_2(t)$ the following hold:

(a) $$q(t) \geqq 0 , \quad w(t) \geqq 0 , \quad r(t) \geqq 0$$

(b) $$q(t) = \int_t^\infty r(u)\, du$$

(c.1) $$e^{-\epsilon t} \hat{c}(t) = \hat{\ell}_2(t) [w(t) + r(t) \hat{\delta}_2(t)]$$

(c.2) $$e^{-\epsilon t} c(t) \leqq \ell_2(t) [w(t) + r(t) \delta_2(t)]$$

(d.1) $$q(t)[\dot{\hat{\delta}} + \Theta \hat{\delta}] = \hat{\ell}_1(t) [w(t) + r(t) \hat{\delta}_1(t)]$$

(d.2) $$q(t)[\dot{\delta} + \Theta \delta] \leqq \ell_1(t) [w(t) + r(t) \delta(t)]$$

(e) $$w(t) = 0 \qquad \text{if } \hat{\ell}_1(t) + \hat{\ell}_2(t) < 1$$

(f) $$r(t) = 0 \qquad \text{if } \hat{\ell}_1(t) \hat{\delta}_1(t) + \hat{\ell}_2(t)\hat{\delta}_2(t) < \delta(t)$$

(g) $$\lim_{t \to \infty} \left\{ e^{\Theta t} q(t) \hat{\delta}(t) \right\} = 0 ,$$ then the

solution with ^ on the variables is optimal.

<u>Proof:</u> From (e), (f)′ and (II.5) it follows that

$$w(t) \quad = \quad w(t)\ [\hat{\ell}_1(t) + \hat{\ell}_2(t)] \tag{II.7}$$

$$r(t)\hat{\delta}(t) \quad = \quad r(t)\ [\hat{\ell}_1(t)\ \hat{\delta}_1(t) + \hat{\ell}_2(t)\ \hat{\delta}_2(t)] \tag{II.8}$$

Using (c.1), (d.1), (II.7) and (II.8) we have

$$e^{-\epsilon t}\ \hat{c}(t) + q(t)\ [\dot{\hat{\delta}} + \Theta\ \hat{\delta}] \ = \ w(t) + r(t)\ \hat{\delta}(t) \tag{II.9}$$

From (b) we know that $\dot{q}(t) = -\, r(t)$. Hence

$$w(t) = e^{-\epsilon t}\ \hat{c}(t) + e^{-\Theta t}\ \frac{d}{dt}\ [e^{\Theta t}\ q(t)\ \hat{\delta}(t)] \tag{II.10}$$

In view of (a), (II.6), (c.2) and (d.2) we have

$$e^{-\epsilon t}c(t) + q(t)\ [\dot{\delta} + \Theta\ \delta] \leqq w(t)[\ell_1(t) + \ell_2(t)] + r(t)[\ell_1(t)\ \delta_1(t) + \ell_2(t)\delta_2(t)]$$

$$\leqq w(t) + r(t)\delta(t)$$

Hence $e^{-\epsilon t}c(t) \ \leqq \ w(t) - e^{-\Theta t}\frac{d}{dt}\ [e^{\Theta t}\ q(t)\ \delta(t)]$ (II.11)

Using (II.10) in (II.11) we have

$$e^{-\epsilon t}c(t) - e^{-\epsilon t}\ \hat{c}(t) \ \leqq \ e^{-\Theta t}\left\{\frac{d}{dt}\left(e^{\Theta t}\ q(t)\hat{\delta}(t)\right) - \frac{d}{dt}\ \left(e^{\Theta}\ q(t)\delta(t)\right)\right\} \tag{II.12}$$

Since $\epsilon = \rho + \Theta$ we can rewrite (II.12) as

$$e^{-\rho t}c(t) - e^{-\rho t}\hat{c}(t) \leqq \frac{d}{dt}\left(e^{\Theta t} q(t)\,\hat{\delta}(t)\right) - \frac{d}{dt}\left(e^{\Theta t} q(t)\delta(t)\right) \qquad \text{(II.12)}'$$

Multiplying both sides of (II.12)' by -1 , reversing the inequality sign, integrating both sides over [0, T] and utilizing the condition that $\hat{\delta}(0) = \delta(0) = \bar{\delta}(0)$ we have

$$\int_0^T e^{-\rho t}\,\hat{c}(t)dt - \int_0^T e^{-\rho t}\,c(t)dt \geqq e^{\Theta T} q(T)\,\delta(T) - e^{\Theta t} q(T)\,\hat{\delta}(T)$$

$$\geqq - e^{\Theta t} q(T)\,\hat{\delta}(T)$$

Letting $T \to \infty$ and utilizing (g), the optimality of the solution with ^ on the variables is established.

We can provide an interpretation for the auxiliary functions. Let us define $p(t) \equiv e^{-\epsilon t}$. Then $p(t)$ can be interpreted as the value at time zero of a unit of consumer good produced at time t . $q(t)$ is the value at time zero of a unit of capital good produced at t . $w(t)$ and $r(t)$ are the values at time zero of the wage rate and rental rate at time t . With these price interpretations, the meaning of the conditions (a)-(g) of the lemma become clear. (a) states that the prices are non-negative. (b) is the intertemporal efficiency condition, that the value of a unit of capital at t is the sum of the rentals on it from t on. (c.1) and (c.2) state that given the prices $p(t)$, $q(t)$, $w(t)$, and $r(t)$, the profit made in the production of consumer goods is zero under the solution with a ^ on the variables, and nonpositive under any other feasible solution. In other words the solution with a ^ , is profit maximizing for the production of consumer goods, at all t, given the prices. $p(t)$, $w(t)$, $r(t)$. (d.1) and (d.2) are similar profit maximizing conditions on the production of capital goods. (e) and (f) state that if a resource (labor or capital) is not fully used its price is zero. Condition (g) is a boundary condition, arising mainly because of the infinite time horizon postulated in our problem.

The reader might wonder how we were able to prove our Lemma without using the properties (2.1) and (2.2) on $f^1[\delta]$ and $f^2[\delta]$. The answer is simple. We stated as a part of the hypothesis of the Lemma, the existence of the auxiliary functions which satisfied conditions (a)-(g). However, in any given problem, without the assumptions (2.1) and (2.2) we may not in general be able to find such auxiliary functions. In fact, in what follows we shall be using (2.1) and (2.2) extensively besides assumption (3.1).

We are now in a position to establish the optimality of the solutions given Section 3. We shall confine our attention to Cases I and III leaving the proof for Case II to the reader. We distinguished the three cases by first defining two constants $\hat{\delta}_1$, $\hat{\delta}_2$ as follows:

$$f_1^1[\hat{\delta}_1] = \epsilon \qquad \frac{f^1[\hat{\delta}_1]}{f_1^1[\hat{\delta}_1]} - \hat{\delta}_1 = \frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_2]} - \hat{\delta}_2 .$$

The existence and uniqueness of $\hat{\delta}_1$, $\hat{\delta}_2$ follow immediately from assumptions (2.1) and (2.2). We recall that these imply that $f_1^i(\delta)$ is a <u>decreasing function</u> of δ, with $f_1^i(0) = \infty$ and $f_1^i(\infty) = 0$. Also $\dfrac{f^i[\delta]}{f_1^i[\delta]} - \delta$ is an increasing function of δ , approaching 0 as $\delta \to 0$ and ∞ as $\delta \to \infty$. Hence $\hat{\delta}_1$ and $\hat{\delta}_2$ exist and are unique. Further our capital intensity assumption (3.1) implies that $\hat{\delta}_2 > \hat{\delta}_1$.

We can now prove the optimality of the solutions given in Section 3.

<u>Case I</u> $\hat{\delta}_2 \geqq \overline{\delta(0)} \geqq \hat{\delta}_1$. The optimal solution is,

$$\delta_1(t) = \hat{\delta}_1 \qquad \delta_2(t) = \hat{\delta}_2$$

$$\delta(t) = \hat{\delta}(\infty) + \left[\overline{\delta(0)} - \hat{\delta}(\infty)\right] e^{-xt}$$

$$\ell_1(t) = \frac{\hat{\delta}_2 - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1} \qquad \ell_2(t) = 1 - \ell_1(t)$$

$c\ (t) = \ell_2(t)\ f^2[\hat{\delta}_2(t)]$. The auxiliary functions are

$$p(t) = e^{-\epsilon t} \qquad q(t) = \frac{1}{\epsilon} f_1^2[\hat{\delta}_2]\ e^{-\epsilon t}$$

$$w(t) = \left\{ f^2[\hat{\delta}_2] - \hat{\delta}_2 f_1^2(\hat{\delta}_2) \right\} e^{-\epsilon t}, \qquad r(t) = f_1^2[\hat{\delta}_2]\ e^{-\epsilon t}$$

where $$x = \frac{\Theta[\hat{\delta}_2 - \hat{\delta}_1] + f^1(\hat{\delta}_1)}{(\hat{\delta}_2 - \hat{\delta}_1)}, \quad \hat{\delta}(\infty) = \frac{\hat{\delta}_2 f^1(\hat{\delta}_1)}{\Theta(\hat{\delta}_2 - \hat{\delta}_1) + f^1(\hat{\delta}_1)}$$

Given that $\hat{\delta}_2 > \hat{\delta}_1 > 0$, it is clear that $\hat{\delta}_2 > \delta(\infty) > \hat{\delta}_1 > 0$ and that $x > 0$. Hence $\delta(t)$ is monotonic in t and since $\hat{\delta}_2 \geq \overline{\delta(0)} \geqq \hat{\delta}_1$, it follows that $\hat{\delta}_2 \geqq \delta(t) \geqq \hat{\delta}_1$ for all $t \geqq 0$. This means that $\ell_1(t), \ell_2(t) \geqq 0$ for t . It is easy to verify that this solution is feasible.

In order to prove the optimality we apply our Lemma. Conditions (a), (b), (c.1), (d.1), (e), (f) and (g) are easily seen to hold. From the definition of $w(t)$ and $r(t)$, we note that their ratio, $\frac{w(t)}{r(t)}$ is a constant over time and further

$$\frac{w(t)}{r(t)} = \frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_1]} - \hat{\delta}_2 = \frac{f^1[\hat{\delta}_1]}{f_1^1[\hat{\delta}_1]} - \hat{\delta}_1 . \tag{II.13}$$

In view of our constant returns to scale assumption on the two production functions, $\frac{f^i[\delta]}{f_1^i[\delta]} - \delta$ is the ratio of the marginal physical product of labour to that of capital in sector i , if the capital labour ratio δ is used. Now $\frac{w(t)}{r(t)}$ is the ratio of wages to rents. Hence given our assumptions (2.1) and (2.2), (II.13) means that $\hat{\delta}_1$ and $\hat{\delta}_2$ are the unique unit cost minimizing capital-labour ratios in the two sectors. Now the unit cost of production in sector i if capital labour ratio $\delta_i(t)$ is used, is

$$\frac{w(t) + r(t)\,\delta_i(t)}{f^i[\delta_i(t)]} .$$

Hence cost minimization at the capital labour ratios $\hat{\delta}_1$ and $\hat{\delta}_2$ implies that for all $\delta_1(t)$, $\delta_2(t)$,

$$\frac{w(t) + r(t)\ \delta_1(t)}{f^1[\delta_1(t)]} \geqq \frac{w(t) + r(t)\ \hat{\delta}_1}{f^1[\hat{\delta}_1]} = e^{-\epsilon t} \qquad \text{(II.14)}$$

$$\frac{w(t) + r(t)\ \delta_2(t)}{f^2[\delta_2(t)]} \geqq \frac{w(t) + r(t)\ \hat{\delta}_2}{f^2[\hat{\delta}_2]} = q(t) \qquad \text{(II.15}$$

These two inequalities imply that for any feasible solution

$$\ell_1(t)\ [w(t) + r(t)\ \delta_1(t)] \geqq q(t)\ell_1(t)f^1[\delta_1(t)] \geqq q(t)\ [\dot{\delta} + \theta\ \delta]$$

$$\ell_2(t)\ [w(t) + r(t)\ \delta_2(t)] \geqq \ell_2(t)f^2[\delta_2(t)]\ e^{-\epsilon t} = e^{-\epsilon t}\ c(t)\ .$$

Hence (c.2) and (d.2) are seen to hold. Hence our solution is optimal.

<u>Case III</u> $\hat{\delta}_1 > \overline{\delta(0)}$. Let $\tilde{\delta}(t)$ be the solution of the differential equation $\dot{\delta} + \theta\ \delta = f^1[\delta]$ with the initial condition $\tilde{\delta}(0) = \overline{\delta(0)}$. We discussed this differential equation in Appendix I. We found that if $\overline{\delta(0)} < \delta^{**}$, the solution $\tilde{\delta}(t)$ has the property that $\dot{\tilde{\delta}}(t) > 0$ for all t and $\lim_{t\to\infty} \tilde{\delta}(t) = \delta^{**}$ where $f^1[\delta^{**}] = \theta\ \delta^{**}$. We noted also that $\delta^* < \delta^{**}$ where δ^* is such that $f^1_1[\delta^*] = \theta$.

Now $f^1_1[\hat{\delta}_1] = \epsilon = \rho + \theta > \theta$, since $\rho > 0$. This means that $\hat{\delta}_1 < \delta^*$ since $f^1_1(\delta)$ is a decreasing function of δ . Hence

$\overline{\delta(0)} < \hat{\delta}_1 < \delta^* < \delta^{**}$. Also there exists a $\underline{t}$ such that $\tilde{\delta}(\underline{t}) = \hat{\delta}_1$.

The optimal solution for this case is

$$\left.\begin{aligned} \delta_1(t) &= \delta(t) = \tilde{\delta}(t) & \delta_2(t) &= \hat{\delta}_2 \\ \ell_1(t) &= 1 & \ell_2(t) &= 0 \\ c(t) &= 0 \end{aligned}\right\} 0 \leqq t \leq \underline{t}$$

$$\left.\begin{aligned} \delta_1(t) &= \hat{\delta}_1 & \delta_2(t) &= \hat{\delta}_2 \\ \delta(t) &= \hat{\delta}(\infty) + [\hat{\delta}_1 - \hat{\delta}]\, e^{-x(t-\underline{t})} \\ \ell_1(t) &= \frac{\hat{\delta}_2 - \delta(t)}{\hat{\delta}_2 - \hat{\delta}_1} & \ell_2(t) &= 1 - \ell_1(t) \\ c(t) &= \ell_2(t)\, f^2[\hat{\delta}_2] \end{aligned}\right\} t > \underline{t}$$

$$\left.\begin{aligned} p(t) &= e^{-\epsilon t} \qquad w(t) = \left[f^1[\delta_1(t)] - \delta_1(t)\, f_1^1[\delta_1(t)\right] q(t) \\ r(t) &= f_1^1[\delta_1(t)]q(t) \quad \text{where } q(t) \text{ is the} \\ &\text{solution of } \dot{q}(t) = -f_1^1\, [\delta_1(t)\, q(t)] \text{ with the} \\ &\text{condition } \quad q(\underline{t}) = \frac{1}{\epsilon} f_1^2\, [\hat{\delta}_2]\, e^{-\epsilon \underline{t}} \end{aligned}\right\} 0 \leqq t \leq t$$

$p(t)\, q(t)$, $w(t)$ and $r(t)$ have the same values as in Case I for $t > \underline{t}$.

The feasibility of this solution can be verified easily. For $t > \underline{t}$. This solution is essentially the same as in Case I and hence the conditions (a) - (g) of the Lemma continue to hold. For any t in $0 \leqq t \leqq \underline{t}$, conditions (a), (b), (c.1), (d.1), (e), (f) of the Lemma are seen to hold. Condition (g), being an asymptotic property, is not relevant for this initial period. In order to establish the optimality of our solution, we need to verify (c.2) and (d.2).

We note from the definition of $q(t)$, $w(t)$ and $r(t)$ that the following hold for any t in $[0, \underline{t}]$:

$$\frac{w(t)}{r(t)} = \frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t)$$

$$q(t) = \frac{r(t)}{f_1^1[\delta_1(t)]}$$

The first of these, means that $\delta_1(t)$ is that capital labour ratio which equates the ratio of wages over rents to the ratio of marginal physical product of labour to that of capital in the capital goods industry. In other words $\delta_1(t)$ minimizes the unit cost of production of capital goods. The second equation, means that the minimal unit just equals price per unit of capital goods. Hence these two together imply that for any other capital labour ratio, profits will be nonpositive. This verifies (d.2).

In order to verify (c.2), we shall show that given $w(t)$, $r(t)$ the minimal unit cost $\tilde{p}(t)$ of producing consumer goods exceeds its price $p(t)$ and hence it is unprofitable to produce any consumer goods. Let $\tilde{\delta}_2(t)$ be that capital labour ratio which minimizes unit cost of production given $w(t)$ and $r(t)$. Then the following hold:

$$\frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t) = \frac{w(t)}{r(t)} = \frac{f^2[\tilde{\delta}_2(t)]}{f_1^2[\tilde{\delta}_2(t)]} - \tilde{\delta}_2(t)$$

$$\tilde{p}(t) = \frac{r(t)}{f_1^2[\tilde{\delta}_2(t)]}$$

We wish to show that $\tilde{p}(t) > p(t) = e^{-\epsilon t}$ for $0 \leqq t < \underline{t}$.

Consider $\delta_1(t)$ at any t in $[0, \underline{t}]$. We know from the choice of $\delta_1(t)$ that $\delta_1(t) < \hat{\delta}_1$ and $\dot{\delta}_1(t) > 0$. Since $f_1^1[\delta]$ is a decreasing function of δ we have:

$$f_1^1[\delta_1(t)] > f_1^1[\hat{\delta}_1] = \epsilon \text{ . By definition ,}$$

$$\dot{q}(t) = - f_1^1[\delta_1(t)] \, q(t) < - \epsilon \, q(t)$$

$$\frac{\dot{q}(t)}{q(t)} < - \epsilon$$

$$\frac{d}{dt} \operatorname{Log} q(t) < - \epsilon \text{ . Integrating both sides over}$$

$[t, \underline{t}]$ we have

$$\text{Log } q(\underline{t}) - \text{Log } q(t) < -\epsilon\,[\underline{t} - t]$$

or
$$\frac{q(\underline{t})}{q(t)} < e^{-\epsilon\,[\underline{t} - t]}$$

$$q(t) > q(\underline{t})\; e^{\epsilon[\underline{t} - t]} = \frac{f_1^2[\hat{\delta}_2]}{\epsilon}\; e^{-\epsilon t} \qquad \text{(II.16)}$$

Now
$$\tilde{p}(t) - e^{-\epsilon t} = \frac{r(t)}{f_1^2[\tilde{\delta}_2(t)]} - e^{-\epsilon t}$$

$$= q(t)\; \frac{f_1^1[\delta_1(t)]}{f_1^2[\tilde{\delta}_2(t)]} - e^{-\epsilon t}$$

$$> e^{-\epsilon t}\left\{\frac{f_1^2[\hat{\delta}_2]}{\epsilon}\;\; \frac{f_1^1[\delta_1(t)]}{f_1^2(\tilde{\delta}_2(t)} - 1\right\}$$

Let
$$\Phi(t) = \frac{f_1^1[\delta_1(t)]}{f_1^2[\tilde{\delta}_2(t)]} - \frac{\epsilon}{f_1^2[\hat{\delta}_2]} = \frac{f_1^1[\delta_1(t)]}{f_1^2[\tilde{\delta}_2(t)]} - \frac{f_1^1[\hat{\delta}_1]}{f_1^2[\hat{\delta}_2]}$$

Clearly $\Phi(\underline{t}) = 0$. We shall show that $\Phi(t) > 0$ for $t < \underline{t}$ by showing that $\dot{\Phi}(t) < 0$ for t in $[0, \underline{t}]$.

$$\dot{\Phi}(t) = \frac{f_{11}^1[\delta_1(t)]\;\dot{\delta}_1}{f_1^2[\tilde{\delta}_2(t)]} - \frac{f_1^1[\delta_1(t)]}{\left\{f_1^2[\tilde{\delta}_2(t)]\right\}^2} \cdot f_{11}^2[\tilde{\delta}_2(t)]\;\dot{\tilde{\delta}}_2$$

We know that $\frac{w(t)}{r(t)} = \frac{f^2[\tilde{\delta}_2(t)]}{f_1^2[\tilde{\delta}_2(t)} - \tilde{\delta}_2(t) = \frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t)$

Differentiating with respect to t ,

$$- \frac{f^2[\tilde{\delta}_2(t)]}{f_1^2[\tilde{\delta}_2(t)]^2} \quad f_{11}^2 [\tilde{\delta}_2(t)] \dot{\tilde{\delta}}_2 = - \frac{f^1[\delta_1(t)]}{[f_1^1[\delta_1(t)]^2} \cdot f_{11}^1 [\delta_1(t)] \dot{\delta}_1$$

Substituting in the expression for $\dot{\Phi}(t)$ and after simplification we have,

$$\dot{\Phi}(t) = \frac{f_{11}^1[\delta_1(t)] \dot{\delta}_1}{f^2[\tilde{\delta}_2(t)]} \left[\frac{f^2[\tilde{\delta}_2(t)]}{f_1^2[\tilde{\delta}_2(t)]} - \frac{f^1[\delta_1(t)}{f_1^1[\delta_1(t)]} \right]$$

$$= \frac{f_{11}^1[\delta_1(t)] \dot{\delta}_1}{f^2[\tilde{\delta}_2(t)]} \quad [\tilde{\delta}_2(t) - \delta_1(t)]$$

By our capital intensity hypothesis $\delta_1(t) < \tilde{\delta}_2(t)$.

We know also that $f_{11}^1[\delta_1(t)] < 0$, $\dot{\delta}_1 > 0$, $f^2[\tilde{\delta}_2(t)] > 0$.

Hence $\dot{\Phi}(t) < 0$ for $0 \leq t < \underline{t}$.

This implies $\Phi(t) > 0$ for $0 \leq t < \underline{t}$, which in turn implies that $\tilde{p}(t) - e^{-\epsilon t} > 0$ for $0 \leq t < \underline{t}$. This completes the proof of the optimality of our solution for Case III.

APPENDIX III

Our analysis in Section 3 was based on the capital intensity assumption that the consumer goods industry was more capital intensive than the capital goods industry. The role of this assumption becomes apparent if we consider the solution for Case I of Section 3. It will be recalled that the optimal solution for this case was:

$$\delta_1(t) = \hat{\delta}_1 \qquad \delta_2(t) = \hat{\delta}_2$$

$$\delta(t) = \hat{\delta}(\infty) + [\bar{\delta}(0) - \hat{\delta}(\infty)]e^{-xt}$$

$$\ell_1(t) = \frac{\delta_2(t) - \hat{\delta}(\infty)}{\hat{\delta}_2 - \hat{\delta}_1} \qquad \ell_2(t) = 1 - \ell_1(t)$$

where $x = \Theta + \dfrac{f^1[\hat{\delta}_1]}{\hat{\delta}_2 - \hat{\delta}_1}$, $\hat{\delta}(\infty) = \dfrac{\hat{\delta}_2 f^1[\hat{\delta}_1]}{x[\hat{\delta}_2 - \hat{\delta}_1]}$

$$f_1^1[\hat{\delta}_1] = \epsilon , \quad \frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_2]} - \hat{\delta}_2 = \frac{f^1[\hat{\delta}_1]}{\epsilon} - \hat{\delta}_1$$

$\epsilon = \rho + \Theta$, ρ is the given positive discount rate.

The efficiency prices associated with this solution were:

$p(t) = e^{-\epsilon t}$, $q(t) = \frac{1}{\epsilon} f^2[\hat{\delta}_2] e^{-\epsilon t}$, $r(t) = f^2[\hat{\delta}_2] e^{-\epsilon t}$ and

$w(t) = \left[f^2[\hat{\delta}_2] - \hat{\delta}_2 f_1^2[\hat{\delta}_2]\right]e^{-\epsilon t}$. The reasoning that led to this solution is the following.

We searched for a feasible solution and a set of auxiliary functions $p(t)$, $q(t)$, $w(t)$, $r(t)$ that satisfied the conditions of our Lemma. Given that $p(t) \equiv e^{-\epsilon t}$ as per Lemma, profit maximization in the production of consumer goods and capital goods lead to the following relations (assuming non-zero amounts of both goods are produced):

$$r(t) = e^{-\epsilon t} f_1^2[\delta_2(t)] = q(t)\, f_1^1[\delta_1(t)] \qquad \text{(III.1)}$$

$$w(t) = e^{-\epsilon t}\left[f^2[\delta_2(t)] - \delta_2(t)\, f_1^2[\delta_2(t)]\right] = q(t)\left[f^1[\delta_1(t)] - \delta_1(t) f_1^1[\delta_1(t)]\right] \qquad \text{(III.2)}$$

These are nothing but the familiar marginal equalities viz. marginal value products of capital and labour in one sector equal their corresponding values in the other sector. These are static efficiency conditions. We have in addition the intertemporal efficiency condition that $q(t) = \int_t^\infty r(u)du$ or its differential form $\dot{q}(t) = -r(t)$. (III.3)

If in addition we assume that $\delta_1(t)$ and $\delta_2(t)$ are constants over time, these three equations are sufficient to solve uniquely for $q(t)$, $w(t)$, $r(t)$ and the constant values $\hat{\delta}_1$ and $\hat{\delta}_2$. Then it becomes a matter of checking the feasibility of the resulting solution.

The crucial assumption that made the resulting solution feasible was the capital intensity assumption. Given this, it is clear that $\hat{\delta}_2 > \hat{\delta}_1$ and $x > 0$. If the capital intensity assumption is violated, $\hat{\delta}_2 < \hat{\delta}_1$, x becomes negative. If x is negative

$\hat{\delta}(\infty) + [\overline{\delta(0)} - \hat{\delta}(\infty)]\, e^{-xt} \rightarrow$ either $+\infty$ or $-\infty$ (depending on the sign of $\overline{\delta(0)} - \hat{\delta}(\infty)$) except when $\overline{\delta(0)} = \hat{\delta}(\infty)$. In either case the above solution is not feasible.

The above reasoning suggests that when the capital intensity assumption is not met, the optimal solution, will not in general have the property that $\delta_i(t) = \hat{\delta}_i$ for all t. We have not been able to get the optimal solutions in this case for general production functions. However if we are to assume that the production in the two sectors are of the Cobb-Douglas type, optimal solutions can be obtained in certain cases at least. We shall discuss below these solutions, without giving the proof of their optimality.

Let us assume that $f^1[\delta] = \delta^{\alpha}$, and $f^2[\delta] = \delta^{\beta}$, $0 < \alpha \neq \beta < 1$. The capital intensity assumption will be met if $\beta > \alpha$. Let us assume that this is not met, i.e., $\beta < \alpha$. Recalling our definition of $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\delta}(\infty)$, we can verify that in this case

$$\hat{\delta}_1 = \left[\frac{\alpha}{\epsilon}\right]^{\frac{1}{1-\alpha}} \qquad \hat{\delta}_2 = \eta\, \hat{\delta}_1 \qquad \hat{\delta}(\infty) = \frac{\eta\, \epsilon\, \hat{\delta}_1}{\epsilon - \alpha(1-\eta)\Theta} \quad ,$$

$x = \Theta - \frac{\epsilon}{\alpha(1-\eta)}$ where $\eta = \frac{\beta(1-\alpha)}{\alpha(1-\beta)}$. It is clear that

$\eta < 1$ and $\hat{\delta}_1 > \hat{\delta}(\infty) > \hat{\delta}_2$. Let A be a constant defined by the equation

$$\overline{\delta(0)} = \frac{\eta}{1-\eta}\, [A + \frac{\alpha}{\epsilon}]^{\frac{1-\beta}{1-\alpha}} \int_0^\infty e^{xu}\, [Ae^{-yu} + \frac{\alpha}{\epsilon}]^{\frac{\beta}{1-\alpha} - 1}\, du \qquad \text{(III.4)}$$

where $y = \left(\frac{1-\alpha}{1-\beta}\right)\epsilon$. It can be shown that, given $\overline{\delta(0)}$, there exists a unique $A > - \frac{\alpha}{\epsilon}$ satisfying (III.4) .

<u>Case I</u> $\alpha + \beta \leqq 1 \qquad 1 \geqq \overline{\delta(0)}\, [A + \frac{\alpha}{\epsilon}]^{\frac{-1}{1-\alpha}} \geqq \eta$

The solution in this case is

$$\delta_1(t) = [A\, e^{-yt} + \frac{\alpha}{\epsilon}]^{\frac{1}{1-\alpha}} \qquad \delta_2(t) = \eta\, \delta_1(t)$$

$$\delta(t) = \frac{\eta}{1-\eta}\, e^{-xt}\, [\delta_1(t)]^{1-\beta} \int_t^\infty e^{xu}\, [\delta_1(u)]^{\alpha+\beta-1}\, du$$

$$\ell_1(t) = \frac{\delta(t) - \delta_2(t)}{\delta_1(t) - \delta_2(t)} , \quad \ell_2(t) = 1 - \ell_1(t)$$

It will be noted that this solution applies only when $\alpha + \beta \leqq 1$. Even with this assumption, it is not possible to obtain the optimal solutions for all initial values $\overline{\delta(0)}$. The subsequent two cases are based on the assumption that $\alpha + \beta = 1$. Given that $\alpha + \beta = 1$, it can be seen that the value of A that satisfies (III.4) is given by

$$-\frac{\eta}{(1-\eta)} \frac{1}{x} \left[A + \frac{\alpha}{\epsilon}\right]^{\frac{\alpha}{1-\alpha}} = \overline{\delta(0)}$$

$$\left[A + \frac{\alpha}{\epsilon}\right]^{\frac{\alpha}{1-\alpha}} = \overline{\delta(0)} \left[-\frac{\eta}{(1-\eta)\ x}\right]$$

$$\overline{\delta(0)} \quad \left[A + \frac{\alpha}{\epsilon}\right]^{-\frac{1}{1-\alpha}} = \left[\overline{\delta(0)}\right]^{1-\frac{1}{\alpha}} \left[\frac{-\eta}{(1-\eta)\ x}\right]^{-\frac{1}{\alpha}}$$

Case I, covered the region $1 \geqq \left[\overline{\delta(0)}\right]^{1-\frac{1}{\alpha}} \left[\frac{-\eta}{(1-\eta)\ x}\right]^{-\frac{1}{\alpha}} \geqq \eta$

Case II $\qquad \eta > \left[\overline{\delta(0)}\right]^{1-\frac{1}{\alpha}} \left[\frac{-\eta}{(1-\eta)\ x}\right]^{\frac{1}{\alpha}}, \quad \alpha + \beta = 1$

The solution is:

$$\left.\begin{array}{l} \delta_2(t) = \delta(t) = \overline{\delta(0)}\, e^{-\theta t} \\ \ell_2(t) = 1, \quad \ell_1(t) = 0 \end{array}\right\} \quad 0 \leq t < \bar{t}$$

$$\left.\begin{aligned}
\delta_1(t) &= \left[B\, e^{-y(t-\bar{t})} + \frac{\alpha}{\epsilon} \right]^{\frac{1}{1-\alpha}}, \quad \delta_2(t) = \eta\, \delta_1(t) \\
\delta(t) &= -\frac{\eta}{(1-\eta)\, x} \left[\delta_1(t)\right]^{1-\beta} \\
\text{where } B &= -\frac{1}{(1-\eta)\, x} - \frac{\alpha}{\epsilon} \\
\ell_1(t) &= \frac{\delta(t) - \delta_2(t)}{\delta_1(t) - \delta_2(t)} \qquad \ell_2(t) = 1 - \ell_1(t)
\end{aligned}\right\} \quad t \geqq \bar{t}$$

where $\bar{t}$ is solved from $\overline{\delta(0)}\ e^{-\Theta \bar{t}} = \eta^{-\frac{\alpha}{1-\alpha}} \left[- \frac{\eta}{(1-\eta)\, x} \right]^{\frac{1}{1-\alpha}}$

<u>Case III</u> $\quad [\overline{\delta(0)}]^{1-\frac{1}{\alpha}} \left[- \frac{\eta}{(1-\eta)\, x} \right]^{\frac{1}{1-\alpha}} > 1\ , \ \alpha + \beta = 1$

The solution is:

$$\left.\begin{aligned}
\delta_1(t) &= \delta(t) = \left[\frac{1}{\Theta} - \left\{ \frac{1}{\Theta} - [\overline{\delta(0)}]^{1-\alpha} \right\} e^{-\Theta(1-\alpha)t} \right]^{\frac{1}{1-\alpha}} \\
\ell_1(t) &= 1 \qquad \ell_2(t) = 0
\end{aligned}\right\} \quad 0 \leq t < \underline{t}$$

$$\left.\begin{aligned}
\delta_1(t) &= \left[C\, e^{-y(t-\underline{t})} + \frac{\alpha}{\epsilon} \right]^{\frac{1}{1-\alpha}} \\
\delta(t) &= -\frac{\eta}{(1-\eta)\, x} \left[\delta_1(t)\right]^{1-\beta} \\
\ell_1(t) &= \frac{\delta(t) - \delta_2(t)}{\delta_1(t) - \delta_2(t)} \qquad \ell_2(t) = 1 - \ell_1(t)
\end{aligned}\right\} \quad t \geqq \underline{t}$$

where $C = - \frac{\eta}{(1-\eta)\,x} - \frac{\alpha}{\epsilon}$ and $\underline{t}$

is solved from $\left[\frac{1}{\theta} - \left\{ \frac{1}{\theta} - [\overline{\delta(0)}]^{1-\alpha} \right\} e^{-\theta(1-\alpha)\underline{t}} \right] = - \frac{\eta}{(1-\eta)\,x}$

It will be observed from the above incomplete discussion of the case when the capital intensity criterion is not met, the solutions have the same asymptotic properties as the solutions for the same discount rate when the criterion is met.