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Capital Risk and Household Consumption:

A Sequential Utility Analysis

Edmund S. Phelps

December 2, 1960

CAPITAL RISK AND HOUSEHOLD CONSUMPTION
A SEQUENTIAL UTILITY ANALYSIS

Edmund S. Phelps*

Consider a consumer having a known lifespan, a fixed rate of non-wealth income and only one storable asset, capital. If not consumed or marketed for consumer goods, this asset is expected to grow but in a random fashion so that its holder is exposed to the risk of a capital loss. After specifying the stochastic process of capital growth and the individual's utility function, we shall investigate the individual's optimal consumption policy. How will his consumption rate depend upon his age and capital stock? How will his "consumption function" vary with changes in such parameters as the fixed income rate, the expected rate of return on capital and the riskiness of capital? (To some of these questions our model will yield no general answer.)

The problem is pertinent most directly to the theory of consumer expenditure. Capital risk may play a part in explaining differences in household expenditure. One would like to know, too, if the usual results in consumption theory need to be qualified upon the admission of capital risk. Among existing deterministic models of consumption, our model resembles Ramsey's [11] more than the contemporary models [7, 10]. Therefore it is largely Ramsey's results which will be modified and extended here.

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The model will now be described in more detail.

1. The behavior of capital.

In this model, capital gains and losses occur in unit amounts. These fluctuations in the capital stock occur at random times and independently of the age of the stock. Thus capital grows according to a discontinuous Markov process. In particular we assume that the growth of capital (in the absence of capital consumption) is described by a stationary, linear birth and death process, with an amendment to be noted. We now review this process briefly.*

* For a thorough discussion see [6], pp. 407-411 and [3], pp. 86-94.

In this process, the probability of a unit gain in the capital stock, x , within a small time interval h is equal to $\lambda xh + o(h)$ and the corresponding probability of a unit capital loss is $\mu xh + o(h)$, where $o(h)$ are terms of higher order than h . The probability of a greater number of gains or losses is also of higher order than h . The corresponding probability of no change in the capital stock is thus

$$1 - (\lambda + \mu)xh - o(h).*$$

* These results are usually derived from assumptions on the corresponding probabilities that individual units of the variable, x , will "split" or "die" (and the further assumption of non-interaction). We shall assume, instead, that capital is continuously divisible and that the rate at which capital may be consumed is continuously variable. Consequently we start from the statements in the text.

We assume that $\lambda > \mu$ so that capital has a positive expected net productivity. Denoting the initial capital stock, $x(0)$, by k , we can state the following well-known results for this process: The probability of ultimate absorption ("ruin") is $(\frac{\mu}{\lambda})^k$. The expected value of the capital stock at time t is $ke^{(\lambda - \mu)t}$. The rate of growth of the mean capital stock is $\lambda - \mu$, independent of the amount of capital. Thus $\lambda - \mu$ is also the expected rate of return on capital.* The variance of the

* The expected value of the relative increase in x per length of time h is

$$\frac{1}{h} [(\lambda kh + o(h)) \left(+\frac{1}{k}\right) + (\mu kh + o(h)) \left(-\frac{1}{k}\right)]$$

The instantaneous expected rate of return is the limit of this as h approaches zero. This limit is equal to $\lambda - \mu$.

distribution of x at time t is [6]

$$k \cdot \frac{\lambda + \mu}{\lambda - \mu} (e^{2(\lambda - \mu)t} - e^{(\lambda - \mu)t})$$

It can be seen that the presence of the second parameter, μ , makes it possible to compare the behavior of processes which have the same expected growth path but different variances and ruin probabilities. If we fix $(\lambda - \mu) > 0$, then the variance will be higher and the probability of ruin higher the larger μ , $0 \leq \mu \leq \lambda - (\lambda - \mu)$.

The last restriction prevents us from considering a process with no risk, in the sense of no variance; because $\mu \geq 0$, $\lambda + \mu > 0$ if $\lambda - \mu > 0$. Therefore we shall amend the birth and death process to include a deterministic trend arising from a certain rate of return, p ; the new expected rate of return is $r = p + \lambda - \mu$. This permits us to make the variance zero (by setting $\lambda = \mu = 0$) while keeping r constant through a compensating increase in p .

Calculating in the manner of the previous footnote, we find the variance of the rate of return on a marginal unit of capital to be $\lambda + \mu$.*

$$\begin{aligned}
 * \quad & \lim_{h \rightarrow 0} \left\{ \begin{array}{l} (ph + 1 - rh)^2 \lambda h \\ + (ph - 1 - rh)^2 \mu h \\ + (ph - rh)^2 (1 - \lambda h - \mu h) \end{array} \right\} \\
 & = \lim_{h \rightarrow 0} \left\{ 1^2 \lambda h + 1^2 \mu h + \sigma(h) \right\} = \lambda + \mu
 \end{aligned}$$

The ruin probability is also increasing in $\lambda + \mu$, $\lambda - \mu$ held constant.

Finally, as a matter of notation, let c_t denote the rate of consumption at time t and let y denote the rate of fixed income. We may think of y as consisting of wages, insurance payments, and so forth.

2. The utility function.

According to the fundamental result of vonNeumann-Morgenstern utility theory [1, 4], if the individual can completely order all the

possible probability mixtures (distributions) of prospects and if his preferences satisfy (1) the Continuity axiom, and (2) the Dominance axiom then the probability mixtures and therefore the prospects can be assigned utility indicators, unique up to an increasing linear transformation, with the properties that (a) of two mixed prospects, the one preferred has the higher utility and (b) the utility of any mixed prospect is the expected value of the utilities of the prospects in the mixture. In the present context, a prospect is to be interpreted as a potential consumption history or time-path. Thus, if as we assume our consumer satisfies the above axioms, his chosen consumption policy may be viewed as maximizing the expected value of $U = U(c)$.

We shall assume that the utility functional takes the form

$$(1) \quad U(c) = \int_0^T e^{-\alpha t} u(c) dt, \quad \alpha \geq 0$$

where the function $u(c)$ is continuous with continuous first derivative.

Until the necessary and sufficient conditions for (1) can be stated, the exact meaning of our analysis will remain unclear. The additive utility function in its discrete time form

$$(1a) \quad U(c_1, \dots, c_n, \dots, c_N) = \sum_1^N a^{n-1} u(c_n), \quad 0 < a \leq 1$$

has been axiomatized for the case in which choices are made under conditions of certainty by Debreu [5]. The meaning of (1a) and a more general form

in which the utility discount is variable has also been studied by Koopmans [8] when $N = \infty$.

When choice takes place under uncertainty, the axioms leading to additivity under certainty have at the least to be recast, and perhaps supplemented, to account for the individual's preferences among probability mixtures of consumption histories. This is still an unsolved problem to our knowledge.*

* Markowitz [9, p. 254] has proposed an assumption to the effect that the individual's preferences among alternative probability mixtures of sub-histories running from $n=2$ to $n=N$ are invariant to the amount of consumption that befalls him in the first period. Markowitz concludes at once that $U(c_1^*, c_2, \dots, c_N)$ and $U(c_1^{**}, c_2, \dots, c_N)$, where c_1^* and c_1^{**} are different first-period consumptions, can differ only by a linear transformation, whence the utility function has the form $U(c_1, \dots, c_N) = u(c_1) + v(c_1)w(c_2, \dots, c_N)$.

It has been proved that, given axioms (1) and (2), every nonlinear, monotonic transformation of U changes the implied ranking of the probability mixtures of consumption histories. It does not seem to be an immediate step from this proposition to the different proposition that when the rankings of probability mixtures of sub-histories are the same despite different complementary histories then the sub-utility functions differ by some linear transformation. If this can be shown, the next step is to postulate the stationarity of preferences [8] so that the utility function becomes like (1a) but with a variable utility discount. An additional axiom will be needed to assure the constancy of the utility discount.

There remains the matter of the shape of the utility function $U(c)$ and the utility-of-consumption function $u(c)$. The same axioms which yield the Neumann-Morgenstern utility indicators also imply that U is bounded from

above and below.* Consequently $u(c)$ must be a bounded function. We

* A proof of boundedness may be found in [1, 4]. The proof uses a generalized St. Petersburg game the idea for which Arrow [1] credits to K. Menger.

denote the upper and lower boundary values by \bar{u} and \underline{u} respectively. Also, let \bar{c} denote the smallest consumption rate for which $u = \bar{u}$; of course, \bar{c} may or may not be finite.

In addition we shall assume that, among all consumption rates smaller than \bar{c} , the individual prefers a higher to a lower consumption rate, and that he is indifferent between all consumption rates greater than or equal to \bar{c} . Consequently $u(c)$ is strictly increasing in the interval $0 \leq c < \bar{c}$ and a constant function for $c > \bar{c}$ if $\bar{c} < \infty$. Finally we assume that the individual is uniformly averse to risk: For every pair of consumption histories $c(t)$ and $c(t)^0$ to which he is not indifferent, he will strictly prefer the certainty of the compromise history $\theta c(t) + (1-\theta)c(t)^0$ to the mixed prospect offering him $c(t)$ with probability θ and $c(t)^0$ with probability $1-\theta$, $0 \leq \theta \leq 1$. Thus $u(c)$ is strictly concave in the region where u is increasing. The indifference curves between present and future consumption therefore have the usual convexity.

3. Derivation of the basic equations.

We want to investigate the consumption policy or strategy - expressible by the "policy" function $c_t = c(x_t, T - t) \equiv c_t(x_t)$ - which, subject to the

stochastic process discussed, maximizes

$$(2) \quad J(c) = \int_{\text{over all paths } x(t)} \int_0^T e^{-\alpha t} u(c_t(x_t)) dt$$

or, as a matter of notation

$$(2a) \quad J(c) = \exp \int_0^T e^{-\alpha t} u(c_t(x_t)) dt = \exp U(c)$$

If we reverse the order of integration in (2) we get

$$(3) \quad J(c) = \int_0^T e^{-\alpha t} \left[\int_0^{\infty} u(c_t(x_t)) dF_t(x_t) \right] dt$$

where $F_t(x_t)$ is the cumulative density function at time t and will depend upon the previous consumption history. In simpler notation,

$$(3a) \quad J(c) = \int_0^T e^{-\alpha t} \left[\exp u(c_t) \right] dt$$

In (3a) c_t is not a deterministic function of time. However the consumption function $c_t = c(x_t, T-t)$ is deterministic and, on certain assumptions to be discussed below, continuous. The time path of the expected value of the state variable, $x(t)$, must then also be continuous. Therefore the bracketed expression in (3a) is also a continuous, integrable function of time.

In what follows we take our inspiration from Chapter 9 of Bellman [2], especially pp. 245-264. We do not know what effect the extension of that technique to this probabilistic problem has upon certain "rigorous details" not discussed there.

First of all, we note the additivity of expected lifetime utility and write

$$(4) \quad J(c) = \int_0^S e^{-\alpha t} [\exp u(c_t)] dt + \int_S^T e^{-\alpha t} [\exp u(c_t)] dt$$

where $0 < S < T$.

Second, we define

$$(5) \quad w(k, T) = \max_c J(c)$$

where the maximum is taken over all admissible consumption policies. The function $w(k, T)$ gives the relation between the expected value of lifetime utility and the individual's initial wealth and remaining lifetime given that he pursues an optimal consumption strategy. We shall assume in the analysis which follows that $u(c)$ and $w(k, T)$ have continuous first derivatives. We may think of $w(k, T)$ as the utility of wealth function of the optimizing consumer.

By (4), (5) and the "principle of optimality" [2] we can express the

following recursive relation

$$(6) \quad w(k, T) = \max_{c[0, S]} \left\{ \int_0^S e^{-\alpha t} [\exp u(c(t))] dt + e^{-\alpha S} \exp w(k(S), T-S) \right\}$$

where $k(S) = x(S)$, $0 < S < T$ and

$$w(k(S), T-S) = \max_{c[S, T]} \int_S^T e^{-\alpha t} [\exp u(c(t))] dt$$

After an elapse of time S the individual is faced with exactly the kind of problem facing him at $t=0$ with the difference that his initial capital at $t=S$ will be $k(S)$ and his remaining lifetime $T-S$. If his lifetime consumption strategy is optimal, certainly it is optimal from $t=S$ onward; consequently the optimizer's utility from S to T is $w(k(S), T-S)$, appropriately discounted. By virtue of the additive utility function, utility in the interval $[0, S]$ can be added to maximum discounted utility in the interval $[S, T]$ to form the maximand.

By virtue of the stochastic process described and (6), we can write for small $S \equiv h$,

$$(7) \quad w(k, T) = \max_{c_0} \left\{ \begin{array}{l} u(c_0)h + o(h) \\ + e^{-\alpha h} \lambda kh w(k+1+(pk+y-c_0)h, T-h) \\ + e^{-\alpha h} \mu kh w(k-1+(pk+y-c_0)h, T-h) \\ + e^{-\alpha h} [1-(\lambda+\mu)kh] w(k+(pk+y-c_0)h, T-h) \end{array} \right\}$$

where $o(h)$ denotes the group of terms of higher order than h , and c_0 denotes the initial rate of consumption.

Expanding the utility-of-wealth terms in the maximand of (7) as a Taylor series and impounding the second and higher order terms in $o(h)$, we get

$$(8) \quad w(k,T) = \max_{c_0} \left\{ \begin{array}{l} u(c_0)h + \sigma(h) \\ + e^{-\alpha h} \lambda kh [w(k+1,T) + (pk+y-c_0)hw_k(k+1,T) - hw_T(k+1,T)] \\ + e^{-\alpha h} \mu kh [w(k-1,T) + (pk+y-c_0)hw_k(k-1,T) - hw_T(k-1,T)] \\ + e^{-\alpha h} [1-(\lambda+\mu)kh] [w(k,T) + (pk+y-c_0)hw_k(k,T) - hw_T(k,T)] \end{array} \right\}$$

If we approximate $e^{-\alpha h}$ by the approximation $1 - \alpha h$, subtract the constant $w(k, T)$ from both sides of (8), and again collect terms on the order of h^2 , we find

$$(9) \quad 0 = \max_{c_0} \left\{ \begin{array}{l} u(c_0)h + \sigma(h) \\ + \lambda kh [w(k+1,T) - w(k,T)] - \mu kh [w(k,T) - w(k-1,T)] \\ + (pk+y - c_0) hw_k(k,T) - hw_T(k,T) - \alpha hw(k,T) \end{array} \right\}$$

If we divide the lefthand side of (9) by h and divide the maximand by h , the equality in (9) continues to hold. In the limit, as h approaches zero, the following partial differential equation is obtained.

$$(10) \quad 0 = \max_{c_0} \left\{ \begin{aligned} &u(c_0) + (pk+y-c_0) w_k(k,T) - w_T(k,T) - \alpha w(k,T) \\ &+ \lambda k[w(k+1,T) - w(k,T)] - \mu k[w(k,T) - w(k-1,T)] \end{aligned} \right\}$$

with the initial condition, for every k

$$(11) \quad w(k,0) = \underline{u} \quad (= \text{the lower bound of } u).$$

From (10) we read that if c_0 is optimal

$$(12) \quad w_T(k, T) = u(c_0) + (pk + y - c_0)w_k(k, T) - \alpha w(k, T) \\ + \lambda k[w(k+1,T) - w(k,T)] - \mu k[w(k,T) - w(k-1,T)]$$

This equation provides an interesting interpretation of the "marginal utility of time." If the individual is given a small reprieve, dT , then he will "earn" at least the expected utility $w(k, T)$ he would have earned without the reprieve but it will be deferred by dT ; rather than expecting to deplete his capital at T he will now plan to deplete it at $T + dT$. This deferment will cost him $\alpha w(k, T)dT$ by virtue of his time preference. However, offsetting this cost is the fact that to earn $w(k, T)$ he need have k not at $t=0$ but only when $t=dT$; in the meantime he can consume the income from his initial capital, k , or save it in order to have more capital than k at $t=dT$. His utility from this extra consumption is

$u(c_0)dT$; the future utility made possible by the additional saving is $(pk + y - c_0)dT w_k(k, T)$. Finally the last two expressions in (12) reflect the increased chance of a capital gain and a capital loss due to the increase, dT , in the period of time during which the initial capital is held.

We shall not pursue it here but the reader can see that equation (12) also provides an interpretation of the optimal "planned" saving rate, $(pk + y - c_0)$, which is a generalization of Keynes' explanation [11, p.547] of Ramsey's result when $\alpha = 0$, $T \rightarrow \infty$, and saving is riskless.

If c_0 attains an interior maximum, the derivative of the maximand in (10) is zero so that

$$(13) \quad w_k(k, T) = u'(c)$$

whence, by (12) and (13), at the optimum,

$$(14) \quad w_T(k, T) = u(c_0) + (pk + y - c_0)u'(c_0) \\ - \alpha w(k, T) + \lambda k[w(k+1, T) - w(k, T)] - \mu k[w(k, T) - w(k-1, T)]$$

Under conditions of certainty, $\lambda = \mu = 0$ so that the last terms in (14) drop out. In this case it is possible to use the method suggested by Bellman [2] for finding $c_0 = c(k, T)$, that is $c_t = c(x_t, T - t)$. Upon differentiating (13) with respect to T , (14) with respect to k and equating the two (Young's theorem) we obtain the following linear partial

differential equation in $c(k, T)$.

$$(15) \quad -u''(c_0) \left\{ \frac{\partial c}{\partial k} (pk - y - c_0) - \frac{\partial c}{\partial T} \right\} = u'(c_0) (p - \alpha)$$

An interesting point about (15) is that it is equivalent to the necessary Euler condition for a maximum

$$(16) \quad \frac{d}{dt} \left\{ -u'(c_t) \right\} = u'(c_t) (p - \alpha)$$

which may be verified by recalling that $c_t = c(x(t), T-t)$.

This method of solution does not, however, carry over to the case of uncertainty in which $\lambda, \mu > 0$. Traces of the last term in (14), which involve the unknown function $w(k, T)$, appear in the counterpart of (15) so that an exact solution for $c(k, T)$ in terms only of the known function $u(c)$ and the parameters p, α and so forth does not seem to be possible. But many of the properties of the solution which are of interest to the economist can be indicated at least in a qualitative way.

4. Properties of the consumption function $c(k, T)$.

Let us investigate first the "structure" of the optimal consumption policy, $c_0 = c(k, T)$.* Subsequently we will examine, in some special cases,

* Since $c(k, T)$ and $c(x_t, T-t)$ are the same functions, knowledge of c_0 for all values of k and T is equivalent to knowing the individuals' consumption policy for all t and x_t .

the effect on the consumption function of variations in some of the parameters such as the risk and return on capital.

(a) In investigating the consumption policy we will disregard the constraints on the consumption rate, specifically the nonnegativity of the consumption rate. After having learned something about the solution we will find the restrictions upon $u(c)$ and so forth which are needed to force the solution to satisfy the restraints.

First, we may rewrite (10) in the form

$$(10a) \quad w_{tt}(k, T) + \alpha w(k, T) - \lambda k[w(k+1, T) - w(k, T)] + \mu k[w(k, T) - w(k-1, T)] \\ = \max_{c_0} [u(c_0) + (pk + y - c_0) w_k(k, T)]$$

We have assumed that, in the interval $0 \leq c \leq \bar{c} \leq \infty$, $u(c)$ is increasing and strictly concave and that if \bar{c} is finite, $u(c) = \bar{u}$, $c \geq \bar{c}$. The only interesting case is that in which y is not so large and initial capital not so abundant that, if an optimal policy is pursued, the expected value of c_t equals \bar{c} , for every t ; in this case the marginal utility of wealth, $w_k(k, T)$, is greater than zero. Thus, in this case, the function in the maximand, $u(c_0) + (pk + y - c_0) w_k(k, T)$ is also strictly concave in c_0 . Consequently the maximizing c_0 is unique for every k and T . (The uniqueness of $w(k, T)$ is also simply shown [2, Ch. 9]).

We have also assumed that $u'(c)$ and $w_k(k, T)$ are continuous. Consequently the maximizing c_0 is a continuous function of k and T .

Equating the derivative of the maximand in (10a) to zero gives (12) which can be regarded as an implicit equation in c_0 , k and T . Taking the total differential of (12), we find

$$(17) \quad c_k(k, T) = \frac{w_{kk}(k, T)}{u''(c_0)}$$

$$(18) \quad c_T(k, T) = \frac{w_{kT}(k, T)}{u''(c_0)}$$

Since $u''(c_0) < 0$ by virtue of the second-order conditions for a maximum, $c_k(k, T) \geq 0$ if and only if $w_{kk}(k, T) \leq 0$ with the corresponding statement holding with respect to the strict inequality.

It has been shown [2] of discrete-time dynamic programming processes of this sort that $w(k, T)$ is (strictly) increasing and (strictly) concave in k , whatever the duration of the process, if $u(c)$ is (strictly) increasing and (strictly) concave. In other words, if the marginal utility of consumption is positive and diminishing, so is the marginal utility of wealth. Thus, if $u''(c) < 0$ for all values of c -- meaning that the upper utility bound is reached only in the limit as c approaches infinity -- then, since our continuous-time process can be viewed as the limiting case of a discrete-time process in which the length of each period goes to zero while the number

of discrete periods goes to infinity, $w_{kk}(k, T) < 0$ for all values of k and $T > 0$. In this case, $c_k(k, T) > 0$ for all values of k and $T > 0$. Noting that $w_k(k, T) \rightarrow 0^+$ as $k \rightarrow \infty$, by virtue of the boundedness of $w(k, T)$, it is easy to show that $c(k, T) \rightarrow \infty$ as $k \rightarrow \infty$.

Now we must admit the possibility that $u(c)$ is strictly concave only up to some finite \bar{c} . It is at least conceivable then that $w(k, T)$ is strictly increasing and strictly concave in k only up to some finite \tilde{k} . But we can say that if $c(k, T) < \bar{c}$ then $w(k, T)$ must be increasing in k , whence strictly concave in k . Therefore, $c_k(k, T) > 0$ if $c(k, T) < \bar{c}$; $c_k(k, T) \geq 0$ if $c(k, T) \geq \bar{c}$. In terms of k , if $0 \leq k < \tilde{k}(T)$ then $c(k, T) < \bar{c}$ and $c_k(k, T) > 0$; otherwise, $c(k, T) \geq \bar{c}$ and $c_k(k, T) \geq 0$. $\tilde{k} = \tilde{k}(T)$ need not be finite. For the problem to be interesting, initial capital must be scarce, i.e., $w_k(k, T) > 0$ whence, by (12), $u'(c) > 0$ so that $c_0 < \bar{c}$. Therefore $\tilde{k} > 0$ for all $T > 0$.

Turning now to $c_m(k, T)$ we see from (18) that its sign is that of $-w_{kT}(k, T)$. Thus $c_m(k, T) \leq 0$, as one would expect, only if $w_{kT}(k, T) \geq 0$.

Of course there is a simple economic argument that, setting $\underline{u} = 0$, $w_m(k, T) \geq 0$. The only economic argument we can surmise for $w_{kT}(k, T) \geq 0$ relies on what essentially is to be proved, namely that, given some increment, dT , to his lifetime, the individual will lower his consumption function,

$c_t(x_t)$, for all values of t , $0 \leq t < T$, in order to permit some consumption in the final, added interval, $[T, T + dT]$.

If we differentiate $w_{kT}(k, T)$ in (12) we get (18) again. The equality of $w_{kT}(k, T)$ and $w_{Tk}(k, T)$ suggests differentiating $w_T(k, T)$ in (14) with respect to k , which yields

$$(19) \quad w_{Tk}(k, T) = (p-\alpha)u'(c_0) + c_k(k, T) (pk + y - c_0) u''(c_0) \\ + \lambda[w(k+1, T) - w(k, T)] - \mu[w(k, T) - w(k-1, T)] \\ + \lambda k[w_k(k+1, T) - w_k(k, T)] - \mu k[w_k(k, T) - w_k(k-1, T)]$$

Clearly this equation fails to show that $w_{kT}(k, T) \geq 0$. Of course, as

$T \rightarrow 0$, $w_k(k, T) \rightarrow 0^+$. It can also be shown that, by virtue of the utility

discount rate α , $\lim_{T \rightarrow \infty} w_T(k, T) = 0$ for every k , whence

$\lim_{T \rightarrow \infty} w_{Tk}(k, T) = \lim_{T \rightarrow \infty} w_{kT}(k, T) = 0$. It is plausible therefore that

$w_k(k, T)$ is a monotone increasing function of T with upper bound $w_k(k, \infty) > 0$.

However, we see no way to demonstrate this.* To do so it seems that, at a

* In a discrete-time model of a process similar to this one we can show that $w_{n+1}'(k) > w_n'(k)$ [where n denotes the number of periods remaining in the process] for the special case in which the utility function is determined up to one parameter and bounded from above: $u(c) = \bar{u} - bc^{-\gamma}$, $b, \gamma > 0$. Perhaps this discrete-time result can be extended to any utility function satisfying the boundedness and concavity restrictions.

minimum, some restrictions on p , α , λ , and μ are needed.

Although we cannot analyze the effect of age on consumption over the entire age range, we can say something about the consumption function in two limiting cases, one which is applicable to the young consumer and the other applicable to the consumer approaching the death date. Let us consider the latter case first.

(i) In this stationary model, the approach of t , clock time, toward T , the death date, is of course equivalent to the approach of T , the lifespan, toward zero (for any given capital stock). The limiting value of c_0 as T approaches zero can be determined from the relation, for small T ,

$$(20) \quad w(k, T) = \max_{c_0} [u(c_0)T + o(T)]$$

It can be seen from (20) that as $T \rightarrow 0$, $c_0(k, T)$ approaches the value of c_0 which maximizes $u(c_0)$ without provision for the future.

If the function $u(c)$ approaches its upper bound only asymptotically, this means that $c(k, T-t) \rightarrow \infty$ as $t \rightarrow T$, for every value of k . Consequently, as $t \rightarrow T$, $x(t) \rightarrow 0$. At the death date all capital has been consumed. Of course it does not follow that if $c_t = c(x_t, T-t) \rightarrow \infty$ as $t \rightarrow T$, for any given x_t , then $\exp c_t \rightarrow \infty$ as $t \rightarrow T$. The capital stock, x_t , will presumably approach zero fast enough, in relation to the approach of t toward T , to cause $c(t)$ to approach a finite rate as $t \rightarrow T$.

If the utility function attains its upper bound, \bar{u} , with a finite consumption rate \bar{c} , then $c(k, T-t) \rightarrow \bar{c}$ as $t \rightarrow T$. This can be seen from (20) or from (10a), noticing that $w_k(k, T) \rightarrow 0$ as $T \rightarrow 0$ (by virtue of (11)).

It seems clear that, for all values of k , $w_k(k, T) = 0$ only in the limit. For any finite k and $T-t > 0$ there remains a finite chance of a capital loss even as large as k . Therefore, it seems reasonable to think that $w_k(k, T) > 0$ for all values of k , $T > 0$. On the other hand, $u'(\bar{c}) = 0$. If our conjecture is correct then $c(k, T) < \bar{c}$ for all values of k and $T > 0$.

For the paradox that follows, however, we need only the weaker statement that $c(k, T) \leq \bar{c}$ for all values of k and $T \geq 0$. Obviously, by (10a), $c(k, T) > \bar{c}$ could occur only if $w_k(k, T) = u'(c) = 0$. But in that event the optimum is not unique and it costs the individual nothing if we impose on him the convention that, if $w_k(k, T) = 0$, $c = \bar{c}$.

One consequence of the proposition that $c(k, T) \leq \bar{c}$ for every k and $T > 0$, while $c(k, T) \rightarrow \bar{c}$ as $T \rightarrow 0$, is the possibility that capital will not deliberately be depleted as the end of the process is approached. A look at Figure 1 shows that this can happen if a "run" of capital gains should bring the capital stock above the level $\hat{k} = \frac{\bar{c} - y}{p}$. In this case, despite the approach of t toward T and therefore the presumed, eventual, upward shift in the consumption function toward $c(k, T) = \bar{c}$ for every k , the individual's policy, $c(k, T) \leq \bar{c}$, $k \geq \hat{k}$, $T \geq 0$, implies the expectation

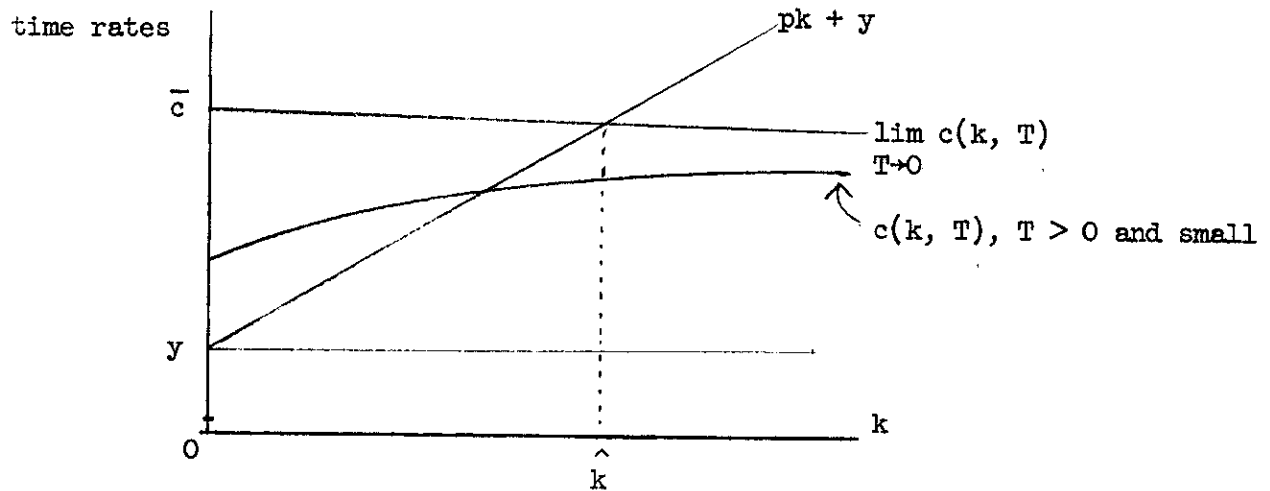


Figure 1

of continued accumulation of capital until $t + T$. Of course a "run" of capital losses may return the capital stock to a level below \hat{k} , which possibility lies behind the optimality of the precautionary accumulation. The probability of this is small and ever decreasing as the capital stock increases; the upper bound on this probability is $(\frac{\mu}{\lambda})^{k-\hat{k}}$ which is approached as $T \rightarrow \infty$ when $p = 0$. Once $k > \hat{k}$, capital may be trapped in this region. Thus terminal capital need not be zero.*

* This result depends upon the presence of uncertainty. If the outcome of any consumption strategy were known, the individual would never adopt a strategy calling for so much capital accumulation that total income would equal or exceed \bar{c} . Such a policy is dominated by another in which c_t reaches \bar{c} at the same point in time and is maintained at this level only through a continuous, positive (and increasing) rate of disinvestment until $t=T$.

If this observation is surprising, it is probably because we are unaccustomed to supposing that a finite consumption rate can produce utility satiation. An alternative conclusion is that \bar{u} is approached only asymptotically.

Before concluding this discussion of the optimal consumption policy for small T , we note that the optimal consumption rate is always positive for sufficiently large t , $t \leq T$. Thus the nonnegativity constraint is not violated for small T . And the consumption rate is not constrained from above if $k > 0$. If $k = 0$, then $c_t \leq y$ applies. This constraint may be troublesome in a broader analysis of the optimal consumption policy. It does not seem to affect the conclusion above that, in the former case, c_T equals some finite rate (which may exceed y) while x_T equals zero, and, in the latter case, c_T equals \bar{c} while x_T may or may not equal zero. To see that $c_T > y$ is possible, consider a "fixed" consumption policy $c(t) = \check{c} > y$ for all $t \geq 0$. If \check{c} is sufficiently large, the individual will eventually run out of capital at an unknown time which we may denote \check{T} . But $c_{\check{T}} = \check{c}$. Of course, if a new planning period were to begin at T with $k(T) = 0$, then the constraint would be potentially binding until some capital was accumulated.

(ii) Consider now the other limiting case in which $T \rightarrow \infty$. Of course, $w_T(k, T) \geq 0$; in this model the individual can always ignore any increment in his lifespan so as to earn the utility $\int_0^T u(0)dt = \underline{u} T = 0$ in addition

to $w(k, T)$. However, $w(k, T)$ is bounded by virtue of the boundedness from above and below of $u(c)$ and the utility discount rate, $\alpha > 0$. Consequently

$$\lim_{T \rightarrow \infty} w_T(k, T) = 0.*$$

* Note that $\frac{d}{dT} \left\{ \int_0^T \exp e^{-\alpha t} u(c) dt \right\} = \exp u(c_T) e^{-\alpha T}$.

This approaches zero as T approaches infinity whatever the expected value of the optimal c_T since $u(c)$ is bounded.

This result and equation (10a) yield

$$(21) \quad \begin{aligned} \omega w(k) - \lambda k[w(k+1) - w(k)] + \mu k[w(k) - w(k-1)] \\ = \max_{c_0} [u(c_0) + (pk + y - c_0) w'(k)] \end{aligned}$$

where $w(k)$ denotes the limiting value of $w(k, T)$ as $T \rightarrow \infty$.

Let us study the consumption function corresponding to (21) initially on the assumption of certainty, in which case $\lambda = \mu = 0$. Subsequently we can indicate the modifications which the presence of uncertainty requires.

First of all we note that, with $T = \infty$, the lefthand side of (19) is identically zero for all values of k , whence, if $\lambda = \mu = 0$

$$(22) \quad -u''(c) (pk+y-c_0) \frac{dc}{dk} = (p - \alpha) u'(c)$$

It is natural to assume that $p > \alpha$; otherwise, all capital is eventually consumed. So long as $c(k)$ is short of the smallest value for which $c(k) = \bar{c}$ -- a value of k previously denoted by \tilde{k} -- then $\frac{dc}{dk} > 0$, $u'(c) > 0$ and $u''(c) < 0$. Capital will be continuously accumulated until "bliss" is attained. Consequently $pk + y - c_0 > 0$ for every k . Obviously \bar{k} exists as a finite number only if \bar{c} is finite. We argued above that \tilde{k} may not be finite whether or not \bar{c} is finite because, in the presence of uncertainty, the individual will always want more capital as a cushion against capital loss. Under certainty, however, \tilde{k} is obviously finite and equal to $\frac{\bar{c} - y}{p}$ if \bar{c} is finite. Our attention is naturally restricted to the case in which $k < \tilde{k}$; there is no problem otherwise.

Second, it follows from (21) that for every value of c_0

$$(23) \quad w'(k) (pk + y - c_0) \leq \alpha w(k) - u(c_0)$$

with strict equality holding only for the optimal c_0 .

Since $pk + y - c_0$ is uniformly positive while $w'(k) > 0$

$$(24) \quad \alpha w(k) - u(c_0) > 0$$

for every c_0 given any value of k .

Putting (12), (23) and (24) together yields, at the optimum,

$$(25) \quad (pk + y - c_0) u'(c_0) = \alpha w(k) - u(c_0) > 0$$

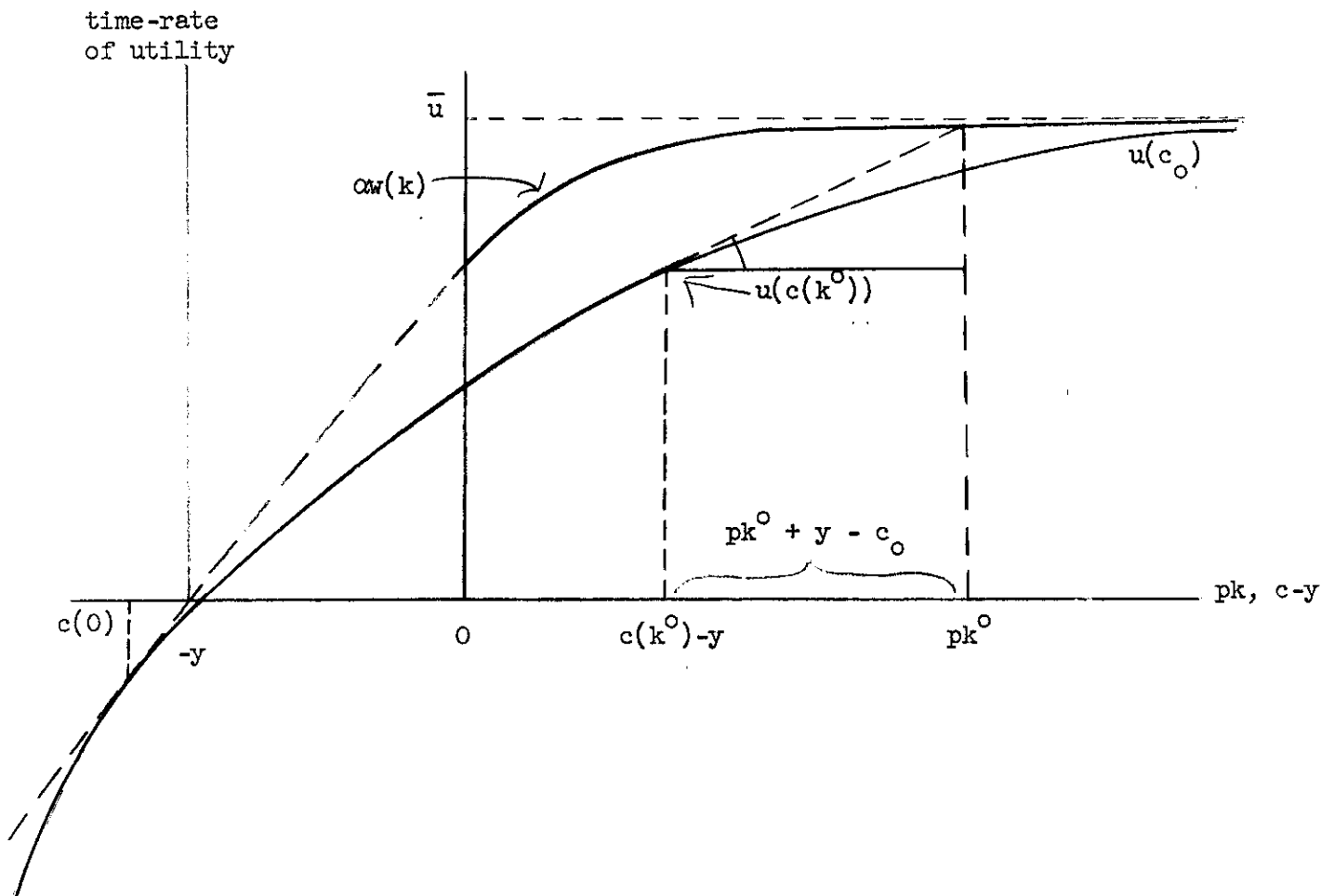


Figure 2.

The diagram in Figure 2 presents the information contained in (25) about the solution. Suppose the initial capital stock is k^0 . (25) tells us that the optimum saving rate is determined by rotating a downward sloping straight line from the point with abscissa k^0 and with ordinate $ow(k^0)$ until it is tangent to the $u(c_0)$ curve. Only that tangency point has the properties implied in (25).

The diagram tells us nothing new about the slope of the consumption function. We relied on the earlier result that $c_k(k, T) > 0$ -- i.e., $c'(k) > 0$ -- in deducing (25) and the diagram. However the diagram does make convenient the point that, if $u(c)$ is comparatively flat in the neighborhood of zero while $ow(0)$ is large and y small, then the solution will call for negative consumption if the initial capital stock is below a certain positive figure. The diagram exhibits this possibility. We cannot say how small $ow(0)$ may be except that it must be smaller than \bar{u} . The following condition is therefore sufficient to guarantee that the solution $c(k)$ will not violate the nonnegativity constraint.

$$(26) \quad u'(0) \geq \frac{\bar{u}}{y}$$

When uncertainty is readmitted, the following equation replaces (22)

$$(27) \quad -u''(c_0) (pk+y-c_0) \frac{dc}{dk} = (p-\alpha) u'(c_0) \\ + \lambda [w(k+1)-w(k)] - \mu[w(k)-w(k-1)] \\ + \lambda k[w'(k+1)-w'(k)] - \mu k[w'(k)-w'(k-1)]$$

Equation (27) shows $p > \alpha$ and $\lambda > \mu$ are not sufficient conditions to guarantee $pk + y - c_0 > 0$ for every value of k . By virtue of the concavity of $w(k)$, the coefficient of μ exceeds that of λ in (27).

Just how large we should require λ/μ to be is not made clearer by the presence of the last group of terms in (27). Since $w'(k)$ is diminishing, the coefficient of μk exceeds the coefficient of λk only if $w'(k)$ is convex; that is, only if the marginal utility of wealth (or consumption) diminishes everywhere at a diminishing (or at least nonincreasing) rate.

It is helpful to define

$$(28) \quad \begin{aligned} w(k+1) - w(k) &= w'(k) - \eta(k), \quad \eta(k) > 0 \\ w(k) - w(k-1) &= w'(k) + \epsilon(k), \quad \epsilon(k) > 0 \end{aligned}$$

The signs of η and ϵ follow from the concavity of $w(k)$. Then we can write (27) in terms of "expected" income and saving.

$$(27a) \quad \begin{aligned} -u''(c_0)[(p+\lambda-\mu)k + y - c_0] c'(k) &= \\ (p + \lambda - \mu - \alpha) u'(c_0) - (\lambda\eta + \mu\epsilon) - (\lambda\eta'(k) + \mu\epsilon'(k))k \end{aligned}$$

If $w'(k)$ is (strictly) convex, then $\eta'(k)$ and $\epsilon'(k)$ are (strictly) negative so that, on that account, the greater λ and μ the greater the righthand side if indeed $w'(k)$ is convex. On the other hand, η and ϵ are positive so that, as one would expect, the greater $\lambda + \mu$ the smaller

will be the righthand side of (27a) and therefore the greater must be $p - \alpha$ and $\lambda - \mu$ in order to insure that the righthand side of the equation is positive. Whatever the net effect of marginal changes in λ and μ , one can clearly make the righthand side of (27a) positive by choosing p large enough and μ and α small enough. Then $(p + \lambda - \mu)k + y - c_0$ will be uniformly positive for all values of k and the previous analysis expressed in terms of expected income and expected saving carries over. Nothing new can be added however. Equation (26) stands.

(b) Let us see now what can be inferred concerning the effects of parameter changes upon the consumption function $c(k, T)$. Consider first the case of small T .

(i) As already noted, we are unable to infer the effect of T , the life-expectancy parameter, upon the consumption function except in the two limiting cases. If capital is scarce initially, then $c(k, T) < \bar{c}$ for all values of k and $T > 0$. But as $T - t$ approaches zero or, equivalently, as T approaches zero, $c(k, T)$ approaches \bar{c} for all values of k . Thus $c_T(k, T) < 0$ for sufficiently small T . Figure 1 shows this.

Of course, such capital parameters as p , λ , and μ have no effect on $c(k, T)$ in this limiting case. In the limit, $c(k, T)$ is the value of c_0 which maximizes $u(c)$. The effects of capital risk and return upon $c_t = c(x_t, T - t)$ are felt entirely through their influence upon $x(t)$ and its time path.

(ii) In the other limiting case, in which $T \rightarrow \infty$, small changes in age have no effect. This follows from the convergence of $c(k, T)$ to $c(k)$ which in turn depends upon the convergence of $w(k, T)$ to $w(k)$.

A scrutiny of (21) indicates that the effect of marginal changes in the aforementioned capital parameters upon the consumption function, $c(k)$, cannot be inferred without knowledge of the effects of such changes upon the terms involving the unknown function $w(k)$. Even if we confine our analysis to structural comparisons of the sort $\lambda, \mu > 0$ versus $\lambda = \mu = 0$ and substitute $u'(c)$ for $w'(k)$, the effect of these global changes on $\alpha w(k)$ have still to be ascertained.

The presumption that setting $\lambda, \mu > 0$ makes $\alpha w(k)$ smaller, rather than helping to make the sign of the effect of this on $c(k)$ determinate, only makes the result ambiguous.

In the end we are driven to the special case in which $\alpha = 0$. This is the special case usually associated with Ramsey's paper. Ramsey was concerned to apply his model to society as a whole and in that application he felt that a utility discount was inappropriate.

When we let α approach zero it might be thought that $\alpha w(k)$ would also approach zero. However $w(k)$ will at the same time approach infinity. Of course, by our earlier result, for every positive value of α , $\alpha w(k)$ is finite and bounded from above by \bar{u} and from below by $u(rk)$. Therefore the limit of $\alpha w(k)$ as α approaches zero is also between these bounds. As a matter of fact, the limit is the upper bound, \bar{u} . This can be seen by considering

directly the case in which $\alpha = 0$ and by letting $T \rightarrow \infty$.

First we have to assume that p and λ are large enough and μ small enough that the righthand side of (27a) is greater than zero for all values of k for which $u'(c(k)) > 0$. Then the individual will always wish to accumulate capital, at every level of wealth, until \bar{u} becomes the optimal utility rate. In the quest for bliss, the individual may be ruined a dozen times. A zero capital stock is not an absorbing barrier due to the positive rate of fixed, nonwealth income y . If we suppose that $u(c)$ is attainable with a finite consumption rate \bar{c} , it seems plausible that capital must eventually grow without bound and that the probability that terminal x will exceed any arbitrary value approaches one as $T \rightarrow \infty$. We hope to be able to show this later. For the present, let us take this for granted. Then the expected value of $u(c_T)$ will approach \bar{u} as $T \rightarrow \infty$. Consequently $w_T(k, T) = \bar{u}$ in the limit, as T goes to infinity.*

From (10a) then, in place of (25), we obtain

$$(29) \quad [(p + \lambda - \mu)k + y - c_0] u'(c) = \bar{u} + (\lambda\eta + \mu\epsilon)k - u(c_0) > 0$$

Figure 3 diagrams the solution.

* If this is not true, no utility maximizing policy for the infinite-time problem will exist. Such a solution exists only if one can find a consumption policy which makes finite the difference between the utility of a lifetime of bliss and the expected value of actual lifetime utility. (Under conditions of certainty, the finiteness of \bar{c} is sufficient though unnecessary.) If the difference can be made finite, an optimal policy is one which minimizes the difference. See [11].

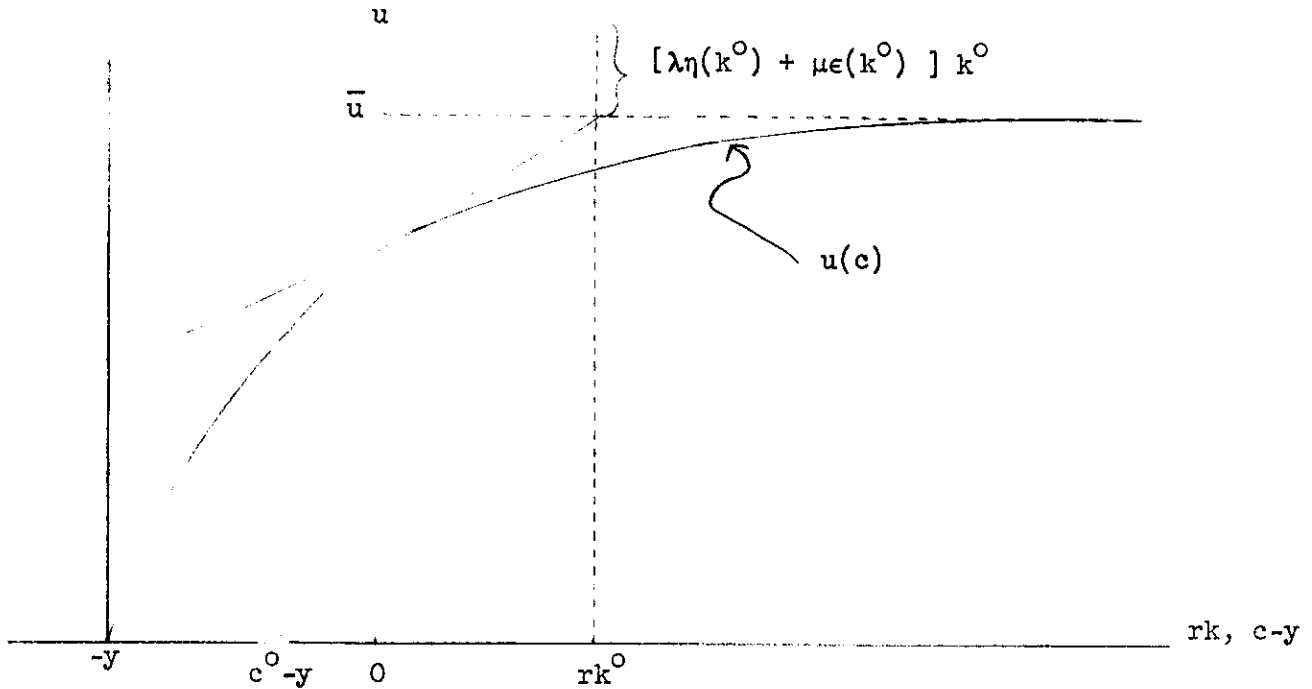


Figure 3

We can see that, in this special case, the presence of risk in the form of the parameters λ and μ acts to discourage consumption for every $k > 0$. Instead of dropping the straight line from the point with coordinates (rk^0, \bar{u}) as must be done in the limiting, no-risk case where $\lambda = \mu = 0$, the straight line must be dropped from a point directly above that point, $(rk^0, [\lambda\eta + \mu\epsilon]k^0 + \bar{u})$. However we cannot tell the effect of marginal increases in $\lambda + \mu$, $\lambda - \mu = \text{constant}$, since $\eta(k)$ and $\epsilon(k)$ depend on the unknown function $w(k)$ which depends in turn upon the parameters λ and μ .

Since an increase in risk discourages consumption, $k > 0$, one might expect an increase in the expected return, $r = p + \lambda - \mu$, $\lambda + \mu$ constant, to encourage consumption. In order to find only the pure "interest rate" effect of such a change, we must make a compensating change in y , $\Delta y = (\Delta r)k$, so as to keep constant both total expected income and initial capital (whence total risk).

For any given saving-consumption policy $c(k)$, this increase in capital's expected return can be expected to increase the amount of capital (and hence the rate of consumption and utility) that will exist at any time in the future. Presumably this "income effect" provides an incentive to consume more in the present. On the other hand, the greater expected productivity of capital creates through the "substitution effect" a stimulus to save more in the present in order to consume still more in the future. When few restrictions are placed on the utility function, the outcome of this conflict usually depends upon the values of the parameters in the case at hand.

It was a peculiar feature of the special undiscounted, untruncated version of Ramsey's model of riskless accumulation that these two opposing effects just balance for all parameter values. This unusual result can be verified by looking again at Figure 3 and equating λ and μ to zero. The reader can see that any combination of y , r and k which leads to the same initial rate of income corresponds to the same consumption rate. The interest rate makes a difference only insofar as it affects the rate of income.

From the same diagram we can see that this result extends to the case of uncertainty only if, when we increase r and decrease y so as to keep expected income constant, the terms $\eta(k)$ and $\epsilon(k)$ are unchanged. However the constancy of these terms cannot be shown and there seems to be no presumption that they will be unaffected by the variation described. Indeed, if the increase in r comes about through a reduction of μ with an equal increase in λ (so as to keep risk constant) then, even if $\eta(k)$ and $\epsilon(k)$ were unchanged, they would also have to be equal to insure an absence of effect on the optimum consumption rate. In fact, these terms are equal only if $w(k)$ is a quadratic function. In general, therefore, variations in the expected rate of return on capital do influence the optimum consumption rate, even when expected income and the riskiness of capital are held constant. The hypothesis that an increase in the expected return on capital has the effect of increasing rather than decreasing the consumption rate has to remain unsubstantiated.

The last parameter of interest is y , the rate of fixed, non-wealth income. The reader will have noticed the previous discussion's implication that, in the limiting, no-risk case, any combination of capital income, rk , and fixed income, y , which adds up to a given rate of total income yields the same consumption rate. Given r , this result also extends to the riskless, untruncated case with utility discounting.* In Keynesian terms, it means that

* Equation (22) shows that, provided $p - \alpha > 0$, capital increases continuously until bliss is reached. Imposing condition (26) on the utility function, we can say that the individual, at every point in time will lend up to the point where he is indifferent between an extra dollar of capital and an increment in y whose capitalized value, y/r , also equals one dollar. In this case, his utility of wealth, whence his consumption rate, will be a function only of his net worth, $k + y/r$. The effect of a dollar increase in y , therefore, must be $1/r$ times greater than an equal increase in k .

the marginal propensity to consume with respect to nonwealth income is equal to the marginal propensity to consume out of capital income, so that consumption is a function only of total income, perhaps the rate of interest, but not wealth (given income).

Once the rate of return on capital is regarded as uncertain, total income cannot be treated as homogeneous. The consumer will not be indifferent between a y increment whose value when capitalized by r has a value of one dollar and a dollar increment in his capital. This is because the former increment constitutes an additional stream of sure income while the latter increment produces an uncertain stream of income whose expected value is y . One would expect, then, that the former increment would be preferred by our risk averse consumer and that the former increment would have the greater effect on the consumption rate.*

* This is the case in the discrete-time models we have examined.

Inspection of Figure 3 shows that an increase in k and decrease in y which leave $rk + y$ constant will decrease the consumption rate if the ordinate of the point from which the tangency is taken is increased. The ordinate will increase unless the increase in k^0 is offset by a decline in $\eta(k)$ and $\epsilon(k)$. Unfortunately the behavior of these unknown functions eludes analysis.

If the marginal propensity to consume out of nonwealth income cannot be presumed to equal the marginal propensity to consume out of capital income, then we cannot reason, as we can in the riskless case, that the former propensity must be greater than zero because the latter is greater than zero (given r).

Positivity of the capital income propensity follows from (17) and the concavity of the utility of wealth function. Given the expected return on capital, an increase in capital income is equivalent to an increase in capital.

The same total differential of equation (1) which yields (17) and (18) also gives

$$(30) \quad \frac{\partial c(k, T)}{\partial y} = \frac{1}{u''(c_0)} \frac{\partial w_k(k, T)}{\partial y}$$

It is plausible that $\frac{\partial}{\partial y} w_k(k, T) < 0$. For sufficiently small k and y , ($T > 0$), $w_k(k, T) > 0$. For any such k , $w_k(k, T) \rightarrow 0$ as $y \rightarrow \bar{c}$. It seems likely therefore that $w_k(k, T)$ is a monotonic, decreasing function of y , $y \leq \bar{c}$.

As a final note which the reader may have been awaiting, we are aware that nothing has thus been said about the saving function. One would like to know, for example, whether saving is an increasing function of capital, given the individual's remaining lifespan. This is a matter of whether the inequality $c_k(k, T) < r$, is satisfied for all values of k . Nothing more can be said on this point even in the riskless version of Ramsey's model without some restrictions upon the shape of the utility function.

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