Yale University

EliScholar – A Digital Platform for Scholarly Publishing at Yale

Cowles Foundation Discussion Papers | Cowles Foundation

12-1-2017

Continuous Workout Mortgages: Efficient Pricing and Systemic Implications

Robert J. Shiller

Rafal M. Wojakowski

M. Shahid Ebrahim

Mark B. Shackleton

Follow this and additional works at: https://elischolar.library.yale.edu/cowles-discussion-paper-series

 Part of the Economics Commons

Recommended Citation

Shiller, Robert J.; Wojakowski, Rafal M.; Ebrahim, M. Shahid; and Shackleton, Mark B., "Continuous Workout Mortgages: Efficient Pricing and Systemic Implications" (2017). *Cowles Foundation Discussion Papers*. 161.
https://elischolar.library.yale.edu/cowles-discussion-paper-series/161

This Discussion Paper is brought to you for free and open access by the Cowles Foundation at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Cowles Foundation Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.

CONTINUOUS WORKOUT MORTGAGES: EFFICIENT PRICING AND SYSTEMIC IMPLICATIONS

By

Robert J. Shiller, Rafal M. Wojakowski, M. Shahid Ebrahim, and Mark B. Shackleton

December 2017

COWLES FOUNDATION DISCUSSION PAPER NO. 2116



COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Continuous Workout Mortgages: Efficient Pricing and Systemic Implications

Robert J. Shiller[1] Rafal M. Wojakowski[2] M. Shahid Ebrahim[3]
Mark B. Shackleton[4,5]

December 8, 2017

[1]Cowles Foundation for Research in Economics, International Center for Finance, Yale University, New Haven CT 06520-8281; Phone: +1 203 432-3726; Email: robert.shiller@yale.edu

[2]Corresponding author: Surrey Business School, University of Surrey, Guildford GU2 7XH, UK. Phone: +44(1483)683477; Email: r.wojakowski@surrey.ac.uk

[3]Durham University Business School, Durham, UK. Phone: +44(191)3345405, Email: m.s.ebrahim@durham.ac.uk

[4]Lancaster University Management School, Lancaster, UK. Phone: +44(1524)594131, Email: m.shackleton@lancaster.ac.uk

[5]We are grateful to Yildiray Yildirim, the Editor and two anonymous referees for their suggestions which greatly helped us improving the paper. We thank Cristian Badarinza, John Campbell, Peter Carr, Paul Cheshire, Paul Doukhan, John Duca, Chris Gerardi, Vernon Henderson, Christian Hilber, Ranko Jelic, Grzegorz Kula, Kevin Lansing, Gianluca Marcato, Doug McManus, Victor Murinde, David Newton, Ingmar Nolte, Henry Overman, Jean-Luc Prigent, Olmo Silva, Dogan Tirtiroglu, Radu Tunaru, Maisy Wong and seminar participants at Adelaide, Birmingham, Cergy, Lancaster, Warsaw, Kent, London School of Economics, Reading, St. Andrews and conference participants at Durham, AREUEA Reading, EFMA Rome, Bachelier Congress Brussels, EFA Lugano and ASSA Boston for their comments on earlier versions of this paper.

Abstract

This paper studies the Continuous Workout Mortgage (CWM), a two in one product: a fixed rate home loan coupled with negative equity insurance, to advocate its viability in mitigating financial fragility. In order to tackle the many issues that CWMs embrace, we perform a range of tasks. We optimally price CWMs and take a systemic market-based approach, stipulating that mortgage values and payments should be linked to housing prices and adjusted downward to prevent negative equity. We illustrate that amortizing CWMs can be the efficient home financing choice for many households. We price CWMs as American option style, defaulting debt in conjunction with prepayment within a continuous time, analytic framework. We introduce random prepayments via the intensity approach of Jarrow and Turnbull (1995). We also model the optimal embedded option to default whose exercise is motivated by decreasing random house prices. We adapt the Barone-Adesi and Whaley (1987) (BAW) approach to work within amortizing mortgage context. We derive new closed-form and new analytical approximation methodologies which apply both for pricing CWMs, as well as for pricing the standard US 30-year Fixed Rate Mortgage (FRM).

Keywords: Negative equity; House price index indexation; Repayment mortgage; Insurance; Embedded option to default; Prepayment intensity.

JEL: C63, D11, D14, D92, G13, G21, R31.

"Because [systemic] risk cannot be diversified away, investors demand a risk premium if they are exposed to it. There is no magic potion that can mitigate this kind of risk: somebody must hold this risk. Because crisis risk is endemic, it cannot be diversified away. To off-load this kind of risk, you have to purchase insurance, that is, induce others to take the risk off your hands by paying them a risk premium."

Jonathan Berk, in Fabozzi et al., 2010, page 127

1 Introduction

The ad hoc measures taken to resolve the subprime crisis involved expending financial resources to bail out banks without addressing the wave of foreclosures. These short-term amendments negate parts of mortgage contracts and question the disciplining mechanism of finance (Roubini et al. (2009)). Moreover, the increase in volatility of house prices in recent years (see Figure 1) exacerbated the crisis. In contrast to ad hoc approaches, we propose a mortgage contract, the Continuous Workout Mortgage (CWM), which is robust to downturns. We demonstrate how CWMs can be offered to homeowners as an *ex ante* solution to non-anticipated real estate price declines.

The Continuous Workout Mortgage (CWM, Shiller (2008b)) is a two-in-one product: a fixed rate home loan coupled with negative equity insurance. More importantly its payments are linked to home prices and adjusted downward when necessary to prevent negative equity. CWMs eliminate the expensive workout of defaulting on a plain vanilla mortgage. This subsequently reduces the risk exposure of financial institutions and thus the government to bailouts. CWMs share the price risk of a home with the lender and thus provide automatic adjustments for changes in home prices. This feature eliminates the rational incentive to exercise the costly option to default which is embedded in the loan contract. Despite sharing the underlying risk, the lender continues to receive an uninterrupted stream of monthly payments. Moreover, this can occur without multiple and costly negotiations.

There had been no major decline in home prices in the U.S.A. since the Great Depression. The subprime crisis was a rare event leading to negative equity for millions of house-

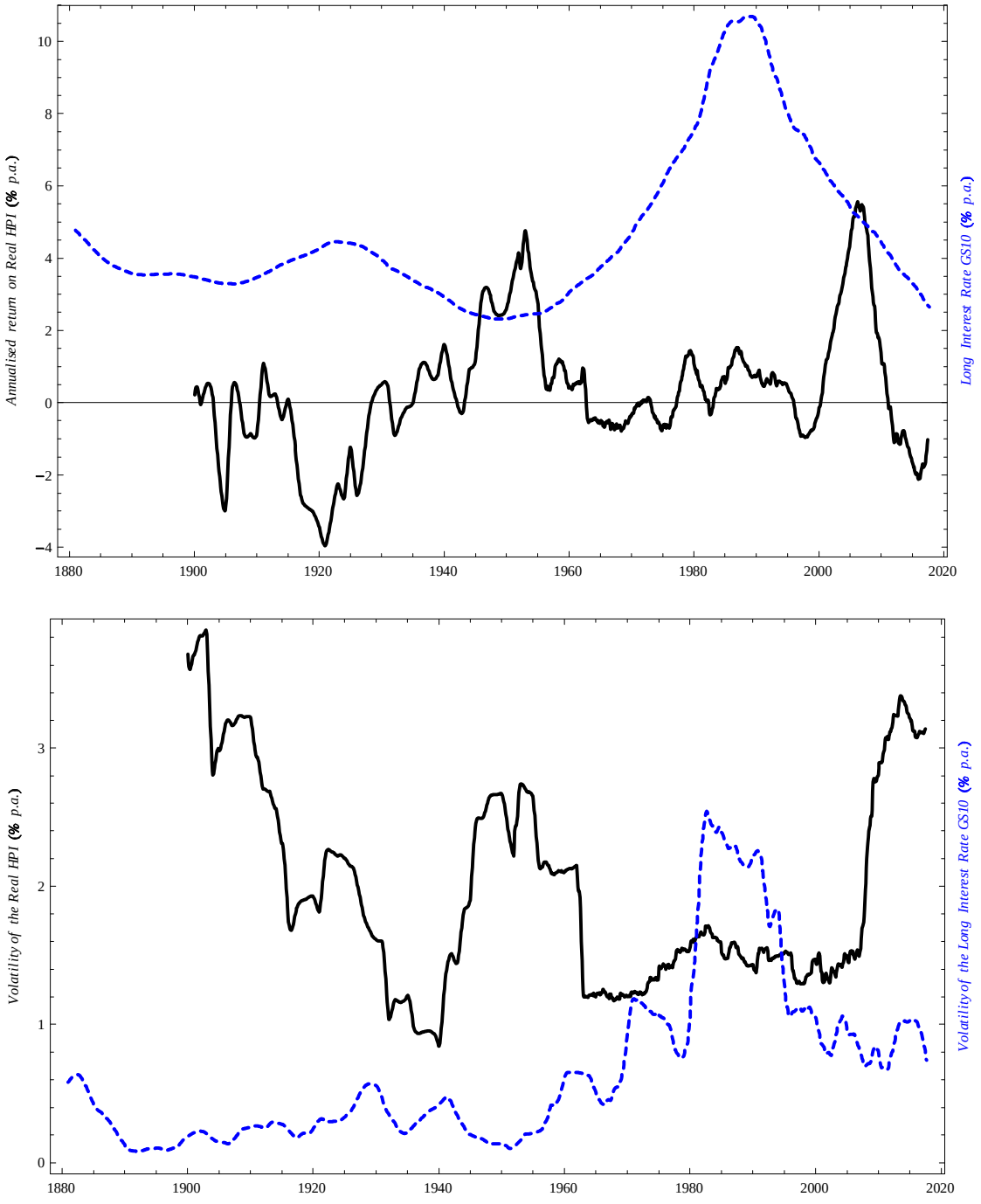


Figure 1: **US house price index (HPI) returns and volatility: decadal moving estimates.** Uses monthly data (post 1953) and interpolated monthly datapoints where finer granularity is not available (before 1953 for housing data). Each monthly estimate uses 120 monthly datapoints spanning the preceding 10 years to date. Housing data begins in 1890, interest rate data in 1871. Last observation: September 2017. Compiled data from various sources available at: `www.econ.yale.edu/~shiller/data/`

holds all at once. Households need to proactively plan ahead to mitigate the impact of catastrophic events with low probability as their impact on society can be devastating. Hyman Minsky (1992) emphasizes how the fragility of the financial system culminates in financial/banking crises. Furthermore, real estate crises are correlated with these financial crises (Piazzesi and Schneider, 2016) and exact a huge toll on the macroeconomy by decreasing the GDP (Renaud, 2003; Hoshi and Kashyap, 2004). These losses occurred because measures taken, such as deposit insurance, exacerbated banking crises instead of mitigating them (Demirgüç-Kunt and Detrajiache, 2002).

Prior to the current crisis, mortgages with repayment schedules contingent on house prices had not been considered. The academic literature, with the exception of Ambrose and Buttimer (2012), has not discussed their mechanics and especially their design. Shiller is the first researcher, who forcefully articulates the exigency of their employment. CWMs were conceived in[1] (Shiller (2008b) and Shiller (2009)) as an extension of the well-known Price-Level Adjusted Mortgages (PLAMs), where the mortgage contract adjusts to a narrow index of local home prices instead of a broad index of consumer prices. In their recent study, Ambrose and Buttimer (2012) numerically investigate the properties of Adjustable Balance Mortgages which bear many similarities to CWMs. Alternatively, Duarte and McManus (2011) suggest creation of derivative instruments written on credit losses of a reference mortgage pool. The model in Shiller et al. (2013) complements the more intricate one of Ambrose and Buttimer (2012). Unlike a numerical grid, it relies on a methodology which allows valuation of optional continuous flows in closed form (see e.g. Carr et al. (2000) and Shackleton and Wojakowski (2007)).

This paper proposes a solution that makes the housing finance system more robust to shocks through the employment of CWMs. The cost of the insurance stemming from the embedded put option in the CWM is quite low. This is due to economies of scale enjoyed by the lender through: (i) hedging by geographic diversification or resorting to futures contracts; and (ii) proactively underwriting CWMs by making the standards more stringent instead of lowering them after a huge run-up in home prices. The second technique is endorsed in Minsky (1992) and Demyanyk and Van Hemert (2011). If financial institutions are able to save the cost of insurance for rare calamities over several real estate cycles, i.e. invest in a fund not correlated with real estate to support higher capital reserves, it will prevent them from being dependent on taxpayers and government to bail

[1] See also Shiller Shiller (2003) for home equity insurance.

them out in crisis (as with deposit insurance).

Continuous workout mortgages need markets and indicators for home prices.[2] These markets and instruments already exist for lenders to hedge risks. The Chicago Mercantile Exchange (CME) for example, offers options and futures on single-family home prices. Furthermore, reduction of moral hazard incentives requires inclusion of the home-price index of the neighborhood into a repayment formula. This is to prevent moral hazard stemming from an individual failing to maintain or, worse, damaging the property in order to reduce mortgage payments.

Finally, we observe evidence showing the change in retirement trends (see Shiller (2014)). More people are planning to sell their house to consume the proceeds in retirement. Notably, the continuing care retirement community (CCRC) is a concept that is growing rapidly around the world. However, as a result of the drop in home prices there is a CCRC crisis in the US today and they have a lot of vacancies. If CWMs had existed, they could have helped to insure house values, thus preserve the welfare gain, and immunize retirement consumption from downside variations in the house price index.

In our approach, the homeowner can choose from a classic 30-year Fixed Rate Mortgage (FRM) or an CWM. We show that, for many households, Continuous Workout Mortgages could be a better form of home financing instrument than FRMs.

The next section describes the contracts: the standard Fixed Rate Mortgage and the Continuous Workout contract. In Section 3 we extend our closed form approach to include prepayments and defaults. In Section 4 we compute the equilibrium contract rates, the embedded options to default and we assess the impact of prepayment risk. Section 5 describes data and deals with calibration of house price index paths. In Section 6 we then conduct our simulations to numerically compare the expected utilities of Continuous Workout and Fixed Rate contracts. Section 7 concludes. Longer mathematical proofs and the floor flow option formula are collected in Appendices A and B, respectively.

[2] In a more general case not considered here CWMs could be made dependent on the levels of individual incomes and, to limit moral hazard, would also need indicators and "macro markets" for specific classes of incomes. See Shiller (1993).

2 The contracts

2.1 The standard Fixed Rate Mortgage (FRM)

A major invention introduced during years of Great Depression were *fully amortizing* repayment mortgages. Typically, these mortgages are analysed in a $T = 30$ year time horizon. For our purposes we use a continuous time representation. In place of the monthly payment we introduce a repayment flow rate R^{FRM} which is constant in time. Because interest rates have low volatility and are at their historical lows (see Figure 1), we motivate prepayments as unpredictable stochastic shocks. Consequently, rates for all maturities are constant and equal to r (see section 3 where we extend our setup to include prepayment and default risks). To achieve full repayment, the mortgage balance Q decreases and becomes zero at maturity. The mortgage balance is equal to the amount owed to the lender at time t and can be computed as the present value of remaining payments

$$Q\left(R^{FRM}, t\right) = \int_t^T e^{-r(s-t)} R^{FRM}\, ds = \frac{R^{FRM}}{r}\left(1 - e^{-r(T-t)}\right) . \tag{1}$$

Differentiating (1) with respect to t we obtain the dynamics of Q

$$\frac{dQ}{dt} = -R^{FRM} e^{-r(T-t)} = -R^{FRM}\left(1 - \frac{rQ}{R^{FRM}}\right) = rQ - R^{FRM} . \tag{2}$$

It is deterministic and is described by the ordinary differential equation (2) with terminal condition $Q\left(R^{FRM}, T\right) = 0$. It says that the balance Q grows at rate r but is progressively repaid by a *constant* mortgage payment flow R^{FRM}. The mortgage can be repaid in full only if the time derivative in (2) is negative, meaning the balance decreases. That is, when the interest accumulation *inflow* to principal, rQ, is *lower* than the coupon *outflow* R^{FRM}. In other words the net flow is negative: $rQ - R^{FRM} < 0$ i.e. funds are transferred back to the lender and the mortgage is being repaid. The advantage and simplicity of FRM is that the required repayment flow parameter R^{FRM} can be computed and set once and for all at the beginning, when the loan contract is signed. To compute R^{FRM} we only need to know three numbers: the initial balance Q_0 (the loan amount), the maturity T of the loan

and the discount rate r. The initial equilibrium condition $Q_0 = Q\left(R^{FRM}, 0\right)$ gives

$$R^{FRM} = \frac{Q_0}{A\left(r,T\right)}\,, \tag{3}$$

where $A\left(r,T\right)$ is the annuity

$$A\left(r,T\right) = \int_0^T e^{-rt}dt = \frac{1-e^{-rT}}{r}\,, \tag{4}$$

i.e. the present value at rate r of a unit flow terminating after T years.

The simplicity of the FRM becomes problematic when a house develops negative equity. This was the case of many households in the US in years following the burst of the housing bubble in 2007. This is illustrated on Figure 2. A $ 500 000 property is financed for 30 years and interest rates are 5%. However, a sharp price decline puts the house in negative equity. House values are well below the line representing the balance. In this example, although the balance in an FRM is set to be fully repaid at maturity, the house is underwater until year 11. In this example the return on original house price is then again negative around year 20 and then year 28, because the value of the property is again below the initial purchase price H_0. However, as most of the initial balance have been repaid, the house is not in negative equity by that time.

2.2 The repayment Continuous Workout Mortgage (CWM)

In this section we argue that a standard repayment FRM can be improved by reducing repayments in bad times. This is illustrated on Figure 3. In this example the household benefits from a substantial reduction, proportional to the drop in the house price, up until year 11 and then again repayments are marginally reduced in years 20 and 28.

For our framework we need a house price index ξ_t which is available for all $t \in [0,T]$ and, without loss of generality, an index which is normalized to one initially, i.e. $\xi_0 = 1$. If HPI_t is the *absolute* value at time t of a real-world house price index (such as the Case-Shiller index in the US or the Countrywide or Halifax index in the UK), we can define ξ_t

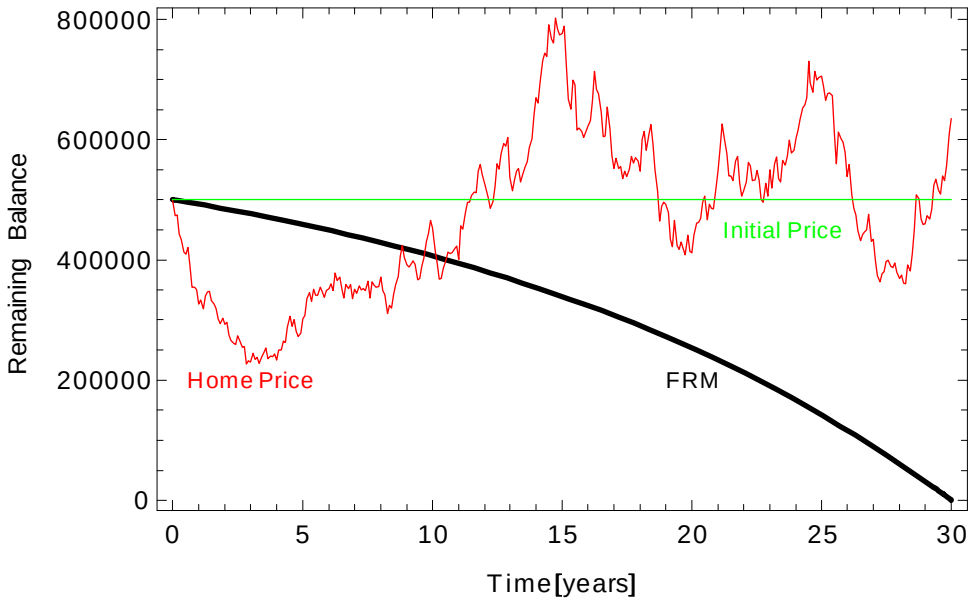


Figure 2: Financing a $500000 property for 30 years with a Fixed Rate Mortgage (FRM) when interest rates are 5%. The drop in home prices puts the house in **negative equity.**

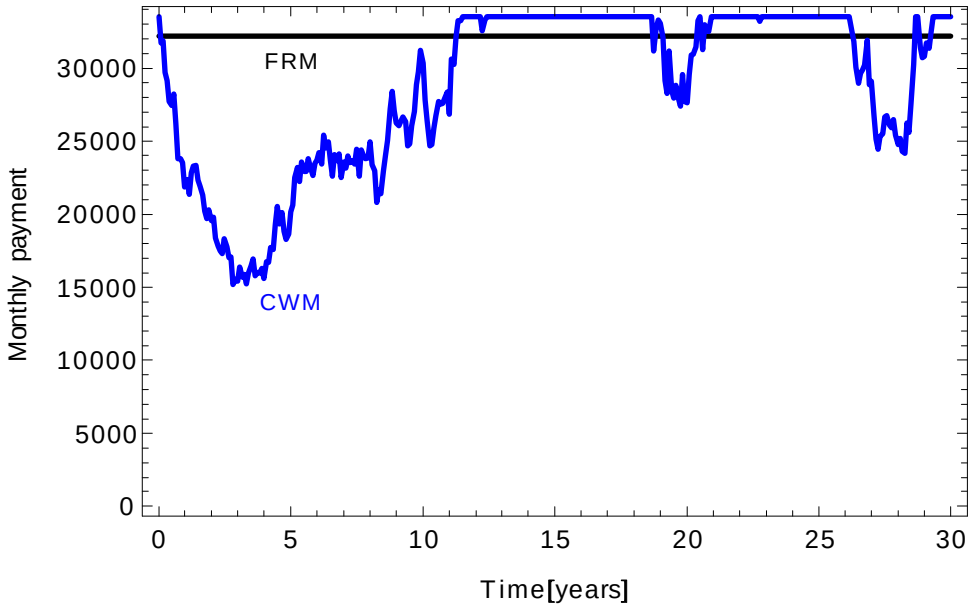


Figure 3: A scenario where the insurance cost of a CWM is less than its benefit: monthly payments are reduced in bad times.

as

$$\xi_t = \frac{HPI_t}{HPI_0} \qquad \Longrightarrow \qquad H_t = H_0 \xi_t. \tag{5}$$

That is, for a house initially worth H_0, the quantity $H_0\xi_t$ is a *proxy* for the value of the house at time $t > 0$.

In good times, when house prices are high, the CWM behaves identically to an FRM. Annual repayments proceed at the maximal annual rate equal to $\bar{R}^{CWM}$.[3] In bad times, the repayment flow $R^{CWM}(\xi_t)$ of a Continuous Workout Mortgage (CWM) depends on the house price index and becomes lower than $\bar{R}^{CWM}$. It decreases when the house price index ξ_t decreases

$$R^{CWM}(\xi_t) = \bar{R}^{CWM} \min\{1, \xi_t\} = \bar{R}^{CWM}\left[1 - (1-\xi_t)^+\right]. \tag{6}$$

Akin to the *constant* annual repayment rate R^{FRM} of an FRM, the *maximal* annual repayment rate $\bar{R}^{CWM}$ of a CWM is an endogenous parameter (see Figure 4), which must be specified upon origination of the mortgage contract.

When $\xi_t < 1$ the return on initial housing purchase is negative and the mortgage risks negative equity if the decline in value is higher than funds repaid so far. In the highly unlikely but not impossible, limiting scenario, when the house price index drops very close to zero $\xi_t \to 0$, the CWM contract automatically produces a full workout i.e. $R^{CWM}(\xi_t) \to 0$. That is, the lender absorbs all losses for as long as ξ_t remains close to this lower bound. This can be inferred from the shape of the payoff function illustrated on Figure 4. If the collateral becomes worthless, the homeowner should be fully compensated and shouldn't pay any interest or repay any principal. In other words there is full insurance against house price declines. In more realistic, intermediate situations when the price declines but not by too much, this contract still provides automatic compensation to homeowners. Repayments of principal and the interest are reduced proportionally to the index. As a consequence it is no longer possible to express the current balance as a function of future payments. In fact, for CWMs, (1) no longer holds. Two quantities emerge which will almost surely deviate when $t > 0$:

[3] In our notation we use a 'bar' over the letter R to remind us that $R^{CWM}(\xi)$ cannot become greater than $\bar{R}^{CWM}$, which caps the annual repayment rate, so as to always have $R^{CWM}(\xi) \leq \bar{R}^{CWM}$.

1. The actual (random) balance to date, which is equal to the present value of the initial balance minus payments done so far

$$Q_t^{CWM-} = Q_0 e^{rt} - \int_0^t R^{CWM}(\xi_s)\, e^{r(t-s)} ds\,; \tag{7}$$

2. The expected payments to occur in future

$$Q_t^{CWM+} = E_t\left[\int_t^T R^{CWM}(\xi_s)\, e^{-r(s-t)} ds\right]. \tag{8}$$

The balance Q_t^{CWM-} is a path-dependent quantity because it is based on past values of the house price index. It can go up as well as down, reflecting the history $\{\xi_s\}_{s=0}^t$ of the house values observed up to time t. If we know this history we can compute the balance as

$$Q_t^{CWM-} = \left[Q_0 - \bar{R}^{CWM} A(r,t)\right] e^{rt} + \bar{R}^{CWM} \int_0^t (1-\xi_s)^+ e^{r(t-s)} ds\,. \tag{9}$$

In particular we note that

$$\left[Q_0 - \bar{R}^{CWM} A(r,t)\right] e^{rt} \le Q_t^{CWM-} \le Q_0 e^{rt}\,, \tag{10}$$

so for $\bar{R}^{CWM} > R^{FRM}$ the lower bound can become negative when $t \to T$ (see Appendix A.1). This occurs when home prices remained high, the workout did not kick in, the loan fraction of the package was repaid earlier and the mortgagee collected the insurance premium over the lifetime of the contract. However, following a sharp decline in house prices the balance can exponentially increase, as shown by the upper bound. Because payments must stop at maturity, the lender bears risk which he was paid to hold. The borrower has been paying a premium to compensate for this shortfall risk.

Repayments under the CWM contract vary randomly with house prices. The expected payments quantity Q_t^{CWM+} should therefore be used for valuation purposes as they reflect the reality a borrower is facing at a given point in time. It also has the advantage of being forward looking, and thus not path-dependent. However, for computations, we need to assume some distribution of the future value of the index $\xi_s : s > t$. Historically, house prices in the mortgage pricing literature had been assumed to be log-normally dis-

tributed (see e.g. Kau et al. (1985), Kau et al. (1992), Azevedo-Pereira et al. (2003), Sharp et al. (2008), Ambrose and Buttimer (2012)) and here we subscribe to this trend.

At origination $t = 0$, we require that the initial balance and the expected payments are equal

$$Q_0^{CWM-} = Q_0^{CWM+} = Q_0 \,. \tag{11}$$

This equilibrium condition is necessary to compute the endogenous repayment flow $\bar{R}^{CWM}$ of a CWM. Subsequently, the fair value of the *insurance premia*, such as $\bar{R}^{CWM} - R^{FRM}$ embedded in a CWM, can also be obtained as well as the CWM and FRM *equilibrium contract rates* (see Section 4).

Proposition 1 *The expected present value of future payments of the Continuous Workout Mortgage (CWM) at time $t \in [0, T]$ is equal to*

$$Q_t^{CWM+} = \bar{R}^{CWM} \left[A\left(r, T-t\right) - P\left(\xi_t, 1, T-t, r, \delta, \sigma\right) \right] \,. \tag{12}$$

*where $P \geq 0$ is the **floor** (i.e. collection of* put *options for all maturities from $t = 0$ to $t = T$) on continuous flow ξ_t capped at 1, expressed in closed form (see Appendix B), and δ, σ are the service flow and the volatility of the house price.*

Proof. See Appendix A.2. ■

The payment flow $\bar{R}^{CWM}$ which appears in (12) is a constant parameter which is computed at origination ($t = 0$) for the duration of the contract. It should not be confused with the mortgage payment $R^{CWM}\left(\xi_t\right)$ given by equation (6) which is a function of randomly changing adjusted house price level ξ_t. Mortgage payments $R^{CWM}\left(\xi_t\right)$ *decrease* when home prices decline, whilst $\bar{R}^{CWM}$ is fixed ex ante.

Proposition 2 *The repayment flow of a CWM is capped by $\bar{R}^{CWM}$, which is an endogenous parameter and can be computed explicitly as*

$$\bar{R}^{CWM} = \frac{Q_0}{A\left(r, T\right) - P\left(1, 1, T, r, \delta, \sigma\right)} > R^{FRM} \,. \tag{13}$$

Proof. Set $t = 0$ in (12) and solve for $\bar{R}^{CWM}$. Inequality obtains because $P\left(1, 1, T, r, \delta, \sigma\right) > 0$ i.e. a sum of put options has strictly positive value for $T > 0$. ■

The endogenous parameter $\bar{R}^{CWM}$ provides the cap on repayment flow under CWM. Clearly, if the mortgage is fairly priced, parameter $\bar{R}^{CWM}$ must be greater than R^{FRM} because the issuer of the insurance must be rewarded. We can think of the difference $\bar{R}^{CWM} - R^{FRM} > 0$ as equal to the price of the insurance to be paid in good states of nature for the continuous workouts to be automatically provided should bad states occur. If there were no risk to be compensated for ($\sigma \to 0$), the present value of insurance puts represented by the floor function P would become zero and $\bar{R}^{CWM}$ would equal R^{FRM}. This is illustrated on Figure 5, where we also consider the case of a partial guarantee which only covers $\alpha = \frac{1}{2}$ of the loss. Equation (13) is paramount for potential originators of continuous workouts with repayment features. This pricing condition helps in evaluating the maximal annual payment for this mortgage. A broker can instantly compute this quantity on a computer screen and make an offer to a customer. Finally, a current estimate could be used and a regulator could mandate an upper bound on volatility σ.

Furthermore, for an "in-progress" CWM mortgage (at time $t > 0$), it is interesting to analyse whether its expected present value of future payments is lower or higher than the balance Q_t^{FRM} of an otherwise identical, standard 30-year FRM. Using (1) and (4) the latter can be written as

$$Q_t^{FRM} = Q\left(R^{FRM}, t\right) = Q_0 \frac{A(r, T-t)}{A(r, T)} \,. \tag{14}$$

Similarly, using (12) and (13) we can represent the expected payments of a CWM as

$$Q_t^{CWM+} = Q_0 \frac{A(r, T-t) - P\left(\xi_t, 1, T-t, r, \delta, \sigma\right)}{A(r, T) - P\left(1, 1, T, r, \delta, \sigma\right)} \,. \tag{15}$$

Clearly, when house prices are low $\xi_t < 1$ and t is relatively low we must have $Q_t^{CWM+} \leq Q_t^{FRM}$. This is because the insurance pays off. However, when house prices are high $\xi_t > 1$ and closer to maturity $t \to T$, we should expect a reversal $Q_t^{CWM+} \geq Q_t^{FRM}$. In this case the insurance puts expire out of the money but a CWM homeowner still has to pay the insurance premia, which results in a slightly higher remaining balance. However, even for very high values of the home price index, expected payments Q_t^{CWM+} are always capped from above by

$$\bar{Q}_t^{CWM} = Q_0 \frac{A(r, T-t)}{A(r, T) - P\left(1, 1, T, r, \delta, \sigma\right)} \geq Q_t^{CWM+} \,, \tag{16}$$

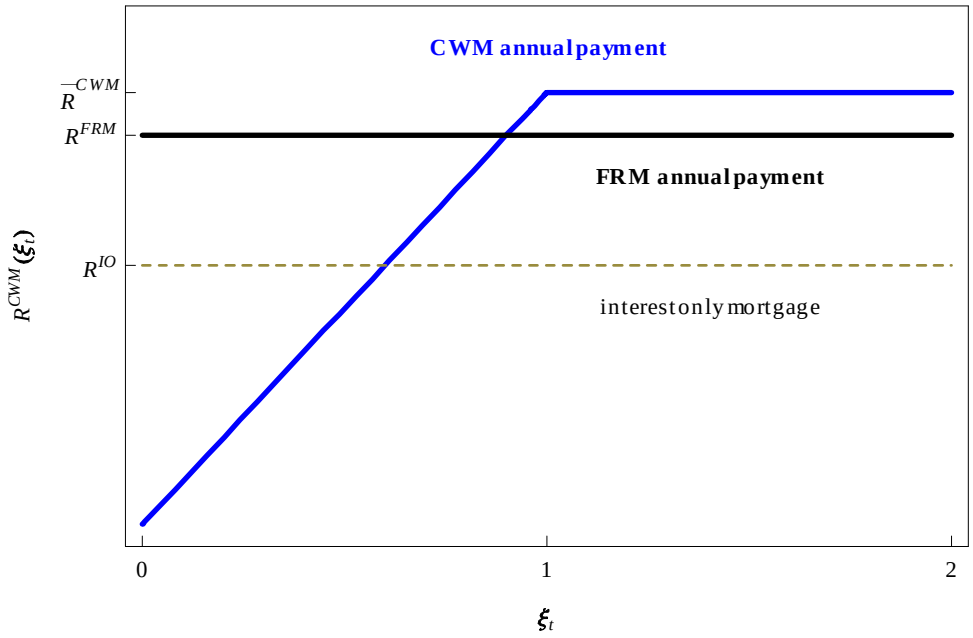


Figure 4: Continuous Workout Mortgage (CWM): The repayment flow $R^{CWM}(\xi_t)$ as a function of the house price index ξ_t.

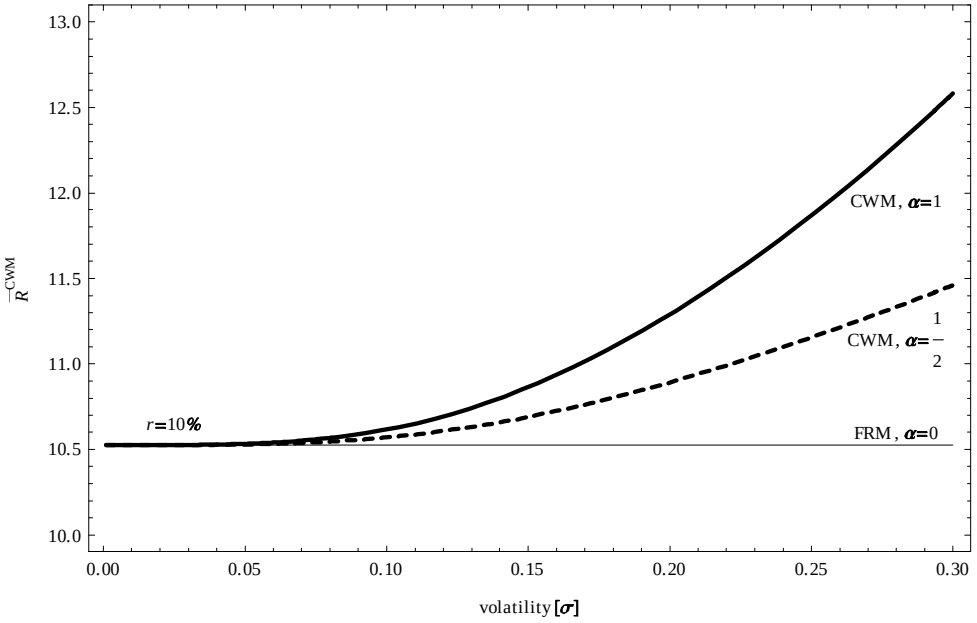


Figure 5: Maximal CWM payment $\bar{R}^{CWM}$ as a function of risk σ. Full workout (CWM, $\alpha = 1$, thick line), reduced workout (CWM, $\alpha = \frac{1}{2}$,dashed line), no workout (FRM, $\alpha = 0$, thin line). Riskless rate $r = 10\%$, maturity $T = 30$ years, service flow rate $\delta = 4\%$. Note: The annual payments R are above $r = 10\%$ because both FRM and CWM mortgages are of repayment type i.e. R is composed of annual interest and annual repayment.

which has been computed as $\lim_{\tilde{\varsigma}\to\infty} Q_t^{CWM+}$. Note that $\bar{Q}_t^{CWM}$ and Q_t^{FRM} are similarly shaped, concave functions of time t, both decreasing to zero at maturity T. However, $\bar{Q}_t^{CWM}$ always dominates Q_t^{FRM}. The difference $\bar{Q}_t^{CWM} - Q_t^{FRM} > 0$ is relatively small and represents a "cushion" area above the standard repayment schedule Q_t^{FRM}. It is in this area where the insurance premium is collected in good times.

Figures 6 and 7 summarize the key differences between FRM and CWM. For FRMs, there is a fixed repayment schedule. In contrast, the repayment schedule for a CWM has a *fixed upper bound*. This upper bound is above (but very close to) the FRM's schedule, forming a "cushion" (see Figure 6). The area immediately below the upper bound $\bar{Q}_t^{CWM}$ (the "cushion") will get quickly "populated" by Q_t^{CWM+} in good times. With reference to FRMs we could therefore say that this extra cushion needs to be there for CWMs to collect the insurance premia in good times. However, in bad times, the CWM's payments outstanding can dive well below FRM's outstanding balance (which is fixed for a given time t). This self adjustment mechanism is key to remain away from negative equity, avoid the house becoming underwater and prevent foreclosures automatically (see Figure 7).

We also investigate the behaviour of a CWM under alternative path scenarios. In the first scenario (Figure 8) prices drop and stay at 40% of initial value until maturity. In the second scenario (Figure 9) prices also drop to 40% of initial value but then jump and stay at \$600000 which represents 110% of the initial value. In both cases expected payments evolve similarly, progressively decreasing to zero at maturity. Obviously, the starting point Q_t^{CWM+} for the second scenario is much higher than for the first. However, this starting point must always lie below the upper bound $\bar{Q}_t^{CWM}$, no matter how much prices appreciate in future.

3 Prepayments and Defaults

Because our approach is analytic, it is relatively straightforward to extend it to include prepayments and defaults.

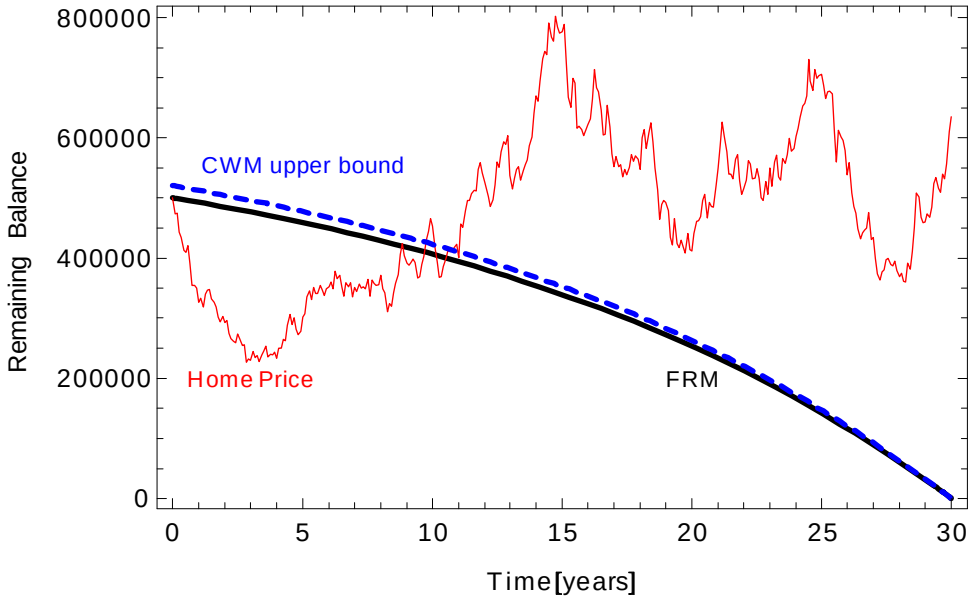


Figure 6: The CWM's upper bound on expected future payments v.s. FRM's scheduled remaining balance.

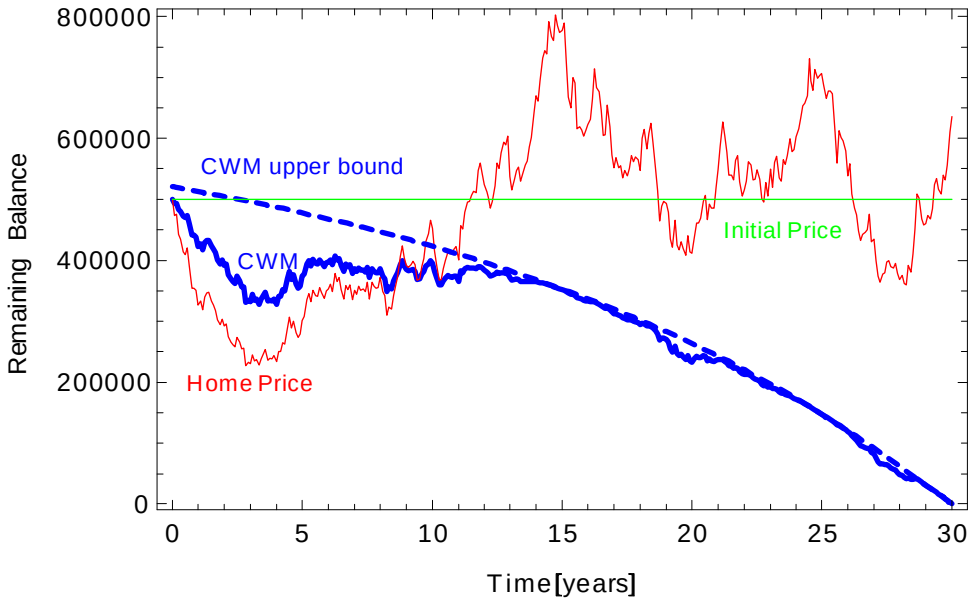


Figure 7: Expected future payments of a CWM, Q_t^{CWM+}, as a function of the current house price levels measured by a Home Price index ξ_t and Initial Price H_0. The dashed line is the CWM upper bound $\bar{Q}_t^{CWM}$.

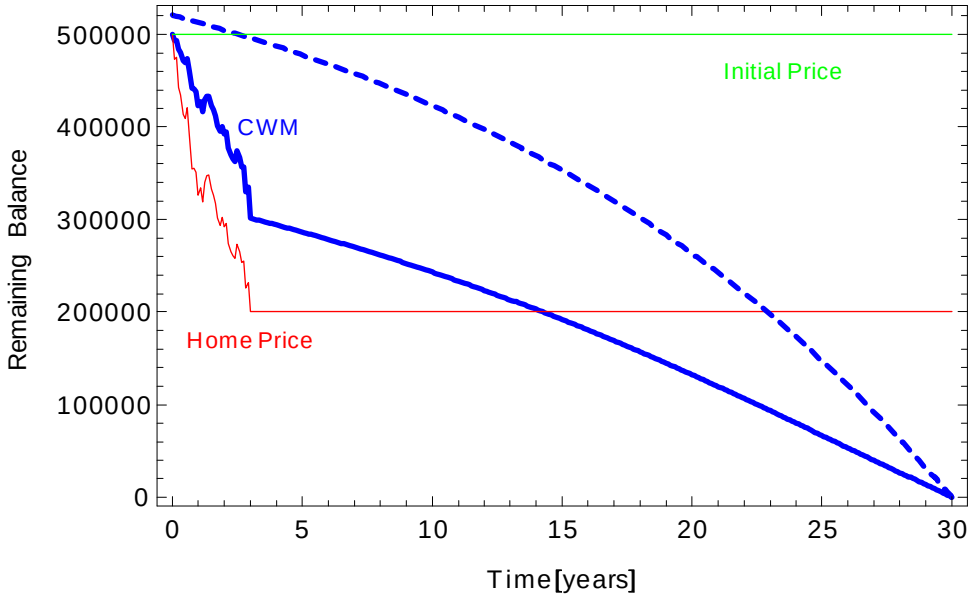


Figure 8: The expected future payments of a CWM when house prices decrease to 40% of initial value and then remain at this level until term.

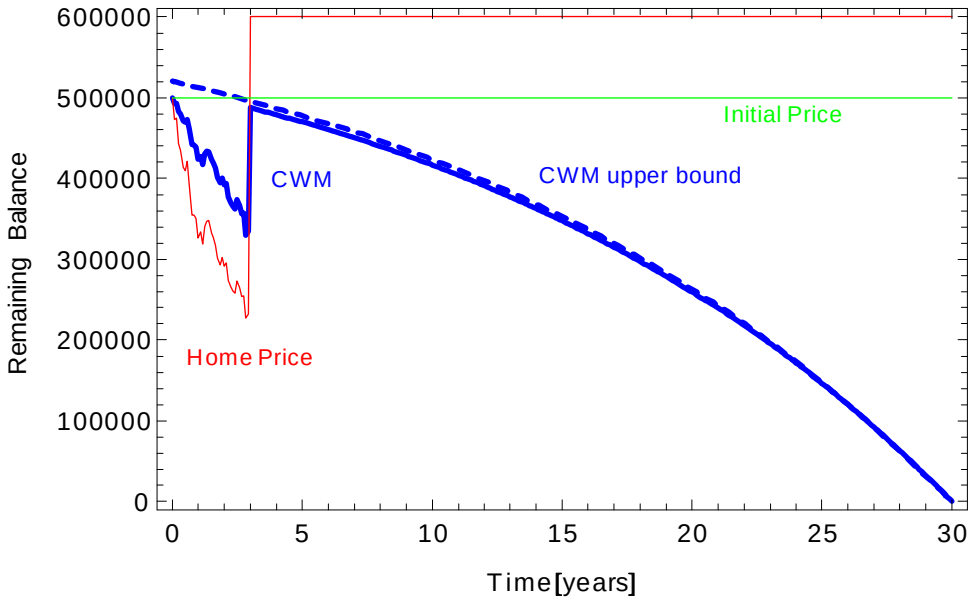


Figure 9: The expected future payments of a CWM when house prices first decrease to 40% of initial value, then instantly appreciate to 110% of initial value and remain at this level until term.

3.1 Prepayments

In the traditional approach to pricing prepayments non-constant interest rates and their effect on prepayment is examined. In particular, the rational early exercise of the American-style prepayment option embedded in a mortgage contract. Modelling these require sophisticated techniques for numerically solving an options pricing problem. In a well-known study Fama (1990) highlights the influence of monetary policy on the behavior of interest rates and thus term structure. It follows that, in the current economic environment, the prevailing view is that the yield curve will continue to flatten and that the Federal Reserve Board will not raise interest rates drastically. This is elaborated, for example, in the following recent articles:

> *"As for skyrocketing long rates, that seems unlikely during the current economic cycle. [...] Since inflation is tamed when real rates rise enough to choke off economic expansion, the lower is the nominal rate necessary to restrain it."*
>
> *"Rates are unlikely to skyrocket, despite pundits predictions"*
>
> *Scott Minerd, FT, July 18, 2017, p. 28*
>
> *"Another month, another impressively low unemployment number, but another flaccid inflation print. No wonder the US Federal Reserve is baffled. Modern macroeconomic theory depends upon the famous Phillips curve, and its pressure cooker model of the inflationary process. Let the economy run too hot, and inflation is sure to follow. Let the pressure drop too low, and wage and price growth will ease. Yet in the US, unemployment is at multiyear lows but inflation is nowhere in sight. In the UK it is hardly better. In Japan it is even worse. Across the developed economies, the Phillips curve has gone ignominiously flat. The world's central bankers are scratching their heads."*
>
> *"Lessons from inflation theorists of the past"*
>
> *Felix Martin, FT, July 28, 2017, p. 11*

On top of rational prepayments becoming less likely, modelling early exercise of the American-style prepayment option is incomplete because it does not reflect prepayments and selling off houses early which are unrelated to interest rate changes; or the other side of prepayments, which is staying in the house longer than usual so as not to lose the mortgage. In what follows we therefore show how our approach can be adapted to introduce prepayments modelled as a one-off occurrence of a random event.

3.1.1 FRM

Let τ be the random refinancing time and $\mathbf{1}_{t<\tau}$ the indicator function equal to 1 if prepayment did not occur at time t and 0 otherwise. Conditional on prepayment not having occurred before time $t \geq 0$, the balance of a fully amortizing FRM mortgage should satisfy

$$\hat{Q}_t^{FRM} = E\left[\underbrace{\int_t^T \mathbf{1}_{\{s<\tau|t<\tau\}} e^{-r(s-t)} R^{FRM}\, ds}_{payments\ until\ \tau>t} + (1+\phi)\underbrace{\mathbf{1}_{t<\tau\leq T} e^{-r(\tau-t)} Q_\tau^{FRM}}_{prepayment\ at\ \tau>t}\right], \quad (17)$$

where ϕ is the percentage prepayment penalty and the expectation E is to be taken under the risk-neutral measure. Cash flows are risky, because of the prepayment risk. The outstanding balance at time $\tau > t$ is given by

$$Q_\tau^{FRM} = \frac{R^{FRM}}{r}\left(1 - e^{-r(T-\tau)}\right). \quad (18)$$

Working out expectations with respect to the random prepayment time τ we obtain

$$\hat{Q}_t^{FRM} = \int_t^T F(s)\, e^{-r(s-t)} R^{FRM}\, ds + (1+\phi)\int_t^T h(s)\, e^{-r(s-t)} Q_s^{FRM} ds, \quad (19)$$

where

$$F(s) = E\left[\mathbf{1}_{\{s<\tau|t<\tau\}}\right] = \Pr(s<\tau|\, t<\tau) = e^{-\lambda(s-t)} \quad (20)$$

is the cumulative probability of the loan surviving beyond time s, conditional on the loan being alive at time $t : s > t$ and λ is the Poisson intensity of the prepayment event. Similarly, the term $h(s)$ is the conditional probability density of the loan being prepaid within time interval $(s, s+ds]$ i.e.

$$h(s) = \frac{d}{ds}(1 - F(s)) = \lambda e^{-\lambda(s-t)}. \quad (21)$$

It follows that

$$\hat{Q}_t^{FRM} = \int_t^T e^{-(r+\lambda)(s-t)} R^{FRM} ds + (1+\phi)\int_t^T \lambda e^{-(r+\lambda)(s-t)} \frac{R^{FRM}}{r}\left(1 - e^{-r(T-s)}\right) ds, \quad (22)$$

which gives the following intermediate condition valid for $t \in [0, T]$ provided that prepayment did not occur

$$\hat{Q}_t^{FRM} = R^{FRM} x(t) = Q_0 \frac{x(t)}{x(0)} \tag{23}$$

where

$$x(t) = A(r, T-t) + \phi [A(r, T-t) - A(r+\lambda, T-t)] \tag{24}$$

For $t = 0$ we have $\hat{Q}_0^{FRM} = Q_0$, which gives the initial equilibrium condition. This condition can be solved to reveal the fair repayment rate of an FRM, which takes into account subsequent prepayment risk and associated costs

$$R^{FRM} = \frac{Q_0}{x_0}. \tag{25}$$

where $x_0 = x(0)$. We can check that in absence of prepayment risk ($\lambda = 0$) or prepayment costs ($\phi = 0$) the above formula reverts to (3) which was computed under no prepayments. We conclude that, from lender's perspective, a repayment FRM is robust to random prepayments, as long as upon the prepayment event the borrower repays the full amount due, given by (18). The CWM which we study in the next subsection should behave in a similar fashion.

3.1.2 CWM

The annual payment of the CWM depends on the house price level. As a result the house price risk and the prepayment risk will interact. In what follows we incorporate the prepayment risk into the equilibrium pricing equation of the CWM. In practice, a mortgage contract is negotiated so that no prepayment should occur immediately. However, the equilibrium value and the ensuing equilibrium contract rate have to incorporate the fact that prepayment can potentially occur at any time. We proceed with the equilibrium condition that the expected future payments (8), with prepayment risk included, must be

equal to the loan balance at time $t \geqslant 0$

$$\hat{Q}_t^{CWM} = E\left[\underbrace{\int_t^T \mathbf{1}_{\{s<\tau|t<\tau\}} e^{-r(s-t)} R^{CWM}(\xi_s)\, ds}_{\text{payments until } \tau>t} + (1+\phi)\underbrace{\mathbf{1}_{t<\tau\leq T} e^{-r(\tau-t)} Q_\tau^{CWM}(\xi_\tau)}_{\text{prepayment at } \tau}\right] \tag{26}$$

Here, cash flows are risky, because of the prepayment risk *and* the house price risk embedded in the house price index ξ. Upon a prepayment event the lender will compute and inform the borrower about the *repayment amount*, $Q_\tau^{CWM}(\xi_\tau)$, akin to *outstanding balance* of an FRM, to be repaid. A *prepayment penalty* is easily added by multiplying the second term by $1+\phi$, where again ϕ is the percentage penalty. Computed in a fair manner, the balance at time τ should take into account the current value of the house price index ξ_τ so that to reflect the present value of the expected future promised payments of the CWM. That means, unlike for FRMs where annual repayments are insensitive to ξ; e.g. if ξ_τ is low upon prepayment, the balance to be repaid should be lowered. We already obtained an explicit formula for a fair $Q_\tau^{CWM}(\xi_\tau)$ which should be equal to $Q_\tau^{CWM+}(\xi_\tau)$ given in equations (12) and (15).

Alternatively, the contract could stipulate the remaining balance as based on the maximal required annual CWM repayment $\bar{R}^{CWM}$. This would be similar to an FRM (for which the prepaid amount is based on R^{FRM}), and could be computed using equation (18) where R^{FRM} should be replaced by $\bar{R}^{CWM}$. In any case, the customer should not be required to pay more than $\bar{R}^{CWM}/r\,(1-\exp\{-r\,(T-t)\})$ plus any prepayment penalty.

Working out expectations with respect to the random prepayment time τ gives

$$\hat{Q}_t^{CWM} = \int_t^T F(s)\, e^{-r(s-t)} E\left[R^{CWM}(\xi_s)\right] ds + (1+\phi)\int_t^T h(s)\, e^{-r(s-t)} E\left[Q_s^{CWM}(\xi_s)\right] ds\,, \tag{27}$$

where the expectation E is to be taken along the house price risk dimension represented by ξ. The first time integral represents the expected present value of continuous annual CWM payments $R^{CWM}(\xi)$ received until prepayment date τ or maturity T, whichever comes first. The second time integral represents the expected present value of the lump sum prepayment received before maturity. The lump sum is assumed equal to estimated

value of the principal outstanding. Using (20), (21) and (27) we obtain

$$\begin{aligned}\hat{Q}_t^{CWM} &= \bar{R}^{CWM}\int_t^T e^{-(\lambda+r)(s-t)}E\left[1-(1-\xi_s)^+\right]ds \\ &\quad +\bar{R}^{CWM}(1+\phi)\int_t^T \lambda e^{-(\lambda+r)(s-t)}E\left[\left[A(r,T-s)-P(\xi_s,1,T-s,r,\delta,\sigma)\right]\right]ds.\end{aligned} \tag{28}$$

It is straightforward to compute the first integral, which contains a *weighted floor* i.e. a weighted time integral of put options. That is λ adds to the rates r and δ

$$\begin{aligned}&\int_t^T e^{-(r+\lambda)(s-t)}E\left[(1-\xi_s)^+\right]ds = \\ &\quad = \int_t^T e^{-(r+\lambda)(s-t)}\times \\ &\qquad \times E\left[\left(1-\xi_t\exp\left\{\left((r+\lambda)-(\delta+\lambda)-\frac{\sigma^2}{2}\right)(s-t)+\sigma(Z_s-Z_t)\right\}\right)^+\right]ds \\ &\quad = P(\xi_t,1,T-t,r+\lambda,\delta+\lambda,\sigma)\ .\end{aligned} \tag{29}$$

This is effectively a weighted sum of floorlets on the normalized house price index ξ struck at 1 over a continuum of maturities $s \in [t,T]$. Each floorlet is weighted by the probability $e^{-\lambda(s-t)}$ that the loan will not be prepaid before $s > t$. By design of the contract, in-the-money floorlets (insurance puts) are *automatically exercised* to lower the annual payment, unless a prepayment happens. Annual payments continue "along" at the rate $\bar{R}^{CWM}$ until a prepayment happens or maturity is reached. That is, their expected present value must also be adjusted by the prepayment intensity parameter λ so that

$$\bar{R}^{CWM}\int_t^T e^{-(r+\lambda)(s-t)}ds = \bar{R}^{CWM}A(r+\lambda,T-t)\ . \tag{30}$$

Integrating the annuity in the second integral is also straightforward

$$\int_t^T e^{-(r+\lambda)(s-t)}A(r,T-s)\,ds = \frac{A(r,T-t)-A(\lambda,T-t)\,e^{-r(T-t)}}{r+\lambda}\ . \tag{31}$$

However, the last element of computation involves an integral of time-t expected values of the floor function P, prepayment-adjusted within interval $s \in [t,T]$. This is in fact a double time integral and a change of integration order gives a closed form (see Appendix

A.3)

$$\begin{aligned}\int_t^T \lambda e^{-(r+\lambda)(s-t)} E\left[P\left(\xi_s, 1, T-s, r, \delta, \sigma\right)\right] ds \\ = P\left(\xi_t, 1, T-t, r, \delta, \sigma\right) - P\left(\xi_t, 1, T-t, r+\lambda, \delta+\lambda, \sigma\right)\end{aligned} \tag{32}$$

We are now ready to state the equilibrium equality that holds at time $t \geq 0$, determining the equilibrium maximal annual payment $\bar{R}^{CWM}$ when prepayments occur with intensity λ. For any $t \in [0, T]$ and $\xi_t > 0$ define the adjusted annuity function

$$\begin{aligned} X\left(\xi_t, t\right) &= A\left(r, T-t\right) - P\left(\xi_t, 1, T-t, r, \delta, \sigma\right) \\ &\quad + \phi\Big[A\left(r, T-t\right) - P\left(\xi_t, 1, T-t, r, \delta, \sigma\right) \\ &\quad - A\left(r+\lambda, T-t\right) + P\left(\xi_t, 1, T-t, r+\lambda, \delta+\lambda, \sigma\right)\Big]\end{aligned} \tag{33}$$

Proposition 3 *In presence of prepayments with intensity λ, the value of promised payments of the CWM at time $t > 0$, with random prepayments included, is equal to*

$$\hat{Q}_t^{CWM+} = \bar{R}^{CWM} X\left(\xi_t, t\right) \tag{34}$$

where X is given by (33). In equilibrium[4] $\hat{Q}_t^{CWM+} = Q_0$ and (34) can be inverted to derive an estimate[5] of the equilibrium maximal mortgage payment $\bar{R}^{CWM}$

$$\bar{R}^{CWM} \approx \frac{Q_0}{X\left(1, 0\right)} \tag{35}$$

Remark 4 *It is easy to check that under no-prepayments regime ($\lambda \to 0$), implying $A\left(\lambda, T\right) \to T$, (35) reduces to (13) from Proposition 2. Similarly, in absence of prepayment penalties ($\phi = 0$), (35) simplifies to (13) and (34) reduces to (15). Therefore, from lender's perspective, the CWM too possesses the "robustness"' property of the FRM, provided that the prepaid amount is fairly calculated, i.e. (34) is used.*

Finally, we note that professional conventions, such as CPR (Conditional Prepayment Rate) and PSA (Public Securities Association) prepayment model, assume that the pre-

[4]For inclusion of arrangement fees π (points) and default risk and their impact on $\bar{R}^{CWM}$ see Section 4.

[5]The estimate considered here embeds prepayment risk. See Section 4for estimates incorporating both prepayments and default risk.

payment rate λ is constant in time and independent of the interest rate r. In practice, however, λ will be influenced by the level of interest rates. Following Gorovoy and Linetsky (2007) we suggest a rate-dependent intensity $\lambda(r)$ equal to a prepayment intensity λ_p plus a refinancing intensity λ_r

$$\lambda(r) = \lambda_p + \lambda_r = \lambda_p + l(\bar{r} - r)^+ . \tag{36}$$

The prepayment component λ_p accounts for prepayments due to exogenous reasons such as relocations. The refinancing component λ_r increases when the interest rate r decreases. This effect can be calibrated using the multiplier $l > 0$ and the threshold $\bar{r} > 0$ parameters. When the interest rate r exceeds the threshold $\bar{r}$, the refinancing component λ_r vanishes, reflecting the fact that homeowners will not rationally refinance when interest rates are high. Initially, we should have $\bar{r} < r$, so that there is no incentive to refinance immediately after the mortgage is originated.

3.2 Defaults

The borrower's decision to default is more difficult to model. Modelling the endogenous decision to default involves pricing an American put option where the underlying variable is the value of the mortgaged property. The standard approach in the literature is to solve the problem using general purpose numerical methods which can be complex to program and time consuming.[6]

With notable exceptions of the singular perturbation approach of Sharp et al. (2008) and the Viegas and Azevedo-Pereira (2012) Richardson's extrapolation technique *à la* Geske and Johnson (1984), research for approximate closed-form solutions for mortgages incorporating the default option have been few. In this section we extend this strand of literature by attaching a default option to our finite-maturity closed-form pricing solution for the CWM we obtained in sections 2.2 and 3.1.

It is well-known from the literature that valuation of American put options is an optimal stopping, free boundary problem, for which no known exact mathematical solution al-

[6]For example Sharp et al. (2008) report computation times of more than 10 hours for computing just one equilibrium contract rate by numerically solving the Partial Differential Equation (PDE) for a given set of parameters.

gorithm exists. Consequently, except for very particular cases, not normally encountered in practice, e.g. perpetual mortgages with infinite maturity, there are no closed-form formulas available. However, it is incorrect to assume that there is no accurate analytical treatment of options where early exercise is permitted (Shaw, 1998, chapter 11). More importantly, rushing to numerical methods forces one to abandon quick and efficient computation of Greeks by ordinary differentiation. These hedge ratios are involved in the calculation of exposures of Mortgage Backed Securities (MBS). Therefore, this particular aspect is paramount for mortgages and has practical implications for issuers when they manage existing portfolios of MBS or introduce new credit products such as CWMs.

In present context it is important to notice that unlike traded American put options, the underlying house price can only be measured approximately. And, unlike a stock price, it cannot be observed very frequently and the futures series should not be trended. Moreover, exogenous factors affect the real magnitude of default options embedded in mortgages and most of these factors are unobservable and unpredictable. For example, a sudden loss of income (which is borrower-specific, and thus rarely modeled in the literature) may force a family to default even when house prices are very high. Therefore, the aim here is to have an analytic rather than an accurate or fast valuation,[7] to assess how the house price dimension affects the decision to default. Therefore, we will approximate the value of the option to default, D_0, by a closed-form approximation.

3.2.1 CWM

We write the value to the lender of the prepayable mortgage, V_t^{CWM}, as equal to

$$V_t^{CWM} = Q_t^{CWM+} - D_t\left(Q_t^{CWM+}, H_t\right) , \tag{37}$$

where Q_t^{CWM+} is the expected present value of outstanding payments of a CWM, exposed to prepayments, as given by equation (34) we obtained in the previous section. The second term on the right hand side, D_t, is the embedded American put option to default. It is a function of the expected outstanding payments Q_t^{CWM+}, too, and of H_t, the current value of the property at time $t \in [0, T]$. Below the optimal default threshold $\underline{H}_t^{CWM}$ the

[7] Numerical methods can be fast and accurate but do not offer the same insight as an analytic approach.

payoff of the put must satisfy the following free boundary condition[8]

$$D_t\left(Q_t^{CWM+}, H_t\right) = Q_t^{CWM+} - H_t \qquad \text{for all } H_t \leq \underline{H}_t^{CWM} \text{ and all } t \tag{38}$$

In the default zone the householder saves the expected future payments net of the current value of the house which reverts to the lender via repossession. For what follows it is useful to express Q_t^{CWM+} and H_t in terms of the house price index ξ_t we introduced in Section 3.1 and introduce the corresponding optimal default threshold, $\underline{\xi}^{CWM}$, for the index ξ. Letting $\xi_t \to \underline{\xi}_t^{CWM}$ then implies in particular (provided that a prepayment did not happen before t) that the optimal exercise boundary, $\underline{\xi}_t^{CWM}$, of the option to default can be expressed as

$$D\left(\underline{\xi}_t^{CWM}, t\right) = H_0\left(\frac{l}{\eta} X\left(\underline{\xi}_t^{CWM}, t\right) - \underline{\xi}_t^{CWM}\right) \qquad \text{for all } t\,, \tag{39}$$

where X is given by (33), l is the initial Loan To Value (LTV) ratio

$$l = Q_0 / H_0 \tag{40}$$

and η is a normalization parameter, independent of t

$$\eta = X\left(1, 0\right)\,. \tag{41}$$

We have also modified notation of the function D so that to emphasize its dependence on time t and the value of the index ξ_t. Condition (39) is the *value-matching* condition. It gives the payoff upon early default. The value of the default option must therefore be equal to the fraction of the initial house price, H_0. The scaling factor is an American put on the house price index ξ_t. Unlike for stock options, the exercise price of this American put is not constant. It is a function of time t and the index ξ_t, but also depends on the loan to value ratio l and other parameters embedded in X not shown here, such as the prepayment intensity λ, the prepayment penalty ϕ, the volatility of the house price index σ, etc. For a full list of dependencies see equation (33). From (39) it is also easy to see that a rational exercise of the default option at origination ($t \to 0$) would never be optimal when $l < 1$, i.e. for LTV ratios below 100%, which are typically encountered in practice

[8] In particular, the initial value of the threshold must satisfy $H_0 > \underline{H}_0$, so that no defaults occur immediately upon origination of a new mortgage to a customer.

and never at maturity ($t \to T$).

The default option must also satisfy the limit condition applicable to any put on underlying ξ

$$\lim_{\xi_t \to \infty} D(\xi_t, t) = 0\,. \tag{42}$$

Among all admissible solutions of the partial differential equation obeyed by D satisfying the above condition and all the corresponding early exercise boundaries $\underline{\xi}^{CWM}$, we should retain the solution pair $\left\{D, \underline{\xi}^{CWM}\right\}$ which maximizes the value of the default option to the householder. For a given t it can be shown, in a way analogous to Merton (1973) who deals with a simpler case of constant exercise price, that this occurs when the following first order optimality condition, obtained by differentiating (39) with respect to ξ *and* letting $\xi_t \to \underline{\xi}_t^{CWM}$, holds

$$\frac{\partial}{\partial \xi} D\left(\underline{\xi}_t^{CWM}, t\right) = H_0 \left(\frac{l}{\eta} \frac{\partial}{\partial \xi} X\left(\underline{\xi}_t^{CWM}, t\right) - 1\right) \tag{43}$$

Note that this *smooth-pasting* condition is substantially different from the standard one for an American put with constant strike, which would just produce $-H_0$ on the right hand side. This is because at the point of exercise we have to smoothly paste the default option not into a straight line but into a non-linear strike function, whose curvature depends on time t (as for the FRM) but also (as opposed to the FRM) on the level of the house price index ξ_t.

We can decompose the value of our American put D into an otherwise identical European default put p plus an early exercise premium ε (see e.g. Carr et al. (1992)), i.e. $D = p + \varepsilon$. However, for mortgages, unlike for stock options, the value of the default option is always zero at maturity. It follows that $p = 0$ i.e. the value of the European default put must be zero throughout and $D = \varepsilon$. The value of the put comes entirely from defaulting on payments before the end of the mortgage i.e. repayment.[9]

MacMillan (1986) and Barone-Adesi and Whaley (1987) obtain their approximations by

[9]Our approach can still be used for designs where the strike price does not converge to zero at maturity, e.g. non amortizing ('flat' strike not depending on time or the HPI ξ) or partly amortizing mortgages, provided that the value of the resulting European put is included in the computation, analogously to inclusion of the European put in the Barone-Adesi and Whaley (1987) decomposition of the plain vanilla American put.

considering solutions to the early exercise premia $\varepsilon(\xi_t, t)$. In numerous numerical experiments they show that their algorithm is very accurate and considerably more computationally efficient than finite-difference, binomial, or compound-option pricing methods. We adapt their multiplicative separation approach to obtain accurate approximations for our repayment CWM mortgage. In contrast, we can impose separability directly on the function D. We require

$$D(\xi_t, t) = f(\xi_t) \cdot g(t) \ , \tag{44}$$

where

$$g(t) = rA(r, T-t) = 1 - e^{-r(T-t)} \ . \tag{45}$$

We substitute (44) into the partial differential equation (see (106), Appendix B) obeyed by D. It follows that

$$\frac{1}{2}\sigma^2\xi^2 D_{\xi\xi} + (r-\delta)\,\xi D_\xi + D_t = rD \ ,$$

where subscripts denote partial derivatives. Substituting (44) and (45) we have

$$\frac{1}{2}\sigma^2\xi^2 f''(\xi)\, g(t) + (r-\delta)\,\xi f'(\xi)\, g(t) + f(\xi)\, g'(t) = rf(\xi)\, g(t) \ . \tag{46}$$

Observing $g'(t) = -re^{-r(T-t)} = r(g(t) - 1)$, dividing by $f(\xi)$ and $g(t)$ and noting the function $g(t)$ as if it were a constant g, we obtain

$$\frac{1}{2}\sigma^2\xi^2\frac{f''(\xi)}{f(\xi)} + (r-\delta)\,\xi\frac{f'(\xi)}{f(\xi)} = \frac{r}{g(t)} \ . \tag{47}$$

This equation is not properly separated because $g(t)$ is not a constant but a function of time t. If $g(t)$ were a constant, i.e. if we could write $g(t) = g$, the solution to the corresponding ordinary differential equation with constant coefficients would have the form

$$f(\xi) = a^{CWM}\xi^q \ , \tag{48}$$

where a^{CWM} is a constant and q is the negative root of the quadratic equation

$$\frac{1}{2}\sigma^2 q\left(q-1\right)+\left(r-\delta\right)q=\frac{r}{g\left(t\right)}\,, \tag{49}$$

so that (42) holds. However, because g is a function of time, so must be q, which gives

$$q\left(t\right)=\frac{1}{2}-\frac{r-\delta}{\sigma^2}-\sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{r}{g\left(t\right)}\frac{2}{\sigma^2}}<0\,, \tag{50}$$

where $g\left(t\right)$ is given by (45). It follows that we can express the option to default as

$$D\left(\xi,t\right)=a^{CWM}\xi^{q(t)}g\left(t\right)\,, \tag{51}$$

where $q\left(t\right)$ is given by (50). The constant a^{CWM} and the optimal exercise boundary $\underline{\xi}_t^{CWM}$ at time t have to be determined from the value-matching (39) and smooth-pasting (43) system of equations

$$\begin{cases} a^{CWM}g\left(t\right)\left(\underline{\xi}_t^{CWM}\right)^{q(t)}=H_0\left[\frac{l}{\eta}X\left(\underline{\xi}_t^{CWM},t\right)-\underline{\xi}_t^{CWM}\right] \\ a^{CWM}g\left(t\right)q\left(t\right)\left(\underline{\xi}_t^{CWM}\right)^{q(t)-1}=H_0\left[\frac{l}{\eta}\frac{\partial}{\partial\xi}X\left(\underline{\xi}_t^{CWM},t\right)-1\right]\,. \end{cases} \tag{52}$$

Dividing side-wise the second equation by the first, a^{CWM} is eliminated and we get

$$\frac{1}{q\left(t\right)}=\frac{1}{\underline{\xi}_t^{CWM}}\cdot\frac{\frac{l}{\eta}X\left(\underline{\xi}_t^{CWM},t\right)-\underline{\xi}_t^{CWM}}{\frac{l}{\eta}\frac{\partial}{\partial\xi}X\left(\underline{\xi}_t^{CWM},t\right)-1}\,, \tag{53}$$

which can easily be solved numerically to obtain the boundary $\underline{\xi}_t^{CWM}$. In particular, both $X\left(\xi,t\right)$ and it's derivative $\frac{\partial}{\partial\xi}X\left(\xi,t\right)$ are available in closed form.[10] Once $\underline{\xi}_t^{CWM}$ is calculated, a^{CWM} can be obtained from either of the equations of the system (52). This ends the valuation of the embedded default put option (51). We note in particular that the final result, (51), *is not* multiplicatively separable in ξ and t which contradicts the initial assumption.[11] However, the method is a *tour de force* as it achieves the stated goal, which

[10] Closed form expressions for partials of the floor $\frac{\partial}{\partial\xi}P\left(\xi,1,T-t,r+\lambda,\delta+\lambda,\sigma\right)$ can be found in Shackleton and Wojakowski (2007). For computations, since $X\left(\xi,t\right)$ is given in closed form (33), we used *Mathematica* computer algebra software to calculate these partials in closed form.

[11] D is of the form $f\left(\xi,t\right)\cdot g\left(t\right)$ instead of $f\left(\xi\right)\cdot g\left(t\right)$ because in f the power of ξ is a function of time, $q\left(t\right)$.

is to obtain accurate values of *both* the option and the free boundary.

Finally, we note that, the *logarithm* of the final result is multiplicatively separable in ξ and t. That is, we have $\ln D = \varphi(\xi) \cdot \chi(t)$ where φ, χ are some functions. Note that if instead we had the initial assumption satisfied all along, i.e. $D = f(\xi) \cdot g(t)$, the Barone-Adesi and Whaley (1987) method would be an exact method for pricing American options and not an approximation.

3.2.2 FRM

The value to the lender of a prepayable, fixed rate mortgage, V_t^{FRM}, can also be decomposed into

$$V_t^{FRM} = \hat{Q}_t^{FRM} - D_t^{FRM}\left(\hat{Q}_t^{FRM}, H_t\right), \tag{54}$$

where $\hat{Q}_t^{FRM}$ is the expected present value of outstanding payments of an FRM, subject to prepayments, as given by equation (23) we obtained in the previous section. Function D_t^{FRM}, is similarly, the embedded American put to default. Below the optimal default threshold $\underline{H}_t^{FRM}$, which will be different from $\underline{H}_t^{CWM}$, must satisfy the initial condition $\underline{H}_t^{FRM} < H_0$ too and the free boundary condition

$$D^{FRM}\left(\hat{Q}_t^{FRM}, H_t\right) = \hat{Q}_t^{FRM} - H_t \qquad \text{for all } H_t \leq \underline{H}_t^{FRM} \text{ and all } t \tag{55}$$

Borrower maximizes value by defaulting immediately after crossing down the boundary $\underline{H}_t^{FRM}$. Using (23) and (5) gives

$$D^{FRM}\left(\hat{Q}_t^{FRM}, H_t\right) = Q_0 \frac{x(t)}{x_0} - H_0 \xi_t, \tag{56}$$

where $x_0 = x(0)$ and $x(t)$ is given by (24). Using notation $l = Q_0/H_0$ for the LTV ratio as before, rearranging, letting $\xi_t \to \underline{\xi}_t^{FRM}$ and conditional on prepayment not happening before $t \geq 0$, the default option can be written as a function of the optimal exercise boundary, $\underline{\xi}_t^{FRM}$

$$D^{FRM}\left(\underline{\xi}_t^{FRM}, t\right) = H_0\left(\frac{l}{x_0} x(t) - \underline{\xi}_t^{FRM}\right) \qquad \text{for all } t\,. \tag{57}$$

Condition (57) is the *value-matching* condition for an FRM and gives the payoff upon early default, a fraction of the initial house price, H_0. The exercise price of the scaling American put on the house price index ξ_t is here a function of time t, loan to value ratio l and early prepayment parameters (intensity λ and penalty ϕ) but is not a function of the index ξ_t or its volatility σ.[12] This is why the *smooth-pasting* condition for FRM here is analogue to a standard American put, where the strike does not depend on the underlying asset, and simplifies to

$$\frac{\partial}{\partial \xi_t} D^{FRM}\left(\underline{\xi}_t^{FRM}, t\right) = -H_0 \tag{58}$$

We decompose D^{FRM} into a European put plus an early default premium: $D^{FRM} = p^{FRM} + \varepsilon^{FRM}$ and notice that $p^{FRM} = 0$. At maturity the loan is fully amortized and default is impossible, implying $D^{FRM} = \varepsilon^{FRM}$. We then follow the Barone-Adesi and Whaley (1987) and MacMillan (1986) as we did for the CWM. We impose separability

$$D^{FRM}(\xi_t, t) = f^{FRM}(\xi_t) \cdot g(t) \ , \tag{59}$$

where $g(t)$ is given by (45), the same as for the CWM. We substitute (59) into the partial differential equation (see again (106), Appendix B) obeyed by D^{FRM} to discover that it follows that same Partial Differential Equation as $D^{CWM}$and must have the same form of the power solution

$$D^{FRM}(\xi, t) = a^{FRM} \xi^{q(t)} g(t) \tag{60}$$

where $q(t)$ is the same, negative solution of the fundamental quadratic, given by (50) and a^{FRM} is a constant proper to an FRM which must be calculated from the system of two equations derived from the value-matching (57) and smooth-pasting (58) boundary conditions

$$\begin{cases} \left(\underline{\xi}_t^{FRM}\right)^{q(t)} a^{FRM} g(t) &= H_0\left(\frac{l}{x_0} x(t) - \underline{\xi}_t^{FRM}\right) \\ a^{FRM} q(t) \left(\underline{\xi}_t^{FRM}\right)^{q(t)-1} g(t) &= -H_0 \ . \end{cases} \tag{61}$$

[12] Here too a rational default at origination ($t \to 0$) or maturity ($t \to T$) would never be optimal for LTV ratios below 100% ($l < 1$). Similarly $\lim_{\xi_t \to \infty} D^{CWM}(\xi_t, t) = 0$.

Eliminating a^{FRM} we get

$$\underline{\xi}_t^{FRM} = \frac{l\frac{x(t)}{x_0}}{1-\frac{1}{q(t)}}$$

Interestingly this reveals that the (approximate) boundary $\underline{\xi}_t^{FRM}$ can be obtained in *closed form* and so can the default option "scale," the constant a^{FRM}

$$a^{FRM} = -\frac{H_0}{g(t)\,q(t)}\left(\underline{\xi}_t^{FRM}\right)^{1-q(t)} = -\frac{H_0}{g(t)\,q(t)}\left(\frac{l\frac{x(t)}{x_0}}{1-\frac{1}{q(t)}}\right)^{1-q(t)} \tag{62}$$

which gives the *default option for FRM in closed form*

$$D^{FRM}(\xi,t) = -\frac{H_0}{q(t)}\left(\frac{l\frac{x(t)}{x_0}}{1-\frac{1}{q(t)}}\right)^{1-q(t)}\xi^{q(t)} \tag{63}$$

Finally note that although not explicit in our notation, $x(t)$ and x_0 do depend on prepayment frequency and penalty parameters λ, ϕ (see equation (24)).

4 Equilibrium contract rate for the CWM and the FRM when default and prepayment risks are present

The point of our paper is that the FRM and the CWM have different risks embedded in them. The FRM places house price risk entirely on the borrower until the point where they would default. The CWM mitigates this risk and, depending on design of how the repayment schedule responds to changes in house prices, transfers all or considerable portions of the house price risk to the lender. The lender is paid a risk premium, in the form of increased repayment rates of the CWM in good states, to bear this risk.

As a result these two contracts will not have the same payment rate in equilibrium. Consequently, their contract rates will differ too. In this section we show how to calculate the correct contract rate of a CWM and how to compare it to the contract rate of an FRM. In order to correctly price the CWM, in the previous section we embedded the prepayment

and the default risk premia in the value of a CWM to the lender (see equation (37)). Using Black Scholes options pricing framework these premia can be computed as expectations under the martingale measure of the expected future CWM and FRM payoffs, discounted using the riskless rate r. As a result, both CWM and FRM are correctly priced, even though pricing by arbitrage requires discounting using the same rate.

We initially expressed our *equilibrium* result in the language of *equilibrium annual payments*, R^{FRM} and $\bar{R}^{CWM}$ (and to be more precise: equilibrium annual *maximal* payment $\bar{R}^{CWM}$, in the case of the CWM). We will now establish a link between these quantities and the equilibrium contract rates for the CWM and the FRM.

4.1 FRM

For a mortgage with initial balance Q_0 the monthly compounded 'contract rate' $\bar{r}_c$ is typically defined via the equation linking the maturity N (expressed as the number of months remaining) and the monthly payment MP

$$Q_0 = \frac{MP}{\frac{\bar{r}_c}{12}} \left(1 - \frac{1}{\left(1 + \frac{\bar{r}_c}{12}\right)^N} \right) \tag{64}$$

It is expressed in percentages per annum and is discretely compounded. Clearly, if all the monthly payments MP are riskless, the right hand side is the present value of a monthly paid annuity and $\bar{r}_c$ must be equal to the riskless rate. This definition is still used in pricing situations where spot riskless rates are stochastic and monthly payments are *implicitly* stochastic i.e. where the monthly payment MP can randomly discontinue before maturity due to prepayment or default, see e.g. Kau et al. (1992). In these papers, for a given initial balance Q_0, term N and a candidate solution contract rate r_c the corresponding monthly 'promised' payment MP is computed via equation (64). It is then checked by solving the pricing PDE numerically (Feynman-Kac), whether the present value (discounted at the riskless rate r) of risk-neutral expectations of the series of monthly payments MP is equal to Q_0. If it is, that means the rate $\bar{r}_c$ is the equilibrium contract rate.

In this section we show that in absence of default and prepayment risks the equilibrium contract rate of a CWM must be greater than the equilibrium contract rate of an FRM. This is because CWM contract incorporates house price risk. For the FRM we use a continuous-

time expression, analogous to (64)

$$Q_0 = \frac{R^{FRM}}{r_c^{FRM}} \left(1 - \exp\left\{-r_c^{FRM} T\right\}\right) \tag{65}$$

where r_c^{FRM} is the equilibrium, continuously compounded FRM contract rate, expressed in percentages per annum. This means that knowing the equilibrium annual repayment rate R^{FRM}, we can figure out the corresponding *equilibrium FRM contract rate* r_c^{FRM}. When all FRM cash flows are riskless, we must have $r_c^{FRM} = r$, where r is the riskless rate. To avoid the contractual arbitrage, the equilibrium condition which must be satisfied at origination is

$$Q_0 (1 - \pi) = V_0^{FRM} = \hat{Q}_0^{FRM} - D^{FRM}(1, 0) \tag{66}$$

That is, the value Q_0 of the home loan granted by the lender minus any *arrangement fees* π (*points*) expressed as a percentage of the loan, should be equal to the value of the mortgage to the lender, i.e. the value of promised payments, considering the possibility of prepayment, minus the option to default. This gives[13]

$$H_0 l (1 - \pi) = R^{FRM} x_0 + \frac{H_0}{q_0} \left(\frac{l}{1 - \frac{1}{q_0}}\right)^{1-q_0} \tag{67}$$

which we can solve for the (promised) annual repayment rate R^{FRM}

$$R^{FRM} = \frac{H_0}{x_0} \left[l (1 - \pi) - \frac{1}{q_0} \left(\frac{l}{1 - \frac{1}{q_0}}\right)^{1-q_0} \right] \tag{68}$$

and obtain the continuously compounded equilibrium contract rate r_c^{FRM} by inverting (65). It turns out that this, again, can be obtained in closed from

$$r_c^{FRM} = \frac{1}{T} \left[\frac{R^{FRM} T}{Q_0} + W\left(-\frac{R^{FRM} T}{Q_0} \exp\left\{ -\frac{R^{FRM} T}{Q_0} \right\} \right) \right], \tag{69}$$

where W is the *Lambert-W* function, also known as *product log*. Because r_c^{FRM} is a continu-

[13] Note that $q(0) = q_0 < 0$.

$\bar{r}_c^{FRM}$ [%]				
$T =$	15	20	25	30
	$\sigma = 5\%$			
Intensity + BAW	10.124	10.112	10.105	6.026
PDE	10.146	10.124	10.146	
Binomial				5.134
	$\sigma = 10\%$			
Intensity + BAW	10.597	10.523	10.478	6.224
PDE	10.028	10.082	10.113	
Binomial				6.22

Table 1: A quick comparison of equilibrium contract rates. Intensity + BAW: this work; PDE: Crank-Nicolson PSOR approach of Sharp et al. (2008) $r = r_0 = 10\%$, $\delta = 7.5\%$; Binomial: twin-tree approach of Ambrose and Buttimer (2012) $r = 6\%$, $r_0 = 4\%$, $\delta = 2\%$.

ously compounded rate, it can be converted to monthly rate via

$$\bar{r}_c^{FRM} = 12\left(\exp\left\{\frac{1}{12}r_c^{FRM}\right\} - 1\right) . \tag{70}$$

We start by checking the output of our model against figures obtained using other methods. From Table 1 it emerges that, when house price volatility is small, our approximate contract rates tend to slightly undervalue the allegedly very high accurate PDE solutions obtained via Crank and Nicolson (1947) implicit PSOR scheme approach of Sharp et al. (2008). When volatility is high the opposite happens. Our estimates also slightly overvalue the Binomial approach of Ambrose and Buttimer (2012), but are very close for $\sigma = 10\%$. However this is less surprising as their term structure is steep (spot rate r starts at 4% and converges to 6% in long term) while the term structure of Sharp et al. (2008) is flat (spot rate r starts at 10% and remains in the vicinity of 10% in the long term). Consistently with the options pricing theory, the higher the volatility of the house price, the higher the equilibrium contract rates obtained from our model. Also consistently with common wisdom (and unlike the PDE results of Sharp et al. (2008) which exhibit a local minimum at $T = 20$ years), the longer the term of the mortgage contract, the lower our model's equilibrium mortgage rates.[14]

[14] Sharp et al. (2008) work from European perspective with maturities typically up to $T = 25$ years in the UK while Ambrose and Buttimer (2012) work with the standard $T = 30$ year maturity US mortgage. We only report contract values for lowest volatilities of interest rate they consider i.e. for $\sigma_r = 5\%$. This is empirically motivated by our observations made in Section 1.

4.2 CWM

In contrast, because cash flows of the CWM are *explicitly* stochastic, defining the corresponding 'contract rate' is not straightforward. However, by analogy with (64) and (65) we posit

$$Q_0 = \frac{\bar{R}^{CWM}}{r_c^{CWM}} \left(1 - \exp\left\{-r_c^{CWM} T\right\}\right) \tag{71}$$

where $\bar{R}^{CWM}$ is the *equilibrium* annual repayment rate of the CWM (which we obtained in closed form) and r_c^{CWM} is the corresponding *equilibrium CWM contract rate.* Because CWM cash flows are all stochastic and capped by $\bar{R}^{CWM}$, which is the equilibrium *maximal* annual repayment rate, r_c^{CWM} is in fact the equilibrium CWM *maximal* contract rate. To obtain $\bar{R}^{CWM}$ we start with the same initial equilibrium condition as we were using for FRM

$$Q_0 (1 - \pi) = V_0^{CWM} = \hat{Q}_0^{CWM} - D^{CWM}(1, 0) \tag{72}$$

where V_0^{CWM} is the value to the lender and π are the points. The major difference is the $D^{CWM}(1,0)$ component which is in closed form too but requires one of the arguments (the early default boundary $\underline{\zeta}_0^{CWM}$) to be solved for numerically. Using (34) we obtain

$$\bar{R}^{CWM} = \frac{Q_0 (1 - \pi) + D^{CWM}(1, 0)}{X(1, 0)} \tag{73}$$

which in conjunction with (71) should be used to obtain the CWM contract rate r_c^{CWM}. Monthly compounded CWM contract rate is obtained from r_c^{CWM} using

$$\bar{r}_c^{CWM} = 12 \left(\exp\left\{\frac{1}{12} r_c^{CWM}\right\} - 1\right) . \tag{74}$$

4.3 CWM v.s. FRM contract rates

It is interesting to compare CWM to FRM annual payments $\bar{R}^{CWM}, R^{FRM}$ and the associated contract rates $\bar{r}_c^{CWM}, r_c^{FRM}$. In absence of prepayments ($\lambda = 0$ or $\phi = 0$) we obtain

$$R^{FRM} = \frac{Q_0 (1-\pi) + D^{FRM}(1,0)}{A(r,T)} \qquad \text{v.s.} \qquad \bar{R}^{CWM} = \frac{Q_0 (1-\pi) + D^{CWM}(1,0)}{A(r,T) - P(1,1,T,r,\delta,\sigma)} \quad (75)$$

There are two opposite effects which determine the CWM maximal annual payment $\bar{R}^{CWM}$. On one hand CWM commands a positive insurance component $P(1,1,T,r,\delta,\sigma)$ which is subtracted from annuity $A(r,T)$ in the denominator. This *increases* the maximal annual payment $\bar{R}^{CWM}$ relative to R^{FRM}. On the other hand we expect the value of the embedded option to default, appearing in the numerator, to be *lower* for the CWM

$$D^{CWM}(1,0) < D^{FRM}(1,0) \quad (76)$$

and to *reduce* the annual payment $\bar{R}^{CWM}$ as a result, relative to the FRM.

Our numerical analysis shows that the insurance effect dominates the default option effect. That is, the maximal annual payment of the CWM is above the constant annual payment of the FRM. However, under many scenarios, the decrease in the embedded option to default is sufficiently prominent for the CWM to offer very affordable contract rates which are only slightly higher than the contract rates of an FRM. This shows that at least some part of the costs of the insurance against default can be financed by reducing the incentives to rationally default when house prices decrease.

Payout rates δ reflect service provided by the property but also the speed of its depreciation relative to the growth of other assets in the economy. Consider a case where interest rates are are low and comparable to the service flow ($r = \delta = 2\%$). For the loan to value ratio $l = 95\%$ (see Table 2) the contract rate of a CWM is only 2.550%, compared to 2.342% for an otherwise identical FRM. When interest rates increase to $r = 12\%$, the attractiveness of the CWM is preserved. Contract rates follow the interest rate to 12.063% and 12.061% for the CWM and the FRM, respectively. Meanwhile, their relative spread decreases by two orders of magnitude, from 20.8 to only 0.2 basis points. Decreasing the loan to value ratio to $l = 90\%$ widens the spread to 0.3 basis points and remains the same for $l = 80\%$.

Consistent with option pricing theory, the spread between CWM and FRM loan rates increases as the volatility of the house prices increases. In our example, when the house price volatility increases from $\sigma = 5\%$ to 15% (see Table 4), FRM and CWM rates increase to 12.384% and 12.457%, respectively, and therefore also become more wide apart (7.3

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	2.342	2.27	1.639	2.55	2.477	1.839
		0.06	3.609	3.532	2.855	6.03	5.943	5.165
		0.12	4.725	4.642	3.921	12.063	11.944	10.837
	0.06	0.02	6.034	5.948	5.169	6.043	5.956	5.178
		0.06	6.328	6.24	5.448	6.608	6.518	5.714
		0.12	7.94	7.843	6.972	12.071	11.952	10.845
	0.12	0.02	12.061	11.942	10.836	12.063	11.944	10.838
		0.06	12.068	11.949	10.843	12.076	11.957	10.85
		0.12	12.367	12.246	11.12	12.751	12.627	11.478
0.9	0.02	0.02	2.256	2.184	1.556	2.534	2.462	1.824
		0.06	3.526	3.449	2.775	6.03	5.943	5.165
		0.12	4.666	4.584	3.865	12.063	11.944	10.837
	0.06	0.02	6.018	5.932	5.154	6.031	5.944	5.166
		0.06	6.229	6.141	5.354	6.603	6.513	5.708
		0.12	7.828	7.732	6.867	12.071	11.952	10.845
	0.12	0.02	12.06	11.941	10.835	12.063	11.944	10.837
		0.06	12.061	11.942	10.836	12.071	11.952	10.845
		0.12	12.244	12.124	11.006	12.751	12.627	11.478
0.8	0.02	0.02	2.136	2.065	1.44	2.534	2.462	1.824
		0.06	3.358	3.282	2.614	6.03	5.943	5.165
		0.12	4.543	4.462	3.748	12.063	11.944	10.837
	0.06	0.02	6.015	5.928	5.151	6.03	5.943	5.165
		0.06	6.108	6.021	5.239	6.603	6.513	5.708
		0.12	7.606	7.511	6.657	12.071	11.952	10.845
	0.12	0.02	12.06	11.941	10.835	12.063	11.944	10.837
		0.06	12.06	11.941	10.835	12.071	11.952	10.845
		0.12	12.121	12.001	10.891	12.751	12.627	11.478

Table 2: **Equilibrium contract rates v.s. prepayment intensity and cost.** $\bar{r}_c[\%]$ p.a. monthly compounded. House price volatility $\sigma = 5\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	2.776	2.702	2.056	3.253	3.178	2.514
		0.06	3.787	3.709	3.025	6.185	6.098	5.313
		0.12	4.793	4.71	3.986	12.097	11.977	10.869
	0.06	0.02	6.255	6.167	5.379	6.335	6.247	5.455
		0.06	6.765	6.674	5.862	7.358	7.264	6.423
		0.12	8.106	8.008	7.128	12.188	12.068	10.954
	0.12	0.02	12.145	12.025	10.914	12.163	12.043	10.931
		0.06	12.251	12.13	11.012	12.31	12.189	11.068
		0.12	12.846	12.721	11.565	13.587	13.458	12.254
0.9	0.02	0.02	2.68	2.606	1.963	3.148	3.073	2.413
		0.06	3.707	3.629	2.949	6.185	6.098	5.313
		0.12	4.736	4.654	3.932	12.097	11.977	10.869
	0.06	0.02	6.166	6.078	5.294	6.254	6.166	5.378
		0.06	6.643	6.553	5.746	7.256	7.163	6.327
		0.12	7.997	7.9	7.026	12.188	12.068	10.954
	0.12	0.02	12.089	11.969	10.862	12.114	11.994	10.885
		0.06	12.155	12.035	10.923	12.229	12.109	10.992
		0.12	12.677	12.554	11.409	13.503	13.375	12.177
0.8	0.02	0.02	2.509	2.436	1.799	3.119	3.044	2.386
		0.06	3.545	3.468	2.794	6.185	6.098	5.313
		0.12	4.616	4.535	3.818	12.097	11.977	10.869
	0.06	0.02	6.069	5.982	5.203	6.194	6.106	5.321
		0.06	6.441	6.352	5.555	7.247	7.154	6.318
		0.12	7.779	7.683	6.82	12.188	12.068	10.954
	0.12	0.02	12.063	11.944	10.838	12.097	11.978	10.87
		0.06	12.081	11.962	10.854	12.19	12.07	10.956
		0.12	12.424	12.303	11.174	13.503	13.375	12.177

Table 3: **Equilibrium contract rates v.s. prepayment intensity and cost.** $\bar{r}_c[\%]$ p.a. monthly compounded. House price volatility $\sigma = 10\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	3.19	3.115	2.453	4.031	3.952	3.259
		0.06	4.017	3.938	3.245	6.578	6.489	5.686
		0.12	4.897	4.813	4.085	12.22	12.1	10.984
	0.06	0.02	6.614	6.524	5.719	6.89	6.799	5.981
		0.06	7.194	7.101	6.268	8.203	8.105	7.221
		0.12	8.333	8.234	7.343	12.508	12.386	11.252
	0.12	0.02	12.384	12.263	11.136	12.457	12.335	11.204
		0.06	12.605	12.483	11.342	12.821	12.697	11.543
		0.12	13.333	13.205	12.017	14.555	14.42	13.151
0.9	0.02	0.02	3.098	3.023	2.365	3.892	3.815	3.127
		0.06	3.943	3.864	3.174	6.566	6.477	5.674
		0.12	4.842	4.759	4.033	12.22	12.1	10.984
	0.06	0.02	6.496	6.406	5.607	6.765	6.675	5.863
		0.06	7.072	6.979	6.152	8.046	7.949	7.074
		0.12	8.229	8.13	7.245	12.508	12.386	11.252
	0.12	0.02	12.258	12.138	11.019	12.339	12.218	11.095
		0.06	12.452	12.33	11.2	12.674	12.551	11.406
		0.12	13.153	13.026	11.85	14.372	14.238	12.982
0.8	0.02	0.02	2.922	2.847	2.196	3.76	3.683	3.
		0.06	3.79	3.712	3.028	6.566	6.477	5.674
		0.12	4.727	4.644	3.923	12.22	12.1	10.984
	0.06	0.02	6.312	6.224	5.433	6.62	6.53	5.725
		0.06	6.847	6.755	5.939	7.951	7.855	6.984
		0.12	8.018	7.92	7.046	12.508	12.386	11.252
	0.12	0.02	12.128	12.009	10.898	12.242	12.122	11.004
		0.06	12.251	12.131	11.013	12.538	12.416	11.28
		0.12	12.843	12.718	11.562	14.323	14.189	12.936

Table 4: **Equilibrium contract rates v.s. prepayment intensity and cost.** $\bar{r}_c[\%]$ p.a. monthly compounded. House price volatility $\sigma = 15\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

basis points, from 0.2). When interest rates fall to 2%, the gap widens (84.1 basis points for $\sigma = 15\%$) and is wider than in the low volatility case (20.8 basis points for $\sigma = 5\%$).

Conversely, when δ is high relative to r, the building loses value at a higher rate and in such circumstances even the FRMs have contract rates significantly above the prevailing interest rates. For example this happens when $\delta = 12\%$ and $r = 2\%$. For $l = 95\%$ the FRM contract rate is 4.725%, so well above the 2%, but increases to 12.063% for a CWM (see again Table 2). This is because in such economic conditions the CWM contract rate predominantly reflects the price of insuring against house price decline and the decline rate is imposed to occur at the rate of 12% per annum via the coefficient δ. Interestingly, by lowering the loan to value l to 80%, it is possible to slightly lower the FRM rate to 4.543%, but the CWM rate stays put at 12.063%.

Tables 2, 3 and 4 also provide the equilibrium contract rates when the prepayment risk is present ($\lambda > 0$) and taken into consideration when designing the loan. The latter is implemented by imposing some prepayment penalties ($\phi > 0$) in the form of percentage of the value prepaid. We consider two scenarios: one where prepayment risk and penalties are moderate ($\lambda = 1/year$, $\phi = 1\%$); and a more extreme situation, where customers are very impatient to prepay ($\lambda = 10/year$) but are strongly penalised when doing so ($\phi = 10\%$). Not surprisingly, we observe that it is possible to impose prepayment penalties high enough so as to offer "attractive" contract rates below the current interest rate level. For example when when the house price volatility is low enough ($\sigma = 5\%$) and when $r = \delta = 2\%$ we get 1.639% and 1.839% for FRM and CWM, respectively. See Table 2. However, when volatility increases to $\sigma = 10\%$ and 15% (see Tables 3 and 4, respectively) contract rates rise above $r = 2\%$, even with imposition of prepayment costs.

4.4 CWM v.s. FRM embedded default option values

In Tables 5, 6 and 7 we compute the values of embedded put options to default as a percentage fraction of the initial value, Q_0, of the loan granted. Because $H_0 = 1$, Q_0 is numerically equal to l, the loan to value ratio, i.e. in tables we effectively report D^{FRM}/l and D^{CWM}/l. This is for easier comparisons, per dollar of loan granted, between scenarios with different l-s. These values are consistent with CWM and FRM contract rates we discovered in the previous section.

CWM's embedded default option values tend to be several orders of magnitude lower than FRM's. This is most prominent where the service flow is high compared to the interest rate, e.g. $\delta = 12\%$ and $r = 2\%$. In this region the put option values are virtually zero for CWMs but highest for FRMs. Moreover, and consistently for all l-s and any prepayment scenario, values of options to default embedded in FRMs always rise with increasing service δ. However, the latter cannot be said of CWMs where the effect of δ is not always monotone, suggesting some non-proportional interplay going on.

To understand these effects we observe that when the service flow is rising, two opposite effects can happen. Higher "rent" δ extracted from the property causes faster house price decline and thus precipitates the rational default. On the other hand, the early default boundary $\underline{\xi}^{FRM}$ retracts because it is now more interesting to keep the property in order to extract that rent, as opposed to rationally default and invest (unpaid) mortgage payments at the riskless rate r.

For example, when $\sigma = 5\%$ and $l = 95\%$ our computations show that for $r = \delta = 2\%$ the early default boundaries at time $t = 0$ are at $\underline{\xi}^{FRM} \approx 0.80$ and $\underline{\xi}^{CWM} \approx 0.76$, respectively. If δ moves to 12%, the FRM boundary retracts to $\underline{\xi}^{FRM} \approx 0.29$. For the FRM, the declining price effect thus dominates because the default option value increases in value 10-fold, from 4.654 to 40.525 percent of the initial loan value (see again Table 5). In contrast, for the CWM, the default option is only worth 0.203 which is almost 20 times less than for the FRM. This is at first puzzling. However, further inspection of the early exercise boundaries reveals that while both start very close together at time $t = 0$, the FRM's boundary declines slowly to reach 0 at maturity $T = 30$, while the CWM's boundary sharply declines to zero within the first year or so (see Figures 10 and 11).[15] This explains the very low value of the CWM's embedded options. With boundary flat at zero any rational default beyond year 1 is highly improbable. This effect becomes larger when we move δ to 12% because the boundary becomes all flat equal to zero from time $t = 0$, meaning that CWM practically eliminates all rational defaults (see Figure 12). However, while early defaults and the embedded put are eliminated, contract rates of the CWM are much higher (see again Table 2). This reflects the high costs of insurance against house price decline when service rate, and thus the rate of decay of the structure, are high, meaning lower house values and rational defaults are more likely in future.

[15] Our finding is consistent with Ambrose and Buttimer (2012) who find zero default option values after 36 months in the quarterly updated variant of their ABM mortgage.

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	4.654	4.654	4.654	0.203	0.203	0.203
		0.06	23.034	23.034	23.034	0.	0.	0.
		0.12	40.525	40.525	40.525	0.	0.	0.
	0.06	0.02	0.206	0.206	0.206	0.144	0.144	0.144
		0.06	3.366	3.366	3.366	0.055	0.055	0.055
		0.12	21.399	21.399	21.399	0.	0.	0.
	0.12	0.02	0.007	0.007	0.007	0.006	0.006	0.006
		0.06	0.06	0.06	0.06	0.039	0.04	0.04
		0.12	2.279	2.279	2.279	0.	0.	0.
0.9	0.02	0.02	3.461	3.461	3.461	0.	0.	0.
		0.06	21.772	21.772	21.772	0.	0.	0.
		0.12	39.582	39.582	39.582	0.	0.	0.
	0.06	0.02	0.035	0.035	0.035	0.015	0.015	0.015
		0.06	2.293	2.293	2.293	0.	0.	0.
		0.12	20.113	20.113	20.113	0.	0.	0.
	0.12	0.02	0.	0.	0.	0.	0.	0.
		0.06	0.004	0.004	0.004	0.001	0.001	0.001
		0.12	1.367	1.367	1.367	0.	0.	0.
0.8	0.02	0.02	1.816	1.816	1.816	0.	0.	0.
		0.06	19.258	19.258	19.258	0.	0.	0.
		0.12	37.603	37.603	37.603	0.	0.	0.
	0.06	0.02	0.001	0.001	0.001	0.	0.	0.
		0.06	0.994	0.994	0.994	0.	0.	0.
		0.12	17.572	17.572	17.572	0.	0.	0.
	0.12	0.02	0.	0.	0.	0.	0.	0.
		0.06	0.	0.	0.	0.	0.	0.
		0.12	0.449	0.449	0.449	0.	0.	0.

Table 5: **Embedded default put option values for different prepayment scenarios.** $D^{FRM} D^{CWM}[\%]$ of the initial loan value Q_0. House price volatility $\sigma = 5\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	10.757	10.757	10.757	1.701	1.702	1.702
		0.06	25.738	25.738	25.738	0.	0.	0.
		0.12	41.637	41.637	41.637	0.	0.	0.
	0.06	0.02	2.577	2.577	2.577	1.587	1.588	1.589
		0.06	8.134	8.134	8.134	1.102	1.103	1.103
		0.12	23.315	23.315	23.315	0.	0.	0.
	0.12	0.02	0.629	0.629	0.629	0.49	0.491	0.491
		0.06	1.415	1.415	1.415	0.9	0.901	0.901
		0.12	5.862	5.862	5.862	0.571	0.573	0.573
0.9	0.02	0.02	9.387	9.387	9.387	0.362	0.363	0.363
		0.06	24.521	24.521	24.521	0.	0.	0.
		0.12	40.715	40.715	40.715	0.	0.	0.
	0.06	0.02	1.613	1.613	1.613	0.723	0.724	0.724
		0.06	6.796	6.796	6.796	0.093	0.093	0.093
		0.12	22.058	22.058	22.058	0.	0.	0.
	0.12	0.02	0.211	0.211	0.211	0.127	0.127	0.127
		0.06	0.704	0.704	0.704	0.304	0.305	0.305
		0.12	4.598	4.598	4.598	0.	0.	0.
0.8	0.02	0.02	6.977	6.977	6.977	0.	0.	0.
		0.06	22.065	22.065	22.065	0.	0.	0.
		0.12	38.777	38.777	38.777	0.	0.	0.
	0.06	0.02	0.582	0.582	0.582	0.087	0.087	0.087
		0.06	4.594	4.594	4.594	0.	0.	0.
		0.12	19.549	19.549	19.549	0.	0.	0.
	0.12	0.02	0.019	0.019	0.019	0.005	0.005	0.005
		0.06	0.154	0.154	0.154	0.016	0.016	0.016
		0.12	2.709	2.709	2.709	0.	0.	0.

Table 6: **Embedded default put option values for different prepayment scenarios.** $D^{FRM}D^{CWM}[\%]$ of the initial loan value Q_0. House price volatility $\sigma = 10\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

			FRM			CWM		
l	r	δ	none	low	high	none	low	high
0.95	0.02	0.02	16.77	16.77	16.77	3.327	3.329	3.329
		0.06	29.295	29.295	29.295	0.125	0.126	0.126
		0.12	43.326	43.326	43.326	0.	0.	0.
	0.06	0.02	6.476	6.476	6.476	3.364	3.366	3.367
		0.06	12.916	12.916	12.916	2.394	2.396	2.397
		0.12	25.96	25.96	25.96	0.	0.	0.
	0.12	0.02	2.406	2.406	2.406	1.745	1.746	1.747
		0.06	4.06	4.06	4.06	2.268	2.271	2.272
		0.12	9.527	9.527	9.527	1.517	1.519	1.521
0.9	0.02	0.02	15.424	15.424	15.424	1.621	1.622	1.623
		0.06	28.141	28.141	28.141	0.	0.	0.
		0.12	42.436	42.436	42.436	0.	0.	0.
	0.06	0.02	5.185	5.185	5.185	2.064	2.065	2.066
		0.06	11.544	11.544	11.544	0.901	0.902	0.903
		0.12	24.747	24.747	24.747	0.	0.	0.
	0.12	0.02	1.471	1.471	1.471	0.875	0.877	0.877
		0.06	2.916	2.916	2.916	1.198	1.2	1.201
		0.12	8.167	8.167	8.167	0.322	0.324	0.324
0.8	0.02	0.02	12.855	12.855	12.855	0.002	0.002	0.002
		0.06	25.781	25.781	25.781	0.	0.	0.
		0.12	40.562	40.562	40.562	0.	0.	0.
	0.06	0.02	3.195	3.195	3.195	0.557	0.558	0.558
		0.06	9.039	9.039	9.039	0.	0.	0.
		0.12	22.297	22.297	22.297	0.	0.	0.
	0.12	0.02	0.504	0.504	0.504	0.163	0.163	0.164
		0.06	1.417	1.417	1.417	0.215	0.216	0.216
		0.12	5.838	5.838	5.838	0.	0.	0.

Table 7: **Embedded default put option values for different prepayment scenarios.** $D^{FRM}D^{CWM}[\%]$ of the initial loan value Q_0. House price volatility $\sigma = 15\%$, mortgage term $T = 30$ years, points $\pi = 0$. Prepayment intensity and cost: $\lambda = 0, \phi = 0$ (none), $\lambda = 1/year, \phi = 1\%$ (low), $\lambda = 10/year, \phi = 10\%$ (high).

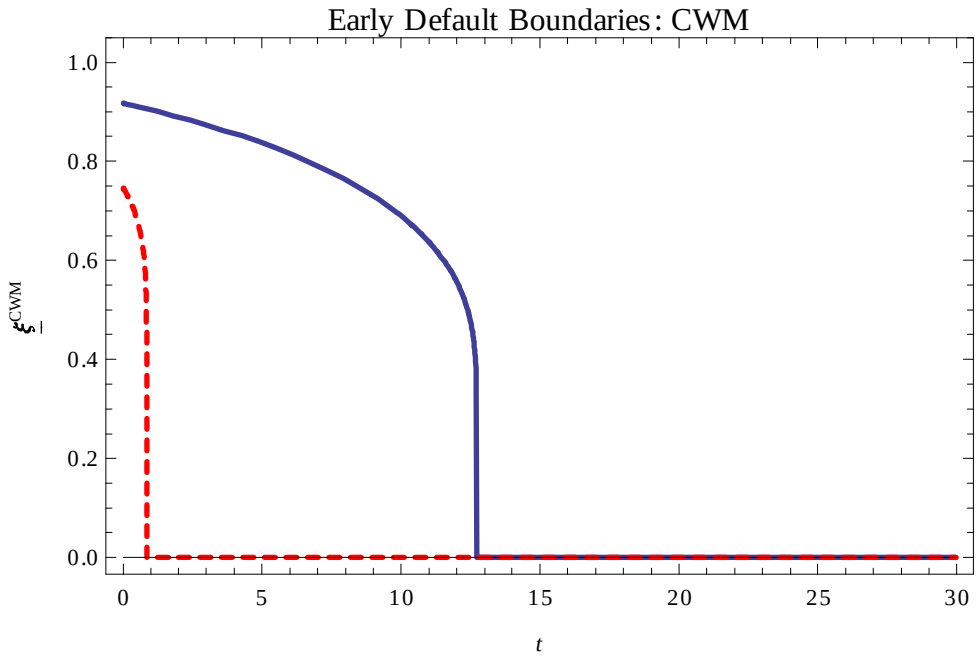


Figure 10: **Early default boundaries of a CWM as a function of time to maturity t.** Service flow ratios: $\delta = 2\%$ (thick line), $\delta = 6\%$ (dashed, thick), $\delta = 12\%$ (thin). Interest rate $r = 6\%$, house price volatility $\sigma = 5\%$, prepayment intensity $\lambda = 0$, prepayment cost $\phi = 0$, points $\pi = 0$.

5 Data and calibration of house price index paths

We use S&P/Case-Shiller home price index. The "10-city" composite 318 not seasonally adjusted monthly observations starts in January 1987. We also check our estimates against the "20-city" composite, which is a shorter series with 162 monthly observations, starting in January 2000. Both series are normalized to 100 in January 2000 and span the period up until July 2013. Before proceeding we check for unit roots. Before using calibrated parameters in simulations we validate them against estimates obtained using a much longer time series data. This series, which we short name "Shiller", has been originally used to produce illustrations in *Irrational Exuberance* (Shiller (2005)) and *Subprime Solution* (Shiller (2008b)) books. It is regularly updated and available for download at `www.econ.yale.edu/~shiller/data/`. It has annual observations of nominal and real home price indices from 1892 till 1952. Observations are then monthly. Either interpolating data pre-1953 or annualizing post-1952 sample we obtain similar results. We therefore only report results from interpolated data.

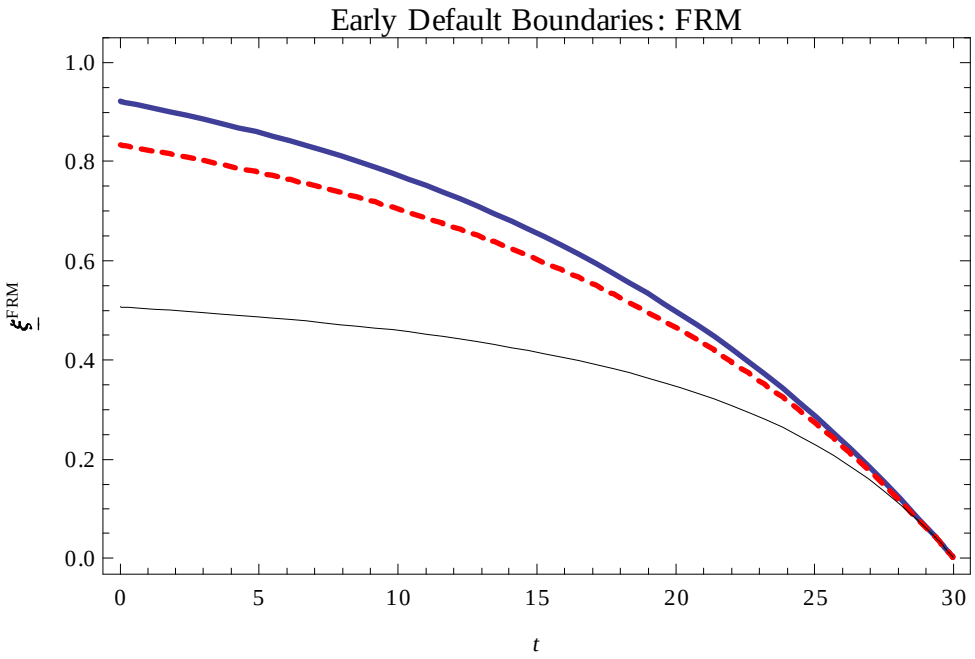


Figure 11: **Early default boundaries of a FRM as a function of time to maturity t.** Service flow ratios: $\delta = 2\%$ (thick line), $\delta = 6\%$ (dashed, thick), $\delta = 12\%$ (thin). Interest rate $r = 6\%$, house price volatility $\sigma = 5\%$, prepayment intensity $\lambda = 0$, prepayment cost $\phi = 0$, points $\pi = 0$.

Using detailed data and tools researchers revealed inefficiencies in house prices (see e.g. Case and Shiller (1989), Tirtiroglu (1992)), structural breaks (see Canarella et al. (2012)), market segmentation (Montañés and Olmos (2013)) and ripple effects (UK: Meen (1999) and US: Canarella et al. (2012)). There is, however, evidence in favour of unit roots in house price index capital gains. Canarella et al. (2012), for example, confirm unit roots in logarithmic differentials of seasonally adjusted regional S&P/Case–Shiller indices for 10 US cities, including the 10-city composite. If unit roots are present in these time series we must have $\phi = 1$ in the following regression

$$\Delta y_t = \alpha + \beta t + (\phi - 1)\, y_{t-1} + \epsilon_t \,. \tag{77}$$

This means that the change in the capital gain $\Delta y_t = y_t - y_{t-1}$, where $y_t = \ln \xi_t$, does *not* depend on the capital gain at $t - 1$. In other words y follows an arithmetic Brownian motion. Taking exponential gives

$$\xi_t \approx \xi_{t-1} e^{\alpha + \beta t + \epsilon_t} \,. \tag{78}$$

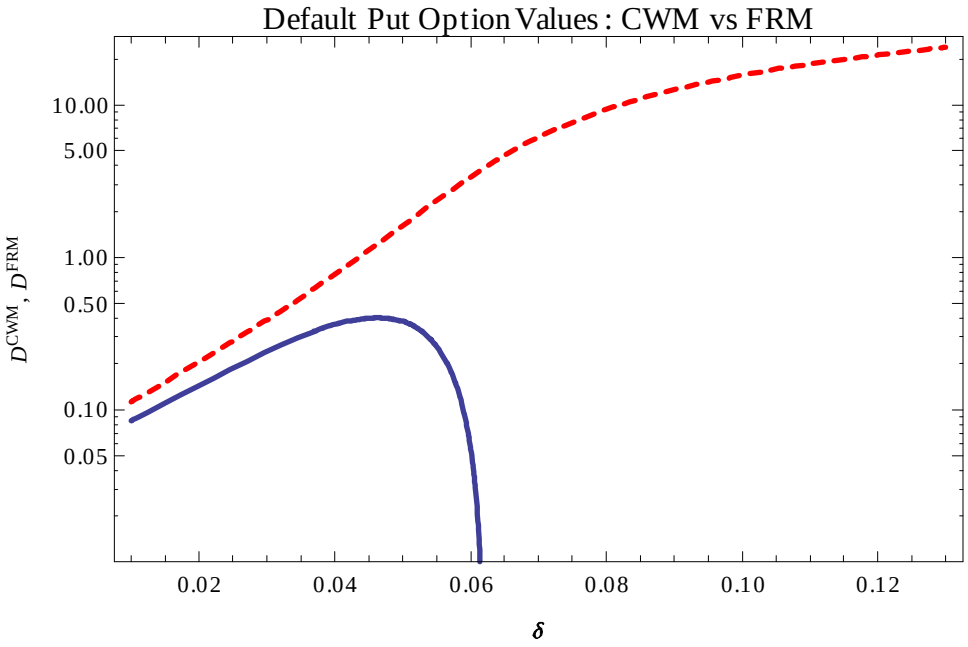


Figure 12: **Embedded default put option values as a function of service flow ratio δ.** Thick line: CWM. Dashed line: FRM. Log-scale. Interest rate $r = 6\%$, house price volatility $\sigma = 5\%$, prepayment intensity $\lambda = 0$, prepayment cost $\phi = 0$, points $\pi = 0$.

	10-city	**20-city**	**Shiller**	Simulations
	1987-	2000-	1890-	
$\hat{\mu}$	0.0395166	0.0367472	0.0310475	0.03
$\hat{\sigma}$	0.0325394	0.0397048	0.0404518	0.01–0.10

Table 8: S&P/Case-Shiller home price index calibration.

Therefore, as a first approximation, we could assume that the house price index is log-normally distributed. Consequently, in a first attempt to model the random behaviour of the house price index we calibrate a geometric Brownian motion for drift and volatility. Also, because of a possible bias toward nonrejection of the unit root hypothesis in seasonally adjusted data (see Ghysels and Perron (1993)), before proceeding we first check for unit roots using non seasonally adjusted and longer time series data. The results of our calibration are reported in Table 8.

We set μ close to the long term trend 3% per annum and we do not retain a particular value for the volatility in our simulations. As observed from a longer perspective, volatility is on average somewhere between 3% and 4% (see Figure 1). However, we adopt a broader range in our simulations. Historically, there were periods of higher and lower

volatility. Using interpolated annual data (pre 1953) and quarterly data (post 1953), a moving volatility estimate is obtained, quarter by quarter. Each estimate is using 40 most recent quarterly observations spanning the last 10 years. This is illustrated on Figure 1.

6 Comparing Continuous Workouts with Fixed Rates

It is difficult to uniquely say what would constitute a good mortgage product. Every prospective homeowner will apply different criteria so it is safer to consider a range of alternative cases and specific situations. To establish when a risk averse mortgagor will be most likely to prefer CWM to FRM we employ a utility-based argument. Borrowers who are unable to dynamically adjust their portfolios, so as to finance their home purchase directly on financial markets, must choose either a CWM or an FRM i.e. they cannot use a mix of two smaller mortgages. Denote this choice by $m \in \{CWM, FRM\}$.

6.1 Intertemporal utilities

Recognizing the fact that mortgage payments occur monthly, intertemporal utilities can be represented as discrete sums. A risk-averse mortgagor maximizes

$$\max_{m\in\{\text{FRM,CWM}\}} \mathcal{U}_m = E\sum_{i=1}^{360} \exp\left\{-\rho\frac{i}{12}\right\} u\left(S - Y_m\left(\frac{i}{12}\right)\right) \tag{79}$$

over the term $T = \frac{360}{12} = 30$ years, where ρ is the subjective discount rate, S are the funds available (salary) and $Y_m(t)$ is the mortgage payment at time $t = \frac{i}{12}$. We also require $S > Y_m(t)$ for all t. This is readily achieved for FRM for which $Y_{\text{FRM}}(t) = R^{FRM}$ by requiring $S > R^{FRM}$ initially. Using R^{FRM} given by (68) we obtain[16]

$$\mathcal{U}_{\text{FRM}} = \sum_{n=1}^{30} \exp\{-\rho n\} u\left(S - R^{FRM}\right) \approx A(\rho, T)\, u\left(S - R^{FRM}\right). \tag{80}$$

Note that because payments are all constant, known in advance from the moment when the contract is signed, FRM removes all randomness other than prepayment and default.

[16]To simplify notation in what follows we use years $n = 1\ldots30$ rather than months $i = 1\ldots360$ as the summation grid.

Both the prepayment and default risk have been included into the computation of the annual payment R^{FRM}.

In contrast, CWM payments are capped by $\bar{R}^{CWM} > R^{FRM}$, but the mortgagor doesn't know their amounts in advance. Therefore, for the CWM we compute the expected utility which is a function of the future evolution of the house price index ξ

$$\mathcal{U}_{\text{CWM}} = E \sum_{n=1}^{30} \exp\{-\rho n\}\, u\left(S - R^{CWM}(\xi_n)\right) . \tag{81}$$

CWM dynamically aligns the outstanding loan balance to changes in collateral value by "controlling" the repayment flow $R^{CWM}(\xi_n)$. Expectations in (81) require the "real world" probability $\mathcal{P}$. This means we specify information about the actual drift μ of house prices. If the utility of consumption is of constant relative risk aversion type

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} , \tag{82}$$

then γ is the risk aversion parameter. For $\gamma = 0$ the individual is risk-neutral and $\gamma = 1$ corresponds to logarithmic utility.

6.2 Simulation results

Expectations in (81) can be evaluated by numerical integration under the real-world probability $\mathcal{P}$. We check our results against monthly simulations over 30 years, where one path requires 360 points drawn from a normal distribution. Each path determines intermediate and terminal cash-flows of the CWM mortgage. After simulating many paths we compute sample average utilities and stop simulations when results stop changing significantly. We repeat expected utility computations for different parameter values.

An individual is planning to acquire a home over a 30-years horizon. There is choice between an FRM and a CWM mortgage, both priced including default and prepayment risks. When computing FRM and CWM contract rates to be offered, respectively r_c^{FRM} and r_c^{CWM}, lenders take into account borrower's optimal default and prepayment decisions.

We set the annual wage S to a constant value of 20% (i.e. 1/5th) of the original house price $H_0 = 1$. The mortgagor relative risk-aversion γ vary in the range from 0.1 to 3.7 and the

rate of intertemporal impatience ρ from 1% to 28%. We compute the net dollar welfare gain, defined as

$$Net\ Gain = \sum_{n}^{30} PV\left(\$\text{-equivalent Gain}\left(n\right)\right) - \sum_{n}^{30} E\left[PV\left(\text{Cost}\left(n,\xi\right)\right)\right] \tag{83}$$

where the \$-equivalent gain

$$\$\text{-equivalent Gain}\left(n\right) = R^{FRM} - \hat{R}^{FRM} \tag{84}$$

refers to the annual dollar amount, in year n, by which the payment of an FRM should be lowered for the fixed rate borrower to enjoy the same welfare level as a holder of a CWM. In our simulations the dollar-equivalent gain (84) turns out to always be positive, for all ranges of parameters considered. We compute $\mathcal{U}_{\text{FRM}}$ (analytically) and $\mathcal{U}_{\text{CWM}}$ (numerically) and then solve for $\hat{R}^{FRM}$. The cost in year n is

$$\text{Cost}\left(n,\xi\right) = R^{CWM}\left(\xi\right) - R^{FRM} \tag{85}$$

That is, the difference between an actual CWM payment and an FRM payment. This cost may become a gain, i.e. $\text{Cost}(n,\xi) < 0$ when house prices are in relative decline. Under such circumstances the house price index ξ is low, $R^{CWM}\left(\xi\right)$ is below R^{FRM} and CWM pays off, compared to an otherwise identical but more expensive FRM. As required, we discount riskless cash flows at the riskless rate r and the risky cash flows, which are all related to the random CWM payments $R^{CWM}\left(\xi\right)$, at the contract rate discount rate r_c^{CWM}.

Below we briefly report and comment on the typical numerical results we obtain within our framework.

Empirical studies have confirmed constant relative risk-aversion. Coefficient estimates consistently range from 1 to 4 with average γ close to 2 (see e.g. Szpiro (1986b), Szpiro (1986a) and Chiappori and Paiella (2011)). In our simulations we change the value of γ in the range $0 < \gamma < 5$. It turns out our results are not very sensitive to the degree of risk aversion, therefore we set γ to 2.1, which is closest to 2 on our grid.

Capital gain or loss on house prices is an important factor for buyers to engage in taking up or defaulting on mortgage loans. Therefore, in our setup, the net gains in equation (83) are sensitive to whether house prices are perceived, in the real world, as relatively

decreasing ($\mu < r - \delta$) or increasing ($\mu > r - \delta$). Therefore we set $r - \delta$ to 4% ($r = 6\%$ and $\delta = 2\%$) and we present two sets of outcomes, one for which $\mu = 1\%$ and another for $\mu = 7\%$. The net gains are also quite sensitive to house price volatility. Therefore, in Figures 13 and 14 we consider σ's equal to 5% (thin line), 10% (dashed line) and 15% (thick line). Figure 13 shows that when house prices are expected to be relatively decreasing ($\mu = 1\%$), the CWM appears to be most attractive. For high levels of volatility ($\sigma = 15\%$) a patient household ($\rho = 1\%$ $p.a.$) expects to achieve more than a $\$30,000$ dollar-equivalent gain by choosing a CWM over an FRM, when purchasing a $\$100,000$ house. Not surprisingly, the relative advantage of the CWM diminishes when the outlook for house prices improves ($\mu = 7\%$). This decreases the net gains (from $\$30,000$ to just below $\$7,000$ when $\sigma = 15\%$) but does not deter even the more myopic households, or when the volatility is low ($\sigma = 5\%$), from preferring CWMs to FRMs. See again Figure 14.

A more realistic numerical simulation would include adding more dimensions to our model, but would invariably render computations less tractable. We leave these important dimensions for future research. They could include: (a) parametrizing how good a saver the prospective homeowner is e.g. the proportion of disposable income saved after paying off the mortgage;[17] (b) parametrizing the housing rental market through the lifetime and modeling its impact on default and/or prepayment decisions; (c) specifying whether or not the homeowner can move back to the starting point (e.g. parent's house where they lived before taking on the mortgage) in case they decide to optimally default; (d) specifying a cap on maximal number of defaults or prepayments during the lifetime and the role of credit rating agencies; (e) specifying the proportion of total population who actually optimally default; (f) specifying whether the borrowers intend to stay in the property or to move into a retirement accommodation after reaching the mortgage maturity; etc.

Finally, we add the following disclaimer as our model encapsulates the key features but in a very simple way. In the real world an increase in volatility of house prices, for example, would impact the level or volatility of incomes (and vice versa). As a result, when incomes are fixed, and especially when the house price volatility is high, the more risk-averse homeowners may, perversely, perceive CWMs as more risky because these contracts require payments which vary with the level of house prices. Instead, some buyers

[17] This could be used, for example, for prepaying and acquiring the property earlier than originally planned, if a random prepayment event occurs before maturity. See Section 4.

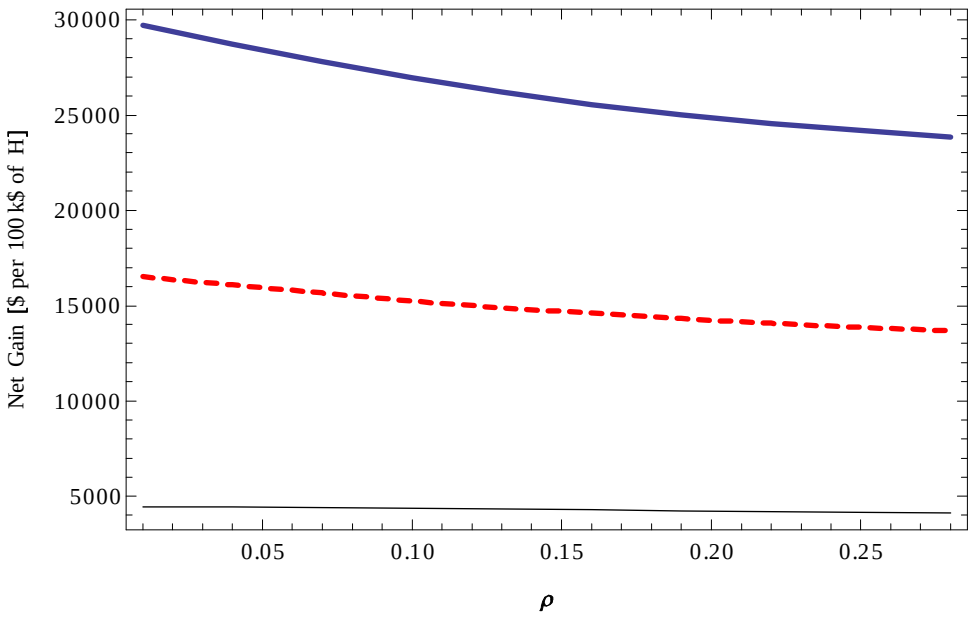


Figure 13: **Net dollar gain of a CWM v.s. FRM: *decreasing* house prices.** House price trend $\mu = 1\%$. Riskless rate $r = 6\%$, service parameter $\delta = 2\%$. House price volatilities: $\sigma = 5\%$ (thin line), 10% (dashed line) and 15% (thick line).

may prefer locking in the comfort of the fixed monthly payments that the FRMs provide.

7 Concluding remarks

This paper shows how to mitigate the fragility of the economy to plain vanilla mortgages by advocating Continuous Workout Mortgages (CWMs). As pointed out by Shiller (2008b), Shiller (2008a) and Shiller (2009), these are related to Price Level Adjusting Mortgages (PLAMs), recommended by Modigliani (1974) for high inflation regimes. CWMs share some positive attributes with PLAMs (such as purchasing power risk) and mitigate other negative attributes (such as default risk, interest rate risk and prepayment risk). The liquidity risk can be alleviated by employing CWMs in sufficient volume to warrant their securitization.

Our approach is consistent with the literature even though we have not used an equilibrium based interest model like the Cox, Ingersoll, and Ross (1985) to motivate prepayments. We adopt the intensity framework of Jarrow and Turnbull (1995) for random prepayments instead. The novelty of our CWM ensues by going beyond the ABM of

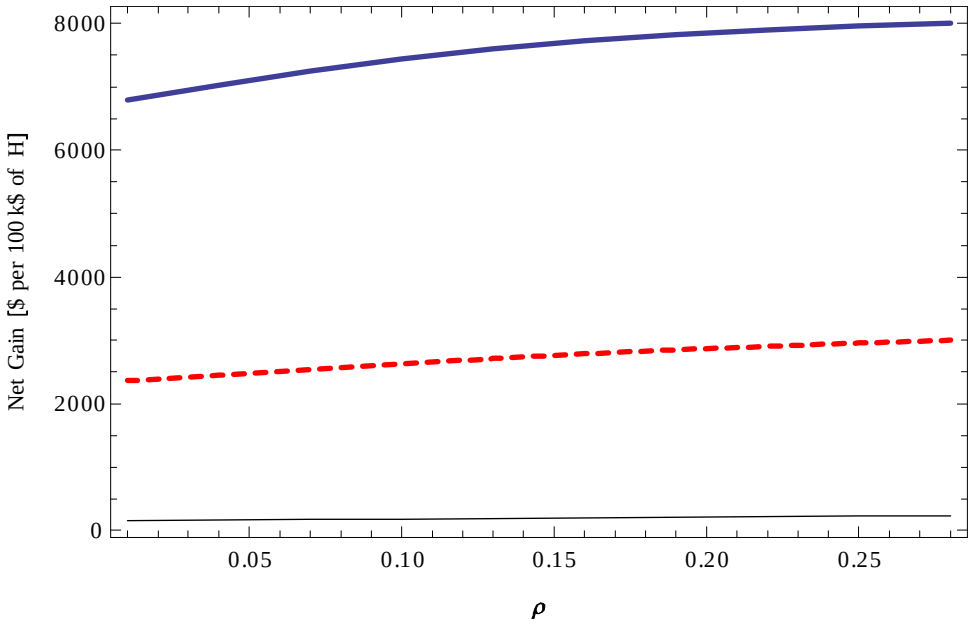


Figure 14: **Net dollar gain of a CWM v.s. FRM: *increasing* house prices.** House price trend $\mu = 7\%$. Riskless rate $r = 6\%$, service parameter $\delta = 2\%$. House price volatilities: $\sigma = 5\%$ (thin line), 10% (dashed line) and 15% (thick line).

Ambrose and Buttimer (2012) by resurrecting it as a non-cumulative income bond.

We evaluate the CWM contract relying on assumptions of option pricing techniques. In this framework speculators are necessary to enable risk to be shifted from hedgers at the correct (arbitrage-free) price. The arbitrage-free price is ensured by a simultaneous presence of arbitrageurs in the market for homes and/or derivative claims contingent on house prices.

Hybrid facilities such as CWMs mitigate risk shifting aspect of agency cost of debt through their structure, and thus deter defaults. Hybrids like preferred stock and income bonds have played a vital role in the development of countries like the USA (Evans Jr., 1929; Fischer and Wilt Jr., 1968; McConnell and Schlarbaum, 1981). Preferred stocks have still thrived through the ages while income bonds have withered (McConnell and Schlarbaum, 1981; Kalberg and Villopuram, 2013). The rationale behind this is given by Stiglitz (1989), who attributes the demise of income bonds is due to the 'contingent income' manipulation by firms. From this perspective, the CWM discussed in our paper can be seen as an attempt to resurrect a non-cumulative income bond in the context of housing finance, by ensuring that there is no underlying asset manipulation by homeowners. We

do this by recommending a CWM contingent on the local house price index (instead of the asset i.e. the borrower's house value).

We evaluated the expected utility of the mortgagor by using the expected utility function (81). This captures the welfare from the demand side of loans. Our results reveal a substantial welfare gain to creating CWMs. In our simulations CWMs appear to be more attractive than FRMs at current volatility levels and for typical values of risk aversion. At these levels agents prefer CWMs to FRMs.

The results of our simulations are encouraging, because measured welfare gains are positive. However, we do not advocate CWMs for everybody. As any insurance product, CWMs may turn out to be costly. If only good times occur in future, the embedded insurance premia will have to be paid making CWMs more expensive ex post, compared to FRMs.

The linear design considered here is appealing by its simplicity and should be easy to explain to a customer. Non-linear and potentially more efficient designs should also be possible.[18] We conjecture that experimenting with these designs should result in an even better immunization against the default risk. However, we leave these experiments for future research.

References

Ambrose, B. W. and R. J. Buttimer. 2012. The adjustable balance mortgage: Reducing the value of the put. *Real Estate Economics* 40: 536–565.

Azevedo-Pereira, J. A., D. P. Newton and D. A. Paxson. 2003. Fixed-rate endowment mortgage and mortgage indemnity valuation. *Joumal of Real Estate Finance and Economics* 26: 197–221.

Barone-Adesi, G. and R. E. Whaley. 1987. Efficient analytic approximation of American option values. *Journal of Finance* 42: 301–320.

Black, F. and M. Scholes. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81: 637–654.

[18] We are grateful to Peter Carr for pointing this out to us.

Canarella, G., S. Miller and S. Pollard. 2012. Unit roots and structural change: An application to US house price indices. *Urban Studies* 49: 757–776.

Carr, P., R. Jarrow and R. Myneni. 1992. Alternative characterizations of American put options. *Mathematical Finance* 2: 87–106.

Carr, P., A. Lipton and D. Madan. 2000. Going with the flow. *Risk* 13: 85–89.

Case, K. E. and R. J. Shiller. 1989. The efficiency of the market for single-family homes. *American Economic Review* 79: 125–37.

Chiappori, P.-A. and M. Paiella. 2011. Relative risk aversion is constant: Evidence from panel data. *Journal of the European Economic Association* 9: 1021–1052.

Cox, J., J. Ingersoll and S. Ross. 1985. A theory of the term structure of interest rates. *Econometrica* 53: 385–407.

Crank, J. and P. Nicolson. 1947. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Mathematical Proceedings of the Cambridge Philosophical Society* 43: 50–67.

Demirgüç-Kunt, A. and E. Detrajiache. 2002. Does deposit insurance increase banking system stability? an empirical investigation. *Journal of Monetary Economics* 49: 1373–1406.

Demyanyk, Y. and O. Van Hemert. 2011. Understanding the subprime mortgage crisis. *Review of Financial Studies* 24: 1848–1880.

Duarte, J. and D. A. McManus. 2011. Residential mortgage credit derivatives. *Real Estate Economics* 39: 671–700.

Evans Jr., G. 1929. The early history of preferred stocks in the united states. *American Economic Review* 19: 43–58.

Fabozzi, F. J., S. M. Focardi and C. Jonas. 2010. *Investment Management after the Global Financial Crisis*, Chapter 9: Looking Ahead, pp. 114–133. The Research Foundation of CFA Institute.

Fama, E. 1990. Term-structure forecasts of interest rates, inflation, and real returns. *Journal of Monetary Economics 25, 59-76.* 25: 59–76.

Fischer, D. and G. Wilt Jr.. 1968. Non-convertible preferred stocks as a financing instrument 1950-1965. *Journal of Finance* 23: 611–624.

Geske, R. and H. E. Johnson. 1984. The American put option valued analytically. *Journal of Finance* 39: 1511–1524.

Ghysels, E. and P. Perron. 1993. The effect of seasonal adjustment filters on tests for a unit root. *Journal of Econometrics* 55: 57–98.

Gorovoy, V. and V. Linetsky. 2007. Intensity-based valuation of residential mortgages: An analytically tractable model. *Mathematical Finance* 17: 541–573.

Hoshi, T. and A. Kashyap. 2004. Japan's financial crisis and economic stagnation. *Journal of Economic Perspectives* 18: 3–26.

Jarrow, R. A. and S. M. Turnbull. 1995. Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance* 50: 53–85.

Kac, M. 1949. On distributions of certain Wiener functionals. *Transactions of the American Mathematical Society* 65: 1–13.

Kalberg, J., L.-C. and S. Villopuram. 2013. Preferred stock: Some insights into capital structure. *Journal of Corporate Finance* 21: 77–86.

Kau, J. B., D. C. Keenan, W. J. Muller and J. F. Epperson. 1985. Rational pricing of adjustable rate mortgages. *Journal of the American Real Estate & Urban Economics Association* 13: 117–128.

Kau, J. B., D. C. Keenan, W. J. Muller and J. F. Epperson. 1992. A generalized valuation model for fixed-rate residential mortgages. *Journal of Money, Credit and Banking* 24: 279–299.

MacMillan, L. W. 1986. Analytic approximation for the American put option. *Advances in Futures and Options Research* 1: 119–139.

Martin, F. 2017. Lessons from inflation theorists of the past. Financial Times (July 28), p. 11.

McConnell, J. and G. Schlarbaum. 1981. Returns, risks, and pricing of income bonds 1956-76 (does money have an odor?). *Journal of Business* 54: 33–63.

Meen, G. 1999. Regional house prices and the ripple effect: A new interpretation. *Housing Studies* 14: 733–753.

Merton, R. C. 1973. The theory of rational option pricing. *Bell Journal of Economics* 4: 141–183.

Minerd, S. 2017. Rates are unlikely to skyrocket, despite pundits' predictions. Financial Times (July 18), p. 28.

Minsky, H. 1992. The financial instability hypothesis. The Jerome Levy Economics Institute Working Paper No. 74.

Modigliani, F. 1974. Some economic implications of the indexing of financial assets with special reference to mortgages. Paper Presented at the Conference on the New Inflation, Milano, Italy, June 24–26.

Montañés, A. and L. Olmos. 2013. Convergence in US house prices. *Economics Letters* 121: 152–155.

Piazzesi, M. and M. Schneider. 2016. Housing and macroeconomics. National Bureau of Economic Research (NBER) Paper Number 22354.

Renaud, B. 2003. Speculative behavior in immature real estate markets, lessons of the 1997 asia financial crisis. *Urban Policy and Research* 21: 151–173.

Roubini, N., S. S. Roach, D. M. Smick, R. J. Shiller and D. Baker. 2009. The worst is yet to come. *Foreign Policy* 170: 62–68.

Shackleton, M. B. and R. M. Wojakowski. 2007. Finite maturity caps and floors on continuous flows. *Journal of Economic Dynamics and Control* 31: 3843 – 3859.

Sharp, N. J., D. P. Newton and P. W. Duck. 2008. An improved fixed-rate mortgage valuation methodology with interacting prepayment and default options. *Journal of Real Estate Finance and Economics* 36: 307–342.

Shaw, W. 1998. *Modelling Financial Derivatives with Mathematica*. Cambridge University Press.

Shiller, R. J. 1993. *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks*. New York: Oxford University Press Inc.

Shiller, R. J. 2003. *The New Financial Order: Risk in the 21st Century*. New Jersey: Princeton University Press.

Shiller, R. J. 2005. *Irrational Exuberance* (2 ed.). Princeton University Press.

Shiller, R. J. 2008a. The mortgages of the future. *The New York Times* September 21:.

Shiller, R. J. 2008b. *Subprime Solution: How Today's Global Financial Crisis Happened and What to Do About It*. Princeton University Press.

Shiller, R. J. 2009. Policies to deal with the implosion in the mortgage market. *The Berkeley Electronic (B.E.) Journal of Economic Analysis and Public Policy* 9: Article 4.

Shiller, R. J. 2014. Why is housing finance still stuck in such a primitive stage? *American Economic Review* 104: 73–76.

Shiller, R. J., R. M. Wojakowski, M. S. Ebrahim and M. B. Shackleton. 2013. Mitigating financial fragility with continuous workout mortgages. *Journal of Economic Behavior & Organization* 85: 269–285.

Stiglitz, J. 1989. Financial markets and development. *Oxford Review of Economic Policy* 5: 55–68.

Szpiro, G. G. 1986a. Measuring risk aversion: An alternative approach. *The Review of Economics and Statistics* 68: 156–159.

Szpiro, G. G. 1986b. Relative risk aversion around the world. *Economics Letters* 20: 19–21.

Tirtiroglu, D. 1992. Efficiency in housing markets: Temporal and spatial dimensions. *Journal of Housing Economics* 2: 276–292.

Viegas, C. and J. Azevedo-Pereira. 2012. Mortgage valuation: a quasi-closed-form solution. *Quantitative Finance* 12: 993–1001.

Appendix

A Proofs

A.1 Proof of equations (9) and (10)

Between $s = 0$ and $s = t$ the dynamics of Q_s^{CWM-} is given by

$$\frac{dQ_s^{CWM-}}{ds} = rQ_s^{CWM-} - R^{CWM}\left(\xi_s\right) \tag{86}$$

For initial condition $Q_0^{CWM} = Q_0$ we have observed ξ_s up to time $t > s$, therefore this ODE has solution

$$\begin{aligned}
Q_t^{CWM-} &= Q_0 e^{rt} - \int_0^t R^{CWM}\left(\xi_s\right) e^{r(t-s)}\, ds && (87)\\
&= Q_0 e^{rt} - \int_0^t \bar{R}^{CWM} \min\left\{1, \xi_s\right\} e^{r(t-s)}\, ds && (88)\\
&= Q_0 e^{rt} - \int_0^t \bar{R}^{CWM} \left[1 - \left(1 - \xi_s\right)^+\right] e^{r(t-s)}\, ds && (89)\\
&= Q_0 e^{rt} - \bar{R}^{CWM} \int_0^t e^{r(t-s)}\, ds + \bar{R}^{CWM} \int_0^t \left(1 - \xi_s\right)^+ e^{r(t-s)} ds && (90)\\
&= Q_0 e^{rt} - \bar{R}^{CWM} \frac{e^{rt} - 1}{r} + \bar{R}^{CWM} \int_0^t \left(1 - \xi_s\right)^+ e^{r(t-s)} ds && (91)\\
&= \left[Q_0 - \bar{R}^{CWM} A\left(r, t\right)\right] e^{rt} + \bar{R}^{CWM} \int_0^t \left(1 - \xi_s\right)^+ e^{r(t-s)} ds && (92)
\end{aligned}$$

In particular note that considering two limiting cases $\xi_s \to 0$ and $\xi_s \geq 1$ (or $\xi_s \to +\infty$) for all $s \in [0, t]$ we conclude that there must be a lower and an upper bound to Q_t^{CWM-}

$$\left[Q_0 - \bar{R}^{CWM} A\left(r, t\right)\right] e^{rt} \leq Q_t^{CWM-} \leq Q_0 e^{rt}\,.$$

When $\bar{R}^{CWM} > R^{FRM} = \frac{Q_0}{A(r,T)}$ and for t close to maturity T, the lower bound can become a negative number, because of the strict inequality in

$$Q_0 - \bar{R}^{CWM} A\left(r, t\right) < Q_0 - \frac{Q_0}{A\left(r, T\right)} A\left(r, t\right) = Q_0 \left[1 - \frac{A\left(r, t\right)}{A\left(r, T\right)}\right] \xrightarrow{t \to T} 0\,. \tag{93}$$

A.2 Proof of equation (12) in Proposition 1

Using (6) in (8) we obtain

$$Q_t^{CWM+} = E_t\left[\int_t^T \bar{R}^{CWM}\left[1-(1-\xi_s)^+\right]e^{-r(s-t)}ds\right] \tag{94}$$

$$= \bar{R}^{CWM}\left\{A\left(r,T-t\right)-\int_t^T e^{-r(s-t)}E_t\left[(1-\xi_s)^+\right]ds\right\}. \tag{95}$$

We notice that the second term is a time integral of time t Black-Scholes put options with maturities within interval $s\in[t,T]$, each with strike price normalized to one. Because ξ_t follows a geometric Brownian motion under the martingale measure, with starting point normalized to $\xi_0=1$, so that

$$\xi_s = \exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)s+\sigma Z_s\right\} = \xi_t\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)(s-t)+\sigma\left(Z_s-Z_t\right)\right\}, \tag{96}$$

it is possible to obtain a closed form solution. For other distributions (e.g. empirically inferred) of the underlying our analysis still holds but the time integral will need to be evaluated numerically. It follows that

$$\int_t^T e^{-r(s-t)}E_t\left[(1-\xi_s)^+\right]ds = P\left(\xi_t,1,T-t,r,\delta,\sigma\right) \tag{97}$$

where P is the floor function: see Appendix B. Combining (97) and (94) gives (12) and proves Proposition 1.

A.3 Proof of equation (32)

We have

$$
\begin{aligned}
&\int_t^T \lambda e^{-(r+\lambda)(s-t)} E_t \left[P\left(\xi_s, 1, T-s, r, \delta, \sigma\right) \right] ds = \qquad (98)\\
&\quad = \int_t^T \lambda e^{-(\lambda+r)(s-t)} E_t \left[\int_s^T e^{-r(u-s)} E_s \left[\left(1-\xi_u\right)^+ \right] du \right] ds \\
&\quad = \int_t^T \lambda e^{-(\lambda+r)(s-t)} E_t \left[E_s \left[\int_s^T e^{-r(u-s)} \left(1-\xi_u\right)^+ du \right] \right] ds \\
&\quad = \lambda E_t \int_t^T e^{-(\lambda+r)(s-t)} \int_s^T e^{-r(u-s)} \left(1-\xi_u\right)^+ du ds \\
&\quad = \lambda e^{(r+\lambda)t} E_t \int_t^T \int_s^T e^{-\lambda s - ru} \left(1-\xi_u\right)^+ du ds \\
&\quad = \lambda e^{(r+\lambda)t} E_t \int_t^T e^{-ru} \left(1-\xi_u\right)^+ \left(\int_t^u e^{-\lambda s} ds \right) du \\
&\quad = \lambda e^{(r+\lambda)t} E_t \int_t^T e^{-ru} \left(1-\xi_u\right)^+ \left[-\frac{e^{-\lambda s}}{\lambda} \right]_t^u du \\
&\quad = \lambda e^{(r+\lambda)t} E_t \int_t^T e^{-ru} \left(1-\xi_u\right)^+ \frac{e^{-\lambda t} - e^{-\lambda u}}{\lambda} du \\
&\quad = e^{(r+\lambda)t} E_t \int_t^T e^{-ru} \left(1-\xi_u\right)^+ \left(e^{-\lambda t} - e^{-\lambda u} \right) du \\
&\quad = E_t \int_t^T e^{-r(u-t)} \left(1-\xi_u\right)^+ du - E_t \int_t^T e^{-(r+\lambda)(u-t)} \left(1-\xi_u\right)^+ du \\
&\quad = P\left(\xi_t, 1, T-t, r, \delta, \sigma\right) - P\left(\xi_t, 1, T-t, r+\lambda, \delta+\lambda, \sigma\right)
\end{aligned}
$$

where we changed the order of integration in double integrals and used the tower law for conditional expectations and the following identity

$$
-\left(\lambda + r\right)\left(s-t\right) - r\left(u-s\right) = \left(r+\lambda\right)t - s\lambda - ru\,. \qquad (99)
$$

B Floor flow option formula

Floors P on flow s with strike flow level k for finite horizon T can be computed using the following closed-form formula (see Shackleton and Wojakowski (2007)):

$$
\begin{aligned}
P\left(s_0, k, T, r, \delta, \sigma\right) = {} & A s_0^a \left(\mathbf{1}_{s_0<k} - N\left(-d_a\right) \right) - \frac{s_0}{\delta} \left(\mathbf{1}_{s_0<k} - e^{-\delta T} N\left(-d_1\right) \right) \qquad (100)\\
& + \frac{k}{r} \left(\mathbf{1}_{s_0<k} - e^{-rT} N\left(-d_0\right) \right) - B s_0^b \left(\mathbf{1}_{s_0<k} - N\left(-d_b\right) \right),
\end{aligned}
$$

where

$$A = \frac{k^{1-a}}{a-b}\left(\frac{b}{r} - \frac{b-1}{\delta}\right), \tag{101}$$
$$B = \frac{k^{1-b}}{a-b}\left(\frac{a}{r} - \frac{a-1}{\delta}\right),$$

and

$$a, b = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}, \tag{102}$$

whereas the cumulative normal integrals $N(\cdot)$ are labelled with parameters d_β

$$d_\beta = \frac{\ln s_0 - \ln k + \left(r - \delta + \left(\beta - \frac{1}{2}\right)\sigma^2\right)T}{\sigma\sqrt{T}} \tag{103}$$

(different to the standard textbook notation) for elasticity β which takes one of four values $\beta \in \{a, b, 0, 1\}$.

Standard Black and Scholes (1973) put on S with strike value of K can be computed using

$$p(S_0, K, T, r, \delta, \sigma) = Ke^{-rT}N(-d_0) - S_0e^{-\delta T}N(-d_1) \tag{104}$$

where d_0 and d_1 can be computed using formula (103) in which *values* S_0 and K can (formally) be used in place of *flows* s_0 and k.

Both floor (100) and put (104) formulae assume that the underlying flow or asset $x \in \{s, S\}$ is an Itô process driven by the equation

$$\frac{dx_t}{x_t} = \mu_x dt + \sigma_x dZ_t \tag{105}$$

with initial value x_0. When $\mu_x = r - \delta$ and $\sigma_x = \sigma$, (105) describes a geometric Brownian motion under the risk-neutral measure where Z_t is the standard Brownian motion, σ is the volatility, r is the riskless rate and δ is the service flow. In particular we assume that (105) then describes the dynamics of the repayment flow s. Similarly, (105) also defines the dynamics of the value S of the real estate property. By the Feynman-Kac theorem (see Kac (1949)) any contingent claim Ψ on a basket of flows or assets $x \in \{x_1 \ldots x_n\}$ must then satisfy the following partial differential

equation

$$\frac{1}{2}\sum_{x}\sum_{y}\gamma_{xy}\frac{\partial^2\Psi}{\partial x\partial y}+\sum_{x}\mu_x\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi}{\partial t}-r\Psi=0 \tag{106}$$

with appropriate boundary conditions, where $\gamma_{xy}=\sum_k\sigma_{xk}\sigma_{ky}:k,y\in\{x_1\dots x_n\}$ and σ_{xy} are elements of the variance-covariance matrix.