6-1-2018

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Giovanni Compiani

Philip A. Haile

Marcelo Sant'Anna

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Giovanni Compiani, Philip Haile, and Marcelo Sant'Anna

June 2018

Revised June 2019

COWLES FOUNDATION DISCUSSION PAPER NO. 2137R

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Common Values, Unobserved Heterogeneity, and Endogenous Entry in U.S. Offshore Oil Lease Auctions*

Giovanni Compiani*, Philip Haile†, and Marcelo Sant'Anna‡

*UC-Berkeley
†Yale University
‡FGV EPGE

June 21, 2019

Abstract

In a "common values" environment, some market participants have private information relevant to others' assessments of their own valuations or costs. Economic theory shows that this type of informational asymmetry can have important implications for market performance and market design. Yet even for the classic example of an oil lease auction, formal evidence on the presence and strength of common values has been limited by the problem of auction-level unobserved heterogeneity that is likely to affect both participation in an auction and bidders' willingness to pay. Here we develop an empirical approach for first-price sealed bid auctions with affiliated values, unobserved heterogeneity, and endogenous bidder entry. We show that important features of the model are nonparametrically identified and apply a semiparametric estimation approach to data from U.S. offshore oil and gas lease auctions. Our empirical results show that common values, affiliated private information, and unobserved heterogeneity—three distinct phenomena with different implications for policy and empirical work—are all present. Failing to account for unobserved heterogeneity obscures the empirical evidence of common values. We examine the implications of our estimates for the classic revenue ranking of sealed bid auction designs, and for the interaction between affiliation, the winner's curse, and the number of bidders in determining the aggressiveness of bidding and seller revenue.

*We are grateful to Ken Hendricks and Rob Porter, who generously provided the data, answered our many questions, and gave valuable feedback on early drafts. We also benefited from the comments and suggestions of Ali Hortaçsu, four anonymous referees, Estelle Cantillon, Xiaohong Chen, and numerous seminar and conference participants.

1 Introduction

In many auction settings it seems likely that important information commonly known among bidders is unobserved by the econometrician. Ignoring such unobserved heterogeneity can lead to a variety of errors. One may infer too much within-auction correlation in bidders' private information, as well as too much cross-auction variation in this information, leading to incorrect conclusions about such issues as bidder market power, the division of surplus, and optimal auction design.[1] In a first-price auction, unobserved heterogeneity presents a particular challenge because standard identification approaches exploit the insight that bidders' equilibrium beliefs about the competition can be inferred from observed distributions of bids;[2] with unobserved auction-level heterogeneity, bidders' beliefs condition on information unavailable to the econometrician. A further problem is that auction-level unobservables are likely to affect not only bids but also bidder participation. Such endogenous bidder entry threatens several identification and testing approaches relying on exogenous variation in the level of competition.[3]

Here we propose an empirical approach for first-price sealed bid auctions with affiliated values, unobserved heterogeneity, and endogenous bidder entry. We show nonparametric identification of key structural features and propose a semiparametric estimation approach. We apply the approach to auctions of offshore oil and gas leases in the United States Outer Continental Shelf ("OCS") in order to evaluate important features of these auctions, including the presence of common values.

Although the term "common values" (or "interdependent values") is often asso-

[1]See, e.g., Krasnokutskaya (2011), Krasnokutskaya and Seim (2011), Athey, Levin, and Seira (2011), and Roberts (2013).

[2]See, e.g., Laffont and Vuong (1993), Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), Hendricks, Pinkse, and Porter (2003), and Athey and Haile (2006, 2007).

[3]See, e.g., Gilley and Karels (1981), Athey and Haile (2002), Haile, Hong, and Shum (2003), Guerre, Perrigne, and Vuong (2009), Campo, Guerre, Perrigne, and Vuong (2011), and Gillen (2010).

ciated with auctions, it refers to a classic form of adverse selection that can arise in a broad range of environments where some parties to a potential transaction have private information relevant to others' assessments of their own valuations or costs (e.g., Akerlof (1970), Arrow (1970), Vincent (1989), Maskin and Tirole (1992), and Deneckere and Liang (2006)). Testing hypotheses about the nature of private information necessarily relies on an indirect approach, and auctions often offer settings in which the maintained assumptions relied upon can be tightly linked to actual market institutions.

But while an auction of drilling rights is frequently cited as an example of a common values environment, formal testing for common values has been hindered by the confounding effects of unobserved heterogeneity. Indeed, we reject private values in favor of common values only when accounting for unobserved heterogeneity and endogenous bidder entry. More broadly, we find that affiliated private information, common values, and common knowledge unobservables—three distinct phenomena with different implications for policy and empirical work[4]—are all present in OCS auctions. We also use our results to quantify Milgrom and Weber's (1982) classic revenue ranking of first- and second-price sealed bid auctions, and to examine the interaction between affiliation, the winner's curse, and the number of bidders in determining the aggressiveness of bidding and seller revenue.

Prior work on testing for common values in auctions includes Paarsch (1992), Athey and Haile (2002), Bajari and Hortaçsu (2003), Haile, Hong, and Shum (2003), Hill and Shneyerov (2014), and Hortaçsu and Kastl (2012). Most of this work, like ours, exploits the fact that in a common values auction the winner's curse becomes more severe as the number of competitors grows (all else equal). Our testing approach

[4]For example, affiliation leads to the "linkage principle" (Milgrom and Weber (1982), Milgrom (1987)) whereas common values leads to the "winner's curse," each with potentially important implications for auction design. Unobserved heterogeneity, which is held fixed in auction theory, implies neither affiliation nor common values but creates challenges for identification.

is most similar to that in Haile, Hong, and Shum (2003) ("HHS"), discussed further below, who studied timber auctions. Our generalizations of their model relax their most restrictive assumptions but make identification substantially more challenging and require a different estimation approach.

Our empirical study of OCS auctions is related to that of Hendricks, Pinkse, and Porter (2003) ("HPP"), who focused on testable implications of a pure common values model.[5] Our work is complementary to theirs. We allow the pure common values model but do not assume it, and we neither exploit nor rely on estimates of realized tract values. HPP point out that tests for common values would be difficult to apply due to likely correlation between bidder entry and auction-level unobservables. Our study also complements Haile, Hendricks, and Porter (2010) and the simultaneous work of Aradillas-Lopez, Haile, Hendricks, and Porter (2019), which focus on implications of competitive (vs. collusive) bidding in OCS auctions after the introduction of "area-wide leasing" in 1983. Here we consider only the period 1954–1983, where the evidence in Haile, Hendricks, and Porter (2010) supports the assumption of competitive bidding, and we focus on methodological and substantive issues driven by the nature of bidders' information—both shared and private.

Haile and Kitamura (2019) review existing econometric approaches to first-price auctions with unobserved heterogeneity. All require compromises of some form. Several (e.g., Krasnokutskaya (2011), Hu, McAdams, and Shum (2013), D'Haultfoeuille and Fevrier (2015)) require that bidders have independent types, enabling all correlation among bids to be attributed to unobserved heterogeneity.[6] Krasnokutskaya and Seim (2011) and Gentry and Li (2014) have extended these methods to models with

[5]Earlier work on OCS auctions includes Gilley and Karels (1981), Hendricks, Porter, and Boudreau (1987), Hendricks and Porter (1988), and Hendricks, Porter, and Spady (1989), as well as Li, Perrigne, and Vuong (2000), who called for formal testing for common values.

[6]See also the partial identification results in Armstrong (2013). An exception among approaches building on the measurement error literature is Balat (2017). His extension of Hu, McAdams, and Shum (2013) exploits observation of potential bidders' entry decisions at two sequential stages.

endogenous entry. Other approaches to unobserved heterogeneity use a control function strategy requiring a one-to-one mapping between the unobserved heterogeneity and an observed auxiliary outcome (often, the number of bidders), allowing one to indirectly condition on the unobservable (e.g., Campo, Perrigne, and Vuong (2003), HHS, Guerre, Perrigne, and Vuong (2009), Roberts (2013)). Simultaneous work by Kitamura and Laage (2018) proposes a finite mixture approach allowing affiliated types but requiring that the unobservable be discrete (cf. HHS and Hu, McAdams, and Shum (2013)) and enter through a separable structure similar to that in Krasnokutskaya (2011). Finally, while control function approaches can provide a strategy for isolating exogenous variation in bidder entry, others generally do not.

Our approach requires compromises as well. We rely on an index assumption similar to that in Krasnokutskaya (2011) and Kitamura and Laage (2018). Like HHS, we require an instrument for entry and rely on a reduced form for the entry outcome in which the auction unobservable is the only latent factor. This rules out selective entry, for example. And our use of a reduced form implies that additional structure would be needed to evaluate interventions that would alter the map from auction characteristics to entry outcomes (see, e.g., section 8.1.).

But our approach also offers advantages. It avoids the requirement of independent bidder types and provides a strategy for exploiting exogenous sources of variation in bidder entry. This combination of features is particularly important in our application. Common values settings generally demand that we allow correlated types (signals), and our test for common values relies on exogenous variation in entry arising through an instrument. We also avoid requiring a bijection (conditional on covariates) between entry and the unobservable (cf. Campo, Perrigne, and Vuong (2003), HHS, and Guerre, Perrigne, and Vuong (2009))—a requirement that can be difficult to rationalize and which limits the support of the unobservable. In contrast, we show that our empirical model can be derived from a two-stage game motivated by our application—an entry stage à la Berry (1992) in which bidders choose whether to acquire a signal of the good's value, followed by competitive bidding à la Milgrom and Weber (1982). In this example, the underlying unobserved heterogeneity may have arbitrary dimension and unrestricted support, may be correlated with observables, may exhibit spatial dependence, and may affect sample selection.

The following section describes our model. In Section 3 we address nonparametric identification. Section 4 describes our proposed estimation method. We then narrow our focus to the OCS auctions, with model estimates discussed in section 5. We examine the tests for common values in section 6, then explore revenue implications of our estimates in section 7. We discuss several caveats, extensions, and directions for future work in section 8 before concluding in section 9.

2 Model

We consider a standard model of first-price sealed bid auctions with symmetric affiliated values, extended to allow for auction-level heterogeneity and endogenous bidder entry. Each auction t is associated with observed characteristics $X_t \in \mathbb{X}$ and a scalar unobservable U_t. Without further loss, we let U_t be uniformly distributed on $[0,1]$. We also assume independence between X_t and U_t.

Assumption 1. $X_t \perp\!\!\!\perp U_t$.

The restriction to a scalar unobservable independent of X_t is less restrictive than it may appear. We show below that this representation can be derived—without loss of generality for most purposes motivating estimation of an auction model—from a model in which auction-level unobservables have arbitrary dimension and arbitrary dependence with X_t. In that model, the weak monotonicity conditions required below are also obtained as results rather than assumptions.

For each auction t we postulate a two-stage process in which entry is followed by bidding. We do not specify a particular model of entry; rather, we posit a reduced form for the entry outcome and assume Bayes Nash equilibrium in the auction stage. The number of bidders entering auction t is denoted by N_t. Bidders are assumed risk neutral. Bidder i's valuation for the good offered is denoted by V_{it}. Upon entering, i observes a private signal $S_{it} \in [\underline{s}, \overline{s}]$ of V_{it}. Let $V_t = (V_{1t}, \dots, V_{N_t t})$, $S_t = (S_{1t}, \dots, S_{N_t t})$, and $S_{-it} = S_t \backslash S_{it}$.

The bidding stage follows Milgrom and Weber (1982). The realizations of (N_t, X_t, U_t) are common knowledge among bidders, as is the distribution of $(S_t, V_t) \,|\, (N_t, X_t, U_t)$.[7] In addition, each bidder i knows the signal S_{it}. Let $F_{SV}\left(S_t, V_t | N_t, X_t, U_t\right)$ denote the joint distribution of signals and valuations conditional on (N_t, X_t, U_t). We make the following standard assumptions on this conditional distribution.

Assumption 2. (i) For all $n \in \text{supp} N_t | \left(X_t, U_t\right)$, $F_{SV}\left(S_t, V_t | n, X_t, U_t\right)$ has a continuously differentiable joint density that is affiliated, exchangeable in the indices $i = 1, \dots, n$, and positive on $(\underline{s}, \overline{s})^n \times (\underline{v}, \overline{v})^n$; (ii) $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ exists and is strictly increasing in S_{it}.

Because the bidding stage involves a standard affiliated values model, it nests a variety of special cases. With *private values*, $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ does not depend on S_{-it}. In our setting this is equivalent to bidders' knowing their valuations, i.e., $S_{it} = V_{it}$. When $E[V_{it} | S_{it}, S_{-it}, N_t, X_t, U_t]$ depends on S_{-it}, we have *common*

[7] Although standard, the assumption that bidders observe the number of competitors may be inappropriate in some applications. In our model of entry outcomes below, N_t is completely determined by the common knowledge auction characteristics X_t, Z_t, U_t. And in our data, we reject the hypothesis that bidders are unaware of the realized n_t in favor of bidding strategies that are more aggressive (given X_t, U_t) for larger n_t (see section 7.1). HPP point out that in OCS auctions, rivals' joint bidding agreements and participation in follow-up seismic surveys are typically known, and that bidders performing a follow-up survey submit bids on roughly 80% of the tracts analyzed. Nonetheless, these empirical findings and institutional features leave room for the possibility that bidders have some uncertainty about the level of competition faced. In general, one challenge in relaxing the assumption that n_t is observed is the need to specify precisely the information available to bidders when forming expectations of the competition. See, e.g., HPP and Gentry and Li (2014).

values (or *interdependent values*). A special case of the common values model is that of *pure common values*, where $V_{it} = \bar{V}_t$ for all i.

A conditional expectation of particular relevance for what follows is

$$w(s_{it}; n_t, x_t, u_t) \equiv E\left[V_{it} \middle| S_{it} = \max_{j \neq i} S_{jt} = s_{it}, N_t = n_t, X_t = x_t, U_t = u_t\right].$$

This is a bidder's expected value of winning the auction conditional on all common knowledge information, the observed private signal, and the event (typically counterfactual) that this signal ties for the highest among all bidders at the auction. This expectation plays an important role in the theory because, when equilibrium bidding strategies turn out to be strictly increasing in signals, tying for the highest signal means that even arbitrarily small deviations from one's equilibrium bid will change the identity of the winner. We therefore refer to $w(s_{it}; n_t, x_t, u_t)$ as bidder i's "pivotal expected value" at auction t. Pivotal expected values also play a central role in our strategy for discriminating between private values and common values.

We impose the following restriction on how the auction characteristics (X_t, U_t) affect bidder valuations.

Assumption 3. (i) $V_{it} = \Gamma(X_t, U_t)\, V_{it}^0$; (ii) conditional on N_t, $(V_{1t}^0, \ldots, V_{N_t t}^0, S_{1t}, \ldots, S_{N_t t})$ is independent of (X_t, U_t); (iii) $\Gamma(X_t, U_t)$ is strictly positive for all (X_t, U_t), bounded, and weakly increasing in U_t.

Assumption 3 is an index restriction requiring multiplicative separability in (X_t, U_t) and weak monotonicity in U_t.[8] An assumption of multiplicative (or additive) separability has often been relied upon in the auctions literature, including for identification in other settings with unobserved heterogeneity. Our identification result will rely on

[8] Without a distributional restriction like that in part (ii) of the assumption, part (i) would have no content. And because more "desirable" realizations of the unobservable state can be labeled with larger values, the monotonicity restriction in (iii) only rules out variation with X_t in the partial order on the unobservable implied by desirability (see also Example 1 and Appendix A).

this assumption as well.[9] Without further loss, we normalize the scale of Γ relative to that of V_{it}^0 by taking an arbitrary point $x^0 \in \mathbb{X}$ and setting

$$\Gamma\left(x^0, 0\right) = 1. \tag{1}$$

We assume initially that the auction is conducted without a binding reserve price, although below we consider an extension allowing a random reserve price. Under Assumption 2, the auction stage of our model admits a unique Bayesian Nash equilibrium in weakly increasing strategies; these strategies, which we denote by $\beta(\cdot; X_t, U_t, N_t) : [\underline{s}, \overline{s}] \to \mathbb{R}$, are symmetric and strictly increasing.[10] Let the random variable $B_{it} = \beta\left(S_{it}; X_t, U_t, N_t\right)$ denote the equilibrium bid of bidder i in auction t, with B_t denoting $(B_{1t}, \ldots, B_{N_t t})$.

A useful fact is that the separability required by Assumption 3 is inherited by the equilibrium bidding strategies.[11] Thus, under Assumptions 2 and 3 we may write

$$\beta\left(S_{it}; X_t, U_t, N_t\right) = \Gamma\left(X_t, U_t\right) \beta^0\left(S_{it}; N_t\right), \tag{2}$$

where β^0 denotes the symmetric Bayesian Nash equilibrium bidding strategy for an N_t-bidder auction at which $\Gamma\left(X_t, U_t\right) = \Gamma\left(x^0, 0\right) = 1$. Following HHS, we refer to $B_{it}^0 = \beta^0\left(S_{it}; N_t\right)$ and V_{it}^0 as "homogenized" bids and valuations.

We link the model of a single auction to the observed sample through Assumption 4. Given Assumption 3, this is the standard assumption that auctions are i.i.d.

[9]For for what follows it is sufficient that the conditional expectations $E[V_{it}|S_t, X_t, U_t, N_t]$ take the multiplicatively separable form. This weaker condition will be more natural when these *ex ante* conditional expectations are positive even though V_{it} may take negative values.

[10]See Theorem 2.1 in Athey and Haile (2007) and the associated references. Milgrom and Weber (1982) characterize the equilibrium strategies.

[11]See, e.g., HHS, Athey and Haile (2007), or Krasnokutskaya (2011).

conditional on auction characteristics, (N_t, X_t, U_t). However, we do not require U_t to be independent across auctions.

Assumption 4. $(V_t^0, S_t) \perp\!\!\!\perp (V_{t'}^0, S_{t'})$ for $t' \neq t$.

Finally, we specify the outcome of the entry stage by supposing that the number of bidders at auction t satisfies

$$N_t = \eta\left(X_t, Z_t, U_t\right) \tag{3}$$

for some function η that is weakly increasing in U_t. Formally, (3) is an assumed reduced form for the entry outcome. The weak monotonicity requirement links the interpretation of the unobservable in the entry and bidding stages: unobservables that make the good for sale more valuable also encourage more entry. The new variable Z_t in (3) is an exogenous auction-specific observable that affects bidder entry but is otherwise excludable from the auction model, as formalized in Assumption 5.

Assumption 5. (i) $Z_t \perp\!\!\!\perp U_t | X_t$; (ii) $Z_t \perp\!\!\!\perp (S_t, V_t^0) | N_t$.

The following example, discussed more fully in Appendix A, describes one fully specified two-stage game leading to the structure assumed above.

Example 1. *Consider a model of entry and bidding for an OCS oil and gas lease, where a standard simultaneous move entry stage à la Berry (1992) precedes a competitive bidding stage à la Milgrom and Weber (1982). Players in the game are firms in the industry. The tract offered for lease is associated with observables X_t, which includes (among other relevant covariates) the number of active leases on neighbor tracts and the sets of bidders for those leases.*[12] *The active neighbor leases are owned by Z_t distinct* ***neighbor firms****. Tract-level unobservables are denoted by E_t. E_t may have*

[12] Tracts s and t are defined to be neighbors if their boundaries coincide at some point.

arbitrary dimension, may be correlated with X_t, and may be spatially correlated. The characteristics X_t and E_t are assumed to scale valuations (multiplicatively) through a bounded index $\lambda(X_t, E_t)$. Firms play a two-stage game. They first choose simultaneously whether to enter, with each entering firm i incurring a signal acquisition cost $c_i(X_t)$. Signal acquisition costs are common knowledge and lower for neighbor firms than other (non-neighbor) firms.[13] *Entrants learn their private signals and the number of entrants, then participate in a first-price sealed bid auction with symmetric affiliated values. Appendix A shows that all pure strategy perfect Bayesian equilibria (with weakly increasing strategies in the bidding stage) can be represented by the model and assumptions above. This representation is obtained by defining $U_t = F_\lambda(\lambda(x_t, E_t)|x_t)$, where $F_\lambda(\cdot|x)$ is the CDF of the random variable $\lambda(x, E_t)$. Observe that in this case the distribution of U_t does not vary with X_t, although its interpretation does.*[14]

3 Nonparametric Identification

In this section we develop sufficient conditions for identification of the entry model, the index function Γ, and key features of the bidding model. We address each of these in turn. Throughout we assume that the observables include X_t, Z_t, N_t and B_t.[15] Let $\mathbb{Y}$ denote the support of (X_t, Z_t), and let $\mathbb{Y}(n)$ denote that conditional on $N_t = n$. Let $\underline{n} \geq 0$ denote the minimum value in the support of N_t; let $\overline{n}$ denote the maximum. Recalling (1), for convenience we take x^0 such that for some z we have $(x^0, z) \in \mathbb{Y}(\underline{n})$.

[13] This structure generalizes that in Hendricks and Porter (1988), where neighbors obtain a private signal for free but non-neighbors face an infinite cost of signal acquisition.

[14] In that case, knowledge of the function Γ will not be sufficient to characterize the effects of a ceteris paribus change in X_t on bidder valuations. See the additional discussion in Appendix A.

[15] In some applications, data may be available only for auctions attracting at least one bidder. In Appendix A we show that, within the fully specified model of Example 1, our maintained assumptions and analysis remain valid in the presence of such sample selection. Our examination of the OCS auction data will exploit this result.

3.1 Identification of the Entry Model

We show identification of the entry model under the following regularity condition on the support of $N_t|(X_t, Z_t)$.

Assumption 6. For all $(x,z) \in \mathbb{Y}$, there exist $\underline{n}(x,z)$ and $\overline{n}(x,z)$ such that $\eta(x,z,U_t)$ has support $(\underline{n}(x,z), \underline{n}(x,z)+1, \ldots, \overline{n}(x,z))$.

Given Assumption 6, for any $(x,z) \in \mathbb{Y}$, the function $\eta(x,z,\cdot)$ is characterized by thresholds

$$\tau_{\underline{n}(x,z)-1}(x,z) \leq \tau_{\underline{n}(x,z)}(x,z) \leq \ldots \leq \tau_{\overline{n}(x,z)},$$

where

$$\tau_{\underline{n}(x,z)-1}(x,z) \equiv 0, \qquad \tau_{\overline{n}(x,z)}(x,z) \equiv 1, \tag{4}$$

and for $n = \{\underline{n}(x,z), \ldots, \overline{n}(x,z)\}$,

$$\tau_{n-1}(x,z) = \inf\{u \in [0,1] : \eta(x,z,u) \geq n\}.$$

With this observation, identification of η follows easily.

Theorem 1. *Under Assumptions 1–6, η is identified.*

Proof. For each (x,z), $\underline{n}(x,z)$ and $\overline{n}(x,z)$ equal $\min\{N_t|X_t = x, Z_t = z\}$ and $\max\{N_t|X_t = x, Z_t = z\}$, respectively. For $n = \underline{n}(x,z), \ldots, \overline{n}(x,z)$, $\tau_n(x,z) = \Pr(N_t \leq n|X_t = x, Z_t = z)$. With (4), this implies the result. □

Identification of η determines the effects of Z_t on bidder entry and provides bounds $\tau_{n_t-1}(x_t,z_t)$ and $\tau_{n_t}(x_t,z_t)$ on the realization of each unobservable U_t. As shown in the following corollary (proved in Appendix B), it also determines the distribution of U_t conditional on (X_t, N_t).

Corollary 1. *Under Assumptions 1–6, the distribution of $U_t|(X_t, N_t)$ is identified.*

3.2 Identification of the Index Function

Define

$$\gamma\left(x,u\right)=\ln\Gamma\left(x,u\right).$$

We first provide conditions sufficient to identify $\gamma\left(x,u\right)$ at each $x\in\mathbb{X}$ and $u\in\mathcal{U}_x$, where

$$\mathcal{U}_x=\bigcup_{\substack{z:(x,z)\in\mathbb{Y}\\ n\in\text{supp }\eta(x,z,U_t)}}\left\{\tau_{n-1}(x,z),\tau_n(x,z)\right\}.$$

We then give additional conditions guaranteeing that $\mathcal{U}_x=[0,1]$ for each x. We begin with the following result, whose proof illustrates a key argument.

Lemma 1. *Under Assumptions 1–6, for all $n\geq\underline{n}$, all $(x,z)\in\mathbb{Y}\left(n\right)$, and all $(x',z')\in\mathbb{Y}\left(n\right)$, the differences $\gamma\left(x',\tau_n\left(x',z'\right)\right)-\gamma\left(x,\tau_n\left(x,z\right)\right)$ and $\gamma\left(x',\tau_{n-1}\left(x',z'\right)\right)-\gamma\left(x,\tau_{n-1}\left(x,z\right)\right)$ are identified.*

Proof. By (2) and monotonicity of the equilibrium bid function,

$$\inf\left\{\ln B_{it}\right|N_t=n,X_t=x,Z_t=z\}=\gamma\left(x,\tau_{n-1}\left(x,z\right)\right)+\ln\beta^0\left(\underline{s};n\right).$$

So under Assumptions 1–6, for any n and all (x,z) and (x',z') in $\mathbb{Y}\left(n\right)$, the differences

$$\gamma\left(x',\tau_{n-1}\left(x',z'\right)\right)-\gamma\left(x,\tau_{n-1}\left(x,z\right)\right)\tag{5}$$

are identified.[16] Similarly, since

$$\sup\left\{\ln B_{it}\right|N_t=n,X_t=x,Z_t=z\}=\gamma\left(x,\tau_n\left(x,z\right)\right)+\ln\beta^0\left(\overline{s};n\right),$$

[16]Because $\beta^0(\underline{s};n)=E[V_{it}^0|S_{jt}=\underline{s},j=1,\ldots,n]$ and $\beta^0(s;n)\leq E[V_{it}^0|S_{jt}=s,j=1,\ldots,n]$ for all s (see, e.g., Milgrom and Weber (1982)), both $\beta^0(\underline{s};n)$ and $\beta^0(\overline{s};n)$ are finite under Assumption 2.

we obtain identification of the differences

$$\gamma\left(x', \tau_n\left(x', z'\right)\right) - \gamma\left(x, \tau_n\left(x, z\right)\right) \tag{6}$$

for all n and all (x, z) and (x', z') in $\mathbb{Y}(n)$. $\square$

Thus far we have not imposed any requirement on the support of Z_t or its effect on entry outcomes. Below we will do so in order to obtain point identification of γ. However, even in the case that no instrument is available, Theorem 1, Corollary 1, and Lemma 1 still hold. And once γ is known, Z_t plays no further role in our identification results. Thus, while we rely on an instrument to obtain point identification, this reliance is formally limited to ensuring that we can move from the partial identification of γ provided by Lemma 1 to point identification of γ.

As a step toward point identification, we introduce two additional assumptions. These allow us to show that the first differences obtained above can be differenced again, cancelling common terms, to obtain a set of first differences sufficient to pin down the value of the index $\gamma(x, u)$ at all x and $u \in \mathcal{U}_x$.

Assumption 7. For all $n \in \{\underline{n}+1, \underline{n}+2, \ldots, \overline{n}\}$, $\mathbb{Y}(n-1) \cap \mathbb{Y}(n)$ is nonempty.

Assumption 8. There exists n^* such that
(i) $\forall n \in \{\underline{n}, \ldots, n^*\}$, $\mathbb{Y}$ contains points $(x(n), z(n))$ and $(x(n), \hat{z}(n))$ such that $\underline{n}(x(n), z(n)) = n$ and $\underline{n}(x(n), \hat{z}(n)) = n+1$; and
(ii) $\forall n \in \{n^*, \ldots, \overline{n}\}$, $\mathbb{Y}$ contains points $(x'(n), z'(n))$ and $(x'(n), \hat{z}'(n))$ such that $\overline{n}(x'(n), z'(n)) = n$ and $\overline{n}(x'(n), \hat{z}'(n)) = n-1$.

Assumption 7 requires variation in U_t that produces local variation in entry. For example, this rules out trivial cases in which U_t has no effect on N_t. Assumption 8 requires variation in the instrument Z_t that can induce local variation in the support of the entry outcomes, at least at some values of X_t.

We prove the following results in Appendix B.

Lemma 2. *Under Assumptions 1–8, for all $n \geq \underline{n}$, all $(x,z) \in \mathbb{Y}(n)$, and all $(x',z') \in \mathbb{Y}(n)$, $\gamma(x,\tau_n(x,z)) - \gamma(x',\tau_{n-1}(x',z'))$ is identified.*

Lemma 3. *Under Assumptions 1–8, for all $n \geq \underline{n}$ and all $(x,z) \in \mathbb{Y}(n)$, the values of $\gamma(x,\tau_{n-1}(x,z))$ and $\gamma(x,\tau_n(x,z))$ are identified.*

By Theorem 1, the values of $\tau_{n-1}(x,z)$ and $\tau_n(x,z)$ are known for all n and $(x,z) \in \mathbb{Y}(n)$. Thus, Lemma 3 demonstrates identification of $\gamma(x,u)$ at each $x \in \mathbb{X}$ and $u \in \mathcal{U}_x$. In general, this may still deliver only partial identification of the index function γ, so that in practice one may rely on parametric structure to interpolate between the points $\{x \in \mathbb{X}, u \in \mathcal{U}_x\}$ at which $\gamma(x,u)$ is nonparametrically point identified. However, the following conditions are sufficient to ensure that no such interpolation is necessary.

Assumption 9. (a) For all $x \in \mathbb{X}$, $\text{supp} Z_t|X_t = x$ is connected.
(b) For all $(x,z,u) \in \mathbb{Y}\times(0,1)$ and all $\delta > 0$ such that $(u-\delta, u+\delta) \subset (0,1)$, there exists $\epsilon > 0$ such that if $\|z'-z\| < \epsilon$ then $\eta(x,z',u') = \eta(x,z,u)$ for some $u' \in (u-\delta,u+\delta)$.

Assumption 10. For every $x \in \mathbb{X}$ there exists a finite partition $0 = \tau^0(x) < \tau^1(x) < \cdots < \tau^{K(x)}(x) = 1$ of the unit interval such that for each $k = 1,\ldots,K(x)$ and some $z(k), z'(k) \in \text{supp} Z_t|X_t = x$, $\eta\left(x,z(k),\tau^{k-1}(x)\right) > \eta\left(x,z'(k),\tau^k(x)\right)$.

Assumption 9 requires continuously distributed instruments and a type of continuous substitution between Z_t and U_t in the "production" of bidder entry: it must be possible to offset the effect (on entry) of a small change in Z_t with a small change in U_t. Assumption 10 requires that variation in Z_t have sufficient effect on participation to offset some discrete variation in the unobservable U_t. A sufficient condition is that for each x there exist z and z' such that $\eta(x,z,\tau_{n-1}(x,z)) > \eta(x,z',\tau_n(x,z))$ for all

$n \in \{\underline{n}(x,z), \ldots, \overline{n}(x,z)\}$; in this case, the set $\left\{\tau_{\underline{n}(x,z)-1}, \ldots, \tau_{\overline{n}(x,z)}\right\}$ could define the partition $\tau^0(x) < \tau^1(x) < \cdots < \tau^K(x)$.

The following lemma, whose proof is provided in Appendix B, leads us to the point identification of γ (and therefore Γ) demonstrated in Theorem 2.

Lemma 4. *Under Assumptions 1–9, $\tau_{n-1}(X_t, Z_t)$ is continuous in Z_t on the pre-image of* $(0,1)$.

Theorem 2. *Under Assumptions 1–10, Γ is identified on* $\mathbb{X} \times [0,1]$.

Proof. We need only show that $\mathcal{U}_x = [0,1]$ for each $x \in \mathbb{X}$. For arbitrary $x \in \mathbb{X}$, let $0 = \tau^0(x) < \tau^1(x) < \cdots < \tau^{K(x)}(x) = 1$ be as in Assumption 10. Take any $k \in \{1, 2, \ldots, K(x)\}$ and let $z = z(k)$ and $z' = z'(k)$ be as in Assumption 10. Let $n = \eta\left(x, z, \tau^{k-1}(x)\right)$. Because $\eta\left(x, z, \tau^{k-1}(x)\right) > \eta\left(x, z', \tau^k(x)\right)$, we have $\eta\left(x, z', \tau^k(x)\right) < n$ and, therefore,

$$\tau_{n-1}(x,z) \leq \tau^{k-1}(x) < \tau^k(x) \leq \tau_{n-1}(x,z').$$

Because the continuous image of a connected set is connected, Lemma 4 and Assumption 9 (part (a)) then imply that for every $\tilde{\tau} \in \left[\tau^{k-1}(x), \tau^k(x)\right]$ there exists $z^{\tilde{\tau}}$ such that $\tau_{n-1}\left(x, z^{\tilde{\tau}}\right) = \tilde{\tau}$. $\square$

3.3 Identification of the Bidding Model

We now demonstrate identification of the joint distribution of the pivotal expected values $(w(S_{1t}; n, x, u), \ldots, w(S_{nt}; n, x, u))$ for all $x \in \mathbb{X}, u \in [0,1]$, and n in the support of $N_t | \{X_t = x, U_t = u\}$. For a private values model, where $V_{it} = S_{it}$, this is equivalent to identification of the joint distribution of bidder valuations conditional on (X_t, N_t, U_t). Thus, Theorem 3 below demonstrates identification of the affiliated private values model. Without the restriction to private values, our result here provides an important form of partial identification.[17] For our empirical application, for example, this is sufficient to allow us to test the hypothesis of equilibrium bidding in the affiliated values model, to test the hypothesis of private values against the alternative of common values, to examine the potential gains from changes in the auction format, and to assess the effects of competition on bidder market power.[18]

Let

$$G_{M|B}\left(m|b,x,u,n\right) = \Pr\left(\max_{j\neq i} B_{jt} \leq m \middle| B_{it}=b, X_t=x, U_t=u, N_t=n\right),$$

and let $g_{M|B}\left(m|b,x,u,n\right)$ denote the associated conditional density (guaranteed to exist by Assumption 2 and strict monotonicity of the equilibrium bid function). Following Laffont and Vuong (1993), Guerre, Perrigne, and Vuong (2000), and Li, Perrigne, and Vuong (2000, 2002), one can characterize the relationship between each realized $w(s_{it}; n_t, x_t, u_t)$ and the associated equilibrium bid $b_{it} = \beta\left(s_{it}; x_t, u_t, n_t\right)$ in terms of the joint distribution of equilibrium bids. In particular, each b_{it} must satisfy the first-order condition[19]

$$w(s_{it}; n_t, x_t, u_t) = b_{it} + \frac{G_{M|B}\left(b_{it}|b_{it}, x_t, u_t, n_t\right)}{g_{M|B}\left(b_{it}|b_{it}, x_t, u_t, n_t\right)}. \tag{7}$$

Although this equation expresses the pivotal expected value $w(s_{it}; n_t, x_t, u_t)$ as a functional of a conditional distribution of bids, the presence of u_t on the right-hand side creates challenges. Because realizations of U_t are not observable or identified, one cannot directly condition on them to identify the functions $G_{M|B}$ and $g_{M|B}$.

[17]Without additional information or structure, common values models are not identified from bidding data. See, e.g., Laffont and Vuong (1993) and Athey and Haile (2002).

[18]Tang (2011) shows how the joint distribution of pivotal expected values can be used to bound counterfactual revenues in standard auctions with binding reserve prices.

[19]See, e.g., Athey and Haile (2007) for a derivation in the affiliated values model.

This precludes obtaining identification directly from (7). With the preceding results, however, we can overcome this problem.

Observe that, like valuations and bids, the pivotal expected values $w(s_{it}; n_t, x_t, u_t)$ will have the separable structure

$$w(s_{it}; n_t, x_t, u_t) = w^0(s_{it}; n_t)\Gamma(x_t, u_t), \tag{8}$$

where

$$w^0(s_{it}; n_t) \equiv E\left[V_{it}^0 \middle| S_{it} = \max_{j \neq i} S_{jt} = s_{it}, N_t = n_t \right].$$

We will refer to $w^0(s_{it}; n_t)$ as bidder i's "homogenized pivotal expected value" at auction t. The first-order condition (7) can then be written as

$$w^0(s_{it}; n_t) = b_{it}^0 + \frac{G_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t)}{g_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t)} \tag{9}$$

where

$$G_{M^0|B^0}(m|b, n_t) = \Pr\left(\max_{j \neq i} B_{jt}^0 \leq m \middle| B_{it}^0 = b, N_t = n \right)$$

and $g_{M^0|B^0}(m|b, n_t)$ is the associated conditional density.

Let $\tilde{B}_{it} = \ln(B_{it})$ and $\tilde{B}_{it}^0 = \ln(B_{it}^0)$. By Assumption 3,

$$\tilde{B}_{it} = \tilde{B}_{it}^0 + \gamma(X_t, U_t). \tag{10}$$

Furthermore, $\tilde{B}_{it}^0$ and $\gamma(X_t, U_t)$ are independent conditional on N_t, X_t. Lemma 5 (proved in Appendix B) shows that under the following regularity condition, Theorem 2 and a standard deconvolution argument yield identification of the joint distribution of $(B_{1t}^0, \ldots, B_{nt}^0)$ for all n.[20]

[20]Under a slight strengthening of Assumption 3, requiring the support of $\Gamma(x, U_t)$ to be compact at some x, Assumption 11 is guaranteed to hold (see, e.g., Krasnokutskaya (2011)).

Assumption 11. For some $x \in \mathbb{X}$ the random variable $\gamma(x, U_t)$ has nonzero characteristic function almost everywhere.

Lemma 5. *Under Assumptions 1–11, conditional on any $N_t = n$, the joint density of $(B^0_{1t}, \ldots, B^0_{nt})$ is identified.*

This leads directly to our main identification result.

Theorem 3. *Let Assumptions 1–11 hold. Then for all $x \in \mathbb{X}, u \in [0,1]$, and $n \geq 2$ in the support of $N_t|\{X_t = x\}$, the joint distribution of $(w(S_{1t}; n, x, u), \ldots, w(S_{nt}; n, x, u))$ is identified.*

Proof. Fix n. From (9), we have

$$w^0(S_{it}; n) = \xi(B^0_{it}; n) \equiv B^0_{it} + \frac{G_{M^0|B^0}(B^0_{it}|B^0_{it}, n)}{g_{M^0|B^0}(B^0_{it}|B^0_{it}, n)}. \tag{11}$$

By Lemma 5, the joint distribution $(\xi(B^0_{1t}; n), \ldots, \xi(B^0_{nt}; n))$ is known. This implies identification of the joint distribution of $(w^0(S_{1t}; n), \ldots, w^0(S_{nt}; n))$. The result then follows immediately from (8) and Theorem 2. $\square$

4 Estimation

We propose a two-stage semiparametric estimation strategy. The first stage involves semiparametric sieve quasi-maximum likelihood estimation (QMLE) of the entry thresholds $\tau_\ell(x, z)$, the index function γ, and the joint distributions of homogenized equilibrium bids. In the second stage, for each level of bidder entry, we estimate the joint distribution of homogenized pivotal expected values by plugging draws from the estimated distribution of homogenized bids into the auction first-order condition and constructing the empirical distribution of the resulting pseudo-sample.

4.1 Stage 1: Sieve-QMLE

Let θ_τ denote the parameters of the entry model, θ_γ the parameters of the index function γ, and θ_B the parameters of the joint distributions of log homogenized bids. Let

$$L_{1t}(n_t; \theta_\tau) = \Pr(N_t = n_t | X_t = x_t, Z_t = z_t; \theta_\tau)$$

denote the (conditional on (x_t, z_t)) likelihood for the entry outcome in auction t. Let

$$L_{2t}(b_t | n_t; \theta_\gamma, \theta_B, \theta_\tau)$$

denote the likelihood of the observed bids at auction t, conditional on the entry outcome n_t (and on (x_t, z_t)). Defining $\theta = (\theta_\tau, \theta_\gamma, \theta_B)$, the conditional quasi-likelihood[21] function for the observed outcomes $\{(n_t, b_t)\}_{t=1}^T$ can be written

$$\mathcal{L}(\theta) = \prod_t L_{1t}(n_t; \theta_\tau) L_{2t}(b_t | n_t; \theta_\gamma, \theta_B, \theta_\tau).$$

We give details of our empirical specification and the two components of the quasi-likelihood in sections 4.1.1 and 4.1.2 below. Estimates of the parameter vector θ can be obtained by maximizing $\mathcal{L}(\theta)$. Because θ_τ is identified from the entry outcomes alone, it is also possible to split the QMLE stage, first maximizing $\prod_t \mathcal{L}_{1t}(\theta_\tau)$ to estimate θ_τ, then maximizing $\prod_t L_{2t}(\theta_\gamma, \theta_B, \hat{\theta}_\tau)$ conditional on $\hat{\theta}_\tau$. In our data, the two approaches yield very similar estimates. However, because we found the two-step QMLE procedure to be more numerically stable in bootstrap samples, below we will report results using the two-step version.

Consistency can be confirmed by adapting the results of White and Wooldridge (1991) for sieve-extremum estimators with weakly dependent time series data to the

[21] Recall that we permit spatial dependence.

case of weak spatial dependence.[22] To conduct inference, we use a nonparametric block bootstrap procedure that captures both dependence among bids within an auction and spatial dependence between the unobservables U_t across auctions. Specifically, we re-sample auctions with replacement, taking all bids from the selected auction, and including in the bootstrap sample all auctions on neighbor tracts as well. Following HPP, we re-sample weighting auctions by factors inversely proportional to the number of auctions in the neighborhood of the tract.[23]

4.1.1 Entry Thresholds

Our entry model above reduces to an ordered response model where, given $X_t = x$ and $Z_t = z$, we have $N_t = n$ if and only if $U_t \in (\tau_{n-1}(x,z), \tau_n(x,z))$. Given any strictly increasing univariate CDF H we can rewrite this as

$$\{N_t = n | X_t = x, Z_t = z\} \iff \{A_t \in (\alpha_{n-1}(x,z), \alpha_n(x,z))\},$$

where $A_t \sim H$ and $\alpha_n(x,z) = H^{-1}(\tau_n(x,z))$.

We specify a linear threshold function

$$\alpha_n(x,z) = \alpha_n - x\alpha_x - z\alpha_z$$

[22]In particular, we represent tract locations by points in $\mathbb{Z}^2$. Then, under a standard "expanding domain" asymptotics, White and Wooldridge's uniform consistency result for stationary α-mixing time series data (Corollary 2.6) can be extended using a Bernstein-type inequality for α-mixing random fields on $\mathbb{Z}^2$ (e.g., Yao (2003)).

[23]Similar results are obtained without weighting. Applying the results of van der Vaart and Wellner (1996) and Lahiri (2003), one can verify validity of the bootstrap when we interpret our finite-sample estimator as that for a parametric model. General conditions for consistency of bootstrap inference procedures for sieve M-estimators in the i.i.d. setting can be found, e.g., in Ma and Kosorok (2005) and Chen and Pouzo (2009). See also Chen and Liao (2014), and Chen, Liao, and Sun (2014) in the case of time series data.

and specify H as the standard normal CDF, yielding an ordered probit model.[24] Letting $\theta_\tau = (\{\alpha_n\}_{n=\underline{n}}^{\overline{n}-1}, \alpha_x, \alpha_z)$, we then have

$$L_{1t}(n; \theta_\tau) = H\left(\alpha_n\left(x_t, z_t; \theta_\tau\right); \theta_\tau\right) - H\left(\alpha_{n-1}\left(x_t, z_t; \theta_\tau\right); \theta_\tau\right).$$

4.1.2 Index Function and Homogenized Bid Distribution

Given our focus on testing for common values, we prioritize flexibility in how the joint distribution and density of bids can vary with n when specifying the second part of the quasi-likelihood. We specify the index function γ parametrically as $\gamma(\cdot, \cdot; \theta_\gamma)$; we will use a linear specification below. For each value of n, the joint density of log homogenized bids is specified semiparametrically, using a parametric copula and a nonparametric (Bernstein polynomial sieve) specification of the common marginal distribution.[25] Below we specify a Gaussian copula, with separate covariance parameter ρ_n for each n.[26]

We specify the marginal density of a generic bidder's log homogenized bid in an n-bidder auction as

$$\tilde{g}_{B_i^0}(\tilde{b}^0; \theta_b, n) = \sum_{j=0}^{m} \theta_{b,n}^{(j)}\, q_{j,m}\left(\Phi\left(\tilde{b}^0\right)\right) \phi\left(\tilde{b}^0\right), \tag{12}$$

[24] Thus, although the theoretical model of entry in Example 1 follows Berry (1992), the induced empirical model of the entry stage is similar to that of Bresnahan and Reiss (1991). In Appendix D we consider a variation in which H is specified using a Hermite polynomial series approximation, following Gallant and Nychka (1987).

[25] See, e.g., Chen, Fan, and Tsyrennikov (2006). Prior work using Bernstein polynomials in estimation of auction models includes Komarova (2017) and Kong (2017a,b). Hubbard, Paarsch, and Li (2012) have previously combined a parametric copula and nonparametric (kernel) specification of marginal densities to estimate auction models.

[26] Our use of a parametric copula reflects in part our choice to let the distribution of bids be fully flexible (i.e., even in finite sample) with respect to the number of bidders n_t. In other applications one might specify a Bernstein copula and account for the effects of n_t within the sieve approximation. As we discuss in Appendix H, the Gaussian copula provides a substantial computational advantage when we transform our estimated joint distributions and densities to the conditional distributions and densities that are plugged into bidders' first-order conditions.

where $q_{j,m}(\nu) = \binom{m}{j}\nu^j (1-\nu)^{m-j}$ and $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal distribution and density functions, respectively. Here m is a parameter, growing with the sample size, that determines the order of the Bernstein polynomial approximation. Let $\theta_{b,n} = \{\theta_{b,n}^{(j)}\}_{j=0}^{m}$. Thus, the parameter vector θ_b in (12) represents $\{\theta_{b,n}\}_{n=\underline{n}}^{\overline{n}}$.

Because Bernstein polynomials approximate functions with domain $[0,1]$, in (12) we use Bernstein polynomials to approximate the marginal density of the transformed variable $\Phi(\tilde{b}^0)$.[27] This transformation is useful not only for standardizing the domain but also for ensuring that the nonparametric estimator will offer sensible approximations even in modest sample sizes. When $m = 0$, for example, the distribution of log-bids will be normal. Thus, the nonparametric component of our specification is based on a sequence of approximating models that starts with a natural (lognormal) parametric specification and adds flexibility as permitted by the sample size.

Let $\tilde{G}_{B_i^0}(\tilde{b}^0;\theta_b,n)$ denote the CDF associated with $\tilde{g}_{B_i^0}(\tilde{b}^0;\theta_b,n)$. Let $\chi(\cdot;\rho_n)$ denote the symmetric Gaussian copula density with covariance parameter ρ_n.[28] We specify the joint density of the log homogenized bids in n-bidder auctions as

$$\tilde{g}_{B^0}\left(\tilde{b}_1^0,\ldots,\tilde{b}_n^0;\theta_{b,n},\rho_n,n\right) = \chi\left(\tilde{G}_{B_i^0}(\tilde{b}_1^0;\theta_{b,n},n),\ldots,\tilde{G}_{B_i^0}(\tilde{b}_n^0;\theta_{b,n},n);\rho_n\right)\tilde{g}_{B_i^0}(\tilde{b}_1^0;\theta_{b,n},n)\ldots\tilde{g}_{B_i^0}(\tilde{b}_n^0;\theta_{b,n},n). \quad (13)$$

[27] We use the normalization $\gamma(0,0) = 0$ (recall (1)). For estimation purposes, however, we add intercepts $\gamma_{n_t}^0$ for each value of n_t to the index function $\gamma(x_t,u_t;\theta_\gamma)$, implying that in this step the joint densities we estimate are actually those of *centered* log homogenized bids $\tilde{b}_t^0 - \gamma_{n_t}^0$. We then adjust the location of each estimated density by the appropriate intercept estimate to obtain the density of (uncentered) log homogenized bids that is relevant to bidders' first-order conditions. When homogenization is performed via OLS, this type of centering procedure is required for consistency (see HHS and Athey and Haile (2007)). In our case this is not essential but offers several practical advantages by ensuring that the log homogenized bids are centered at zero prior to transformation by the normal CDF, ensuring that the location of the estimated bid distribution can move freely with n_t, and freeing the Bernstein coefficients to capture features of the marginal density other than its location.

[28] The symmetric Gaussian copula density $\chi(h_1,\ldots,h_n;\rho_n)$ is given by $\frac{1}{|\Upsilon|^{1/2}}\exp\left(-\frac{1}{2}\varphi(h)'\left(\Upsilon^{-1}-I\right)\varphi(h)\right)$, where $\varphi(h) = \left\{\Phi^{-1}(h_1),\ldots,\Phi^{-1}(h_n)\right\}$ and the matrix Υ has ones on the diagonal and covariance ρ_n in all off-diagonal entries.

We discuss computational details in Appendix H.

Letting $\rho = \{\rho_n\}_{n=\underline{n}}^{\overline{n}}$ and $\theta_B = (\theta_b, \rho)$, we have

$$L_{2t}(b_t|n_t;\theta) = \int_{\tau_{n_t-1}(x_t,z_t;\theta_\tau)}^{\tau_{n_t}(x_t,z_t;\theta_\tau)} \frac{\tilde{g}_{b^0}\left(\tilde{b}_{1t} - \gamma(x_t,u;\theta_\gamma),\ldots,\tilde{b}_{n_t t} - \gamma(x_t,u;\theta_\gamma);\theta_{b,n_t},\rho_{n_t},n_t\right)}{\tau_{n_t}(x_t,z_t;\theta_\tau) - \tau_{n_t-1}(x_t,z_t;\theta_\tau)} du, \tag{14}$$

where $\tau_{n_t-1}(x_t,z_t;\theta_\tau)$ and $\tau_{n_t}(x_t,z_t;\theta_\tau)$ denote the bounds on u_t implied by the entry model parameters θ_τ.[29] We approximate the integral by Monte Carlo simulation.

4.2 Stage 2: Invert Equilibrium First-Order Conditions

Given the first-stage estimates of the index function γ and joint distribution of homogenized bids, estimation of the relevant auction primitives is straightforward and does not involve further use of the data. Here we use the equilibrium first-order condition (9), which can be written in terms of the distribution of log homogenized bids as

$$w^0(s_{it};n_t) = \exp\left(\tilde{b}_{it}^0\right)\left(1 + \frac{\tilde{G}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0,n_t)}{\tilde{g}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0,n_t)}\right), \tag{15}$$

where $\tilde{G}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0,n_t)$ and $\tilde{g}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0,n_t)$ are, respectively, the CDF and pdf of $\max_{j\neq i}\tilde{B}_{jt}^0$ conditional on $\tilde{B}_{it}^0 = \tilde{b}_{it}^0$.

For each value of $N_t = n$, we transform the estimated joint distributions and densities obtained from stage 1 to construct the conditional distributions and densities appearing (15) (see Appendix H). Then we draw log homogenized bids from their estimated marginal distributions and plug these into (15), yielding pseudo-samples

[29] When $n_t = 1$ one may set $L_{2t}(b_t|n_t;\theta) = 1$ by convention, since our baseline model of competitive bidding does not imply an interpretation of the quantity $\tilde{b}_{it} - \gamma(x_t,u_t)$ in that case. Alternatively, because we specify different parameters $\theta_{b,n}$ for each value of n, (14) gives a correct expression for $L_{2t}(b_t|n_t;\theta)$ for $n_t = 1$ whenever bids in 1-bidder auctions are assumed to inherit the separable structure required of valuations. Below we develop an extension incorporating a random reserve price, where this separability is an implication of equilibrium behavior. Therefore, given the large number of one-bidder auctions in our sample, we include these in the quasi-maximum likelihood estimation using (14) for all specifications.

of the vectors $(w^0(s_{1t};n),\ldots,w^0(s_{nt};n))$ for many simulated auctions t. The empirical distribution of these pseudo-draws provides a consistent estimate of the joint distribution of homogenized pivotal expected values for n-bidder auction. Although these joint distributions will suffice for our application, the pseudo-draws can also be scaled by the estimated value of the index $\Gamma(x,u)$ in order to estimated the joint distributions of (non-homogenized) pivotal expected values given $X_t = x$ and $U_t = u$. In a private values setting, for example, this would yield an estimate of the joint distribution of bidder valuations.

5 OCS Auctions

5.1 Background and Data

We apply our method to first-price sealed bid auctions of oil and gas leases in the U.S. Outer Continental Shelf (OCS) held between 1954 and 1983. Extensive discussion of the OCS auctions can be found in, e.g., Gilley and Karels (1981), Hendricks and Porter (1988), Hendricks, Porter, and Spady (1989), and HPP. For a more complete institutional background we refer readers to that prior work, upon which we rely heavily ourselves. Briefly, however, auctions were held for the right to lease a specified tract for exploration and production of oil, gas, and other minerals. The seller in these auctions was the Mineral Management Service ("MMS"), at that time an agency of the U.S. Department of the Interior. Tracts in the sample typically cover a rectangular area (a "block") of 5,000–5,760 acres in the Gulf of Mexico.[30] Production on a tract was subject to royalty payments from the leaseholder at a pre-specified rate, usually

[30]We limit attention to tracts of at least 4,000 acres. Smaller tracts are typically half-blocks or quarter-blocks, where several of our measures of neighborhood characteristics would have interpretations different from those on standard tracts. We also drop auctions with missing values for our covariates.

1/6. Bids at an auction were offers of an additional up-front "bonus" payment for the right to become the leaseholder.

No exploratory drilling was permitted prior to the auction, although in some cases exploration and production would already have occurred on neighbor tracts and would be publicly observable. Bids would also reflect information obtained through evaluation of data from magnetic, gravity, and seismic surveys.[31] Although initial collection of survey data was often funded jointly, firms relied on their own experts for modeling and analysis of the data and often performed follow-up surveys of the tracts on which they intended to bid. Differences in expert assessments of the survey data are likely an important source of heterogeneity in bidder beliefs about the value of a given tract (HPP). These features lead us to treat bidder entry as a decision to acquire a costly signal about the value of the tract (recall Example 1).

We have data on all auctions attracting at least one bidder.[32] Table 1 shows the number of auctions in our sample by number of bidders. We do not separate wildcat, development, and drainage tracts; instead, we account directly for the presence of active neighbor leases and neighbor production, and we allow asymmetry between neighbor vs. non-neighbor costs of signal acquisition in a way that generalizes the structure considered in Hendricks and Porter (1988) (see Example 1).[33] Like Aradillas-Lopez, Haile, Hendricks, and Porter (2019), we model bidders as symmetric conditional on acquisition of a signal. Thus, while bidders may decide not to acquire a signal through analysis of the seismic data and may reach different conclusions from such analysis, the technology producing signals is modeled as symmetric across firms. Formal tests, developed in Appendix C, fail to reject the symmetry assumption.

[31] See, e.g., "G&G Data Acquisition and Analysis" at https://www.boem.gov/Oil-and-Gas-Energy-Program/Resource-Evaluation/Geological-and-Geophysical-Data-Acquisition/GGData-Geophysical-Surveys.aspx, March 24, 2018.

[32] Appendix A demonstrates that with the model of entry and bidding given in Example 1, our maintained assumptions remain valid in the presence of this sample selection.

[33] As a robustness check, we also examine the subset of auctions that excludes drainage tracts.

Table 1: Sample Sizes

n	1	2	3	4	5	6	7	8	9	10	11–17
T_n	814	498	293	229	172	127	100	73	56	56	128

T_n denotes the number of auctions in our sample with n bidders.

The MMS sometimes announced a small minimum acceptable bid of \$10–\$25 per acre; the MMS also retained the option to reject all bids when it deemed the auction to be noncompetitive. Such rejections were rare in our sample except at auctions attracting only one bid (see section D.1). We initially treat both the announced minimum bid and the MMS bid rejection policy as nonbinding, as in Li, Perrigne, and Vuong (2000). However, we also consider a variation of the model in which the MMS bid rejection policy is modeled with a random secret reserve price.

Limited forms of joint bidding were permitted in these auctions. Following the prior literature, we model each bid as coming from a generic "bidder," which might be solo firm or a bidding consortium.[34] Typically a given tract will have eight neighbors, only some (or none) of which will be "active" (under lease). Our measure of the number of neighbor firms (distinct owners of leases on adjacent tracts) accounts for the presence of joint bidding by linking together firms that have bid together previously in the same neighborhood, following the criteria developed by Aradillas-Lopez, Haile, Hendricks, and Porter (2019).

Our tract characteristics X_t, all measured as of the time of the auction, include the number of active neighbor leases, whether the tract is isolated (no active neighbors), the number of firms that bid for neighbor leases, whether the tract was offered previously (attracting no bidders or being relinquished by a prior leaseholder), whether a lease has expired on a neighbor tract, the number of neighbor tracts previously drilled,

[34]Hendricks and Porter (1992) and Hendricks, Porter, and Tan (2008) examine empirical and theoretical aspects of joint bidding in these auctions.

the number of "hits" on neighbor tracts, average water depth (and its square), and the royalty rate associated with the lease. We present a summary of these auction characteristics in Table 2. Below we will also incorporate year fixed effects.

Table 2: Summary Statistics

Variable	Mean	Median	Std Dev
# active neighbor leases	1.48	0.00	1.96
isolated lease indicator	0.59	1.00	0.49
# firms that bid for neighbors	2.37	0.00	3.56
re-offered tract indicator	0.20	0.00	0.40
neighbor expired indicator	0.31	0.00	0.46
# neighbor tracts drilled	1.69	1.00	2.25
# neighbor hits	0.65	0.00	1.30
water depth (thousands of ft)	0.20	0.14	0.28
royalty rate (%)	16.10	16.00	1.35
# neighbor firms	0.89	0.00	1.27

Following Example 1, we consider one instrument for participation: the number of neighbor firms.[35] bidder entry because ownership of a neighbor tract is likely to reduce the cost of assessing the value of the current tract. As discussed in Appendix A, when we condition on the number of neighbor tracts and the set of firms that previously bid for those tracts, variation in the number of neighbor firms is determined entirely by the realizations of bidder signals at prior auctions, and therefore independent of U_t under our maintained assumptions. Table 2 includes summary statistics for this instrument.

[35]Because this instrument has limited discrete support, our identification results suggest that we may rely on functional form to fill in the gaps between the points at which the index function Γ is nonparametrically identified.

5.2 Baseline Model Estimates

Here we report estimates of the entry model, index function, and joint distribution of equilibrium bids, using our baseline empirical specification. We estimate the joint distributions of log homogenized bids using a Bernstein polynomial approximation of order $m = 4$ and Gaussian copula.[36] In general we estimate separate joint distributions for each value of n_t. However, for large values of n_t we observe relatively few auctions (recall Table 1), leading us to assume that all auctions with $n_t \geq 11$ share the same marginal distribution of homogenized bids and the same copula correlation parameter. Combined with the ordered probit specification of the entry model, this leads to a baseline specification with 151 parameters.

5.2.1 Entry Model

Table 3 shows our estimates of the entry model parameters. We also report both the standard parametric standard error estimates ("SE"), which ignore spatial dependence, and standard errors obtained from the spatial block bootstrap ("SE (BB)").[37] As discussed in section 2 (see also Appendix A), the function η characterizes the effect of Z_t on entry but generally will not reveal the effects of X_t. Thus one must interpret the estimated coefficients on X_t with caution. However, the coefficient on Z_t is positive (consistent with the prediction of our motivating example) and statistically significant, supporting its value in providing a source of variation in bidder entry.

[36] Our choice of m reflects our experience with the tradeoff between flexibility an precision using similar sample sizes in our Monte Carlo simulations. We obtain similar results with slightly larger or smaller values of m, although the precision of the estimates declines as we add parameters.

[37] We report results based on 800 bootstrap replications throughout.

Table 3: Entry Model Estimates

		Est.	SE	SE (BB)
# active leases	X	-0.089	0.028	0.033
isolated lease		0.518	0.089	0.100
# firms that bid for neighbors		0.016	0.011	0.012
reoffered tract		-0.201	0.065	0.075
neighbor expired		-0.044	0.070	0.079
# neighbors drilled		-0.046	0.028	0.030
# neighbor hits		0.076	0.028	0.027
depth		-0.558	0.179	0.283
depth squared		0.120	0.068	0.132
royalty rate		-0.002	0.017	0.022
time controls		Sale year dummies		
# neighbor firms	Z	0.1743	0.0414	0.0446

5.2.2 Index Function and Bid Distribution

Table 4 displays estimates of the index parameters γ_x and γ_u, along with their estimated standard errors. We again caution that the estimated coefficients on X_t do not have the usual causal interpretations. However, the estimates indicate a strong effect of the unobserved heterogeneity on bids. Because U_t is normalized to have a uniform distribution, the coefficient implies that a one standard deviation increase in U_t drives up bids (and valuations) by roughly 33%.

We will not report estimates of the Bernstein polynomial parameters. However, Table 5 shows our estimates of the Gaussian copula correlation parameters ρ_n. The point estimates are positive, consistent with our assumption of positive dependence between bidders' private information. Although the point estimates generally suggest greater correlation at higher levels of competition, a Wald test fails to reject the hypothesis of equal copula correlation for all values of n.[38] However, a Wald

[38]While not statistically distinguishable from the others, the estimated correlation parameter for the nine-bidder auctions stands out. A close examination of the data revealed no clear explanation for this. We do not use auctions with more than seven bidders in our tests for common values below, and results are similar when we omit auctions with $n_t > 7$ from the estimation altogether.

Table 4: Index Function Estimates

		Est.	SE	SE (BB)
# active leases	X	0.016	0.018	0.021
isolated lease		0.042	0.059	0.087
# firms that bid for neighbors		0.029	0.008	0.011
reoffered tract		-0.167	0.047	0.067
neighbor expired		-0.280	0.047	0.067
# neighbors drilled		0.049	0.019	0.024
# neighbor hits		-0.022	0.020	0.025
depth		-0.224	0.135	0.269
depth squared		0.092	0.064	0.118
royalty rate		-0.006	0.010	0.017
time controls		Sale year dummies		
unobserved heterogeneity	U	1.136	0.273	0.351

test strongly rejects (p-value < 0.001) the null that all ρ_n are zero. Because homogenized bids are strictly increasing functions of signals, this implies rejection of the hypothesis of independent bidder types. This finding is of some importance on its own. Common knowledge unobservables and correlated private information are two distinct phenomena with different implications for behavior and policy. Often only one of these two sources of correlation between bids has been permitted in applications. Also important, however, is that the estimated correlation is generally small, suggesting modest correlation of bidders' private information. As we discuss below, this can have significant implications for auction design.

Table 5: Copula Correlation Estimates

n	Est.	SE	SE (BB)
2	0.044	0.041	0.050
3	0.031	0.029	0.040
4	0.089	0.031	0.032
5	0.136	0.030	0.041
6	0.112	0.035	0.037
7	0.111	0.044	0.043
8	0.133	0.039	0.040
9	0.295	0.071	0.067
10	0.161	0.052	0.068
11- 18	0.109	0.025	0.020

5.3 Decomposition of Correlation and Variance

Correlated private information is just one reason bids are correlated within an auction: auction observables and unobservables also play a role. Our model estimates allow us to assess the relative contributions of each factor. We present a decomposition in Table 6, which also shows a decomposition of the overall variance of bids.[39] Figures in the first column, labeled "log B_{it}^0," are for the homogenized log bids. Here the pairwise correlation reflects the correlation among signals, the nonlinearity of the bidding strategy (and log transformation), and the fact that, all else equal, bid levels vary with the number of competitors in the auction. Similarly, the variance in this column reflects the variability in bidders' assessments of tract values, as well as variation in bidding strategies across auctions with different numbers of entrants.

A natural way to characterize the contributions of unobservables is with the correlation/variance arising from variation in $\gamma(x, U_t)$ at a representative value of x. With our linear specification of γ, this variation is identical for all x. In the second column, labeled "log $B_{it}^0 + \gamma_u U_t$," we add the contribution of auction-level unobservables.

[39]We measure the "within-auction pairwise correlation" with the Pearson correlation coefficient for pairs of bids within the same auction.

Table 6: Decomposition of Log Bid Correlation and Variance

$\log B^0_{it}$	$\log B^0_{it} + \gamma_u U_t$	$\log B^0_{it} + \gamma_u U_t + X_t'\gamma_x$	$\log B_{it}$
	Within-Auction Pairwise Correlation		
0.133	0.210	0.222	0.425
	Variance		
1.520	1.669	1.693	2.292

After accounting for the contributions of the log homogenized bids and unobservable U_t, all remaining correlation/variance in the log bids reflects auction observables. Again exploiting our linear specification of γ, the third column of the table, labeled "$\log B^0_{it} + \gamma_u U_t + X_t'\gamma_x$," adds only the variation due to auction-level covariates. The final column, labeled "$\log B_{it}$," then adds the contribution of the year fixed effects to yield the total correlation and variance of the equilibrium bids.[40] Given our wide time span, it is not surprising that the fixed effects account for a substantial portion of the correlation and variance. More interesting is a comparison of the contributions of the auction-level covariates and the auction-level unobservables. The the estimated contribution of the unobservables is roughly six times as large as that of the observed covariates. This is particularly noteworthy because we selected covariates X_t from an unusually rich set of observables based in part on explanatory power in descriptive analysis of bids. Unobserved heterogeneity could be even more important in applications where only a limited set of covariates is available.

[40]Unlike the rest of the decomposition, the order of these last two steps could matter, due to correlation between the covariates and fixed effects. Here, reversing the order has virtually no effect on the implied contributions to the within-auction correlation; however it increases the contribution of the covariates to the bid variance: the impact of unobservables is then only three (rather than six) times as large as that of the observed covariates.

6 Tests For Common Values

6.1 Testing Approach

Using the estimates above, we test the null hypothesis of private values against the alternative of common values, relying on the following additional assumption.

Assumption 12. For all $n = 2, \ldots, \overline{n} - 1$,
$F_{S,V^0}(S_{1t}, \ldots, S_{nt}, V^0_{1t}, \ldots, V^0_{nt} | N_t = n) = F_{S,V^0}(S_{1t}, \ldots, S_{nt}, V^0_{1t}, \ldots, V^0_{nt} | N_t = n+1)$.

This is an assumption that U_t is the only source of dependence between the number of bidders at auction t and their valuations/signals. Thus, for example, holding (X_t, U_t) fixed, variation in N_t is not associated with variation in the valuations or in the precision of signals. Given this condition, HHS showed that the homogenized pivotal expected values $w^0(S_{it}; n)$ are unaffected by n in a private values auction but decreasing in n in a common values auction (see also Athey and Haile (2002)). This distinction reflects the winner's curse, which is present in (and only in) common values auctions, and which becomes more severe as the number of competitors increases (all else equal).[41] HHS also pointed out that under the maintained assumptions of the affiliated values auction model, $w^0(S_{it}; n)$ must be weakly decreasing in n. Because violations of this requirement indicate a rejection of at least one of our maintained hypotheses, this allows a test of the maintained assumptions, including that of equilibrium bidding.[42]

To perform the tests, we first construct estimates of the marginal distributions $F_w(\cdot; n)$ of homogenized pivotal expected values conditional on $N_t = n$, following the

[41] We use the term "winner's curse" to describe the adverse selection faced by a bidder competing against others with informative signals, not to errors or regret on the part of a bidder.

[42] Another testable implication of the maintained hypotheses is that the inverse homogenized bid functions—i.e., the right-hand-side of the first order conditions (15)—be strictly increasing as functions of the log homogenized bid (see Guerre, Perrigne, and Vuong (2000)). We find no violation of this requirement, even before allowing for sampling error (see Figure 2 below.)

procedure discussed in section 4. We then subject the estimated distributions to tests of the null hypothesis of equality (private values),

$$H_0: \quad F_w(w;n) = F_w(w;n+1) \quad \forall w, n = 2, \ldots, 6,$$

against the one-sided alterative of first-order stochastic dominance (common values),

$$H_1: \quad F_w(w;n) \leq F_w(w;n+1) \quad \forall w, n = 2, \ldots, 6,$$

with the inequality strict for at least some n and w. We limit attention to auctions with at most seven bidders in part to ensure that we have a sample of at least 100 auctions for each value of n considered.[43] An additional reason, however, is that growth in the severity of the winner's curse with the level of competition tends to diminish quickly as n grows. Intuitively, once a bidder assumes that $n-1$ others have low signals, learning that one additional signal is low conveys little "bad news" unless n is small. Thus, with common values, pivotal expected values are decreasing and, typically, convex in n.[44] This intuitive feature leads us to expect any evidence for common values to be clearest when comparing distributions at the lowest values n to those at higher levels of n.

We compare pairs of distributions using the one-sided Cramér-Von Mises type statistic

$$CVM = \int_{-\infty}^{\infty} \left[\hat{F}_w(w;\mathbb{N}_1) - \hat{F}_w(w;\mathbb{N}_2)\right]_+^2 dw, \tag{16}$$

where $[y]_+ = y \times 1\{y > 0\}$ and $\hat{F}_w(w;\mathbb{N})$ is the estimated distribution of $w^0(S_{it};N_t)$ conditional on N_t lying in a range of values defined by a set $\mathbb{N}$. We focus primarily on

[43] Abusing notation slightly, we let n (without an index) represent the number of bidders in a non-specific auction rather than referring repeatedly to "the number of bidders."

[44] This convexity holds in all examples of symmetric common values auctions we are aware of. An interesting question is whether additional assumptions are needed to prove this as a general property.

sets $\mathbb{N}$ containing two adjacent values of n ("coarse binning"). This pooling is done to reduce the impact of sampling error. However, by combining $n = 2$ and $n = 3$, where we expect the largest change in the severity of the winner's curse, this pooling may hide the strongest evidence of common values. Thus, we will also consider singleton sets ("fine binning"). In addition to pairwise tests, we will construct a single test statistic for the full range $n = 2, \ldots, 7$, based on the maximum statistic (or smoothed maximum) over the pairwise statistics:

$$CVM^{max} = \max_j \int_{-\infty}^{\infty} \left[\hat{F}_w(w; \mathbb{N}_j) - \hat{F}_w(w; \mathbb{N}_{j+1})\right]_+^2 dw \tag{17}$$

Below we report results for the pairwise and "max" tests for coarse binning, as well as the for the max test for fine binning.[45] We construct p-values for the test statistics using the distribution of (re-centered) test statistics from the spatial block bootstrap procedure described above.

6.2 Results: Baseline Specification

In Figure 1a we show the estimated CDFs under coarse binning, where we compare "low"($n \in \{2, 3\}$), "medium" ($n \in \{4, 5\}$), and "high" ($n \in \{6, 7\}$) levels of competition. Under the null, these distributions should differ only due to sampling error, whereas the alternative of common values implies that the CDFs will shift "northwest" as n increases. The estimated distributions shown here exhibit the stochastic ordering implied by the common values model. Further, as expected, the gap between the distributions for low and medium n is substantially larger than that between the distributions for medium and high n. Table 7 shows the relative sizes of these shifts (focusing on percentage changes) for the median homogenized pivotal expected val-

[45] Appendix I provides the complete test results under fine binning. The strongest evidence for common values (a p-value of 0.001) comes from comparisons of $n = 2$ to $n = 3$, where we expect the change in severity of winner's curse to be largest, and where we have the largest samples of auctions.

ues in each of the three bins.[46] As we shift from auctions with $n \in \{2,3\}$ to those with $n \in \{4,5\}$, the median pivotal expected value falls by about 14 percent. When moving from $n \in \{4,5\}$ to $n \in \{6,7\}$, the median falls by just under 3 percent.

Table 7: Competition and Median Pivotal Expected Values

	$n \in \{2,3\}$	$n \in \{4,5\}$	$n \in \{6,7\}$
Median Homogenized Pivotal Exp. Value	1,067,371	915,437	890,112
Change		151,933	25,326
Percent Change		14.2%	2.8%

Median homogenized pivotal expected values in 1982 dollars.

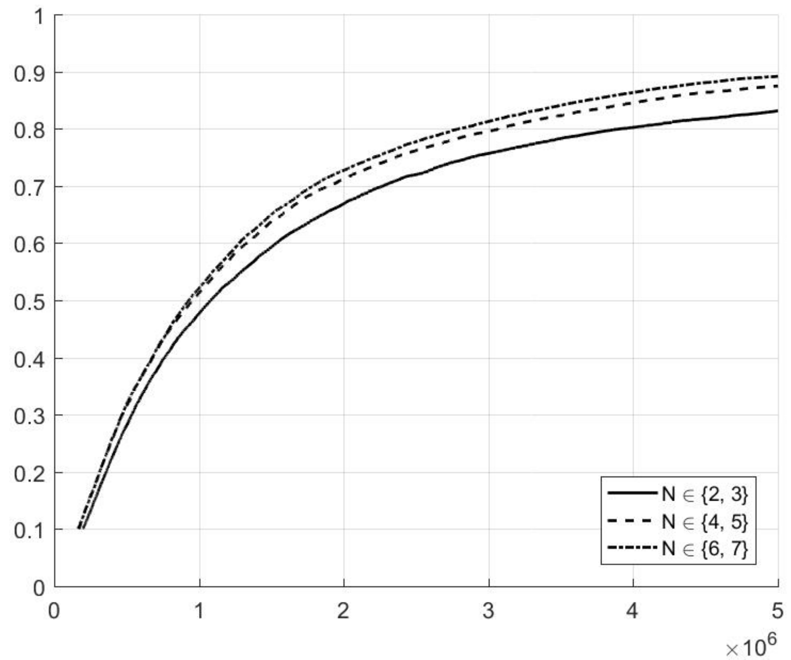
Contrast the patterns in Figure 1a with those in Figure 1b, which shows the estimated distributions obtained when we estimate the model without allowing for unobserved heterogeneity. Here the results suggest stochastic ordering in the direction *opposite* that predicted by common values. This suggests a misspecified model.

Table 8 shows the p-values obtained from the formal tests. These results confirm what was suggested by the figures above. First consider the comparison between low n and medium n when we allow unobserved heterogeneity ("With UH"). The test for common values implies rejection of private values in favor of common values, with a p-value of 0.021. The smaller gap between the estimated CDFs for medium and high n observed in Figure 1 cannot be statistically distinguished. However, the max tests—for both coarse and fine binning—also imply rejection at significance levels around 2 percent. Consistent with Figure 1, the specification test yields no evidence suggesting misspecification in the model with unobserved heterogeneity.

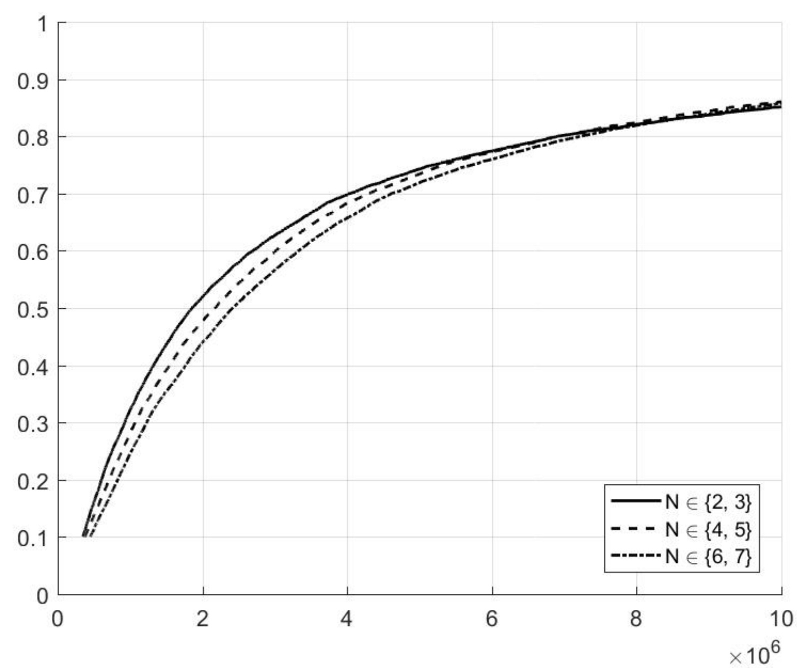
[46]Note that percentage changes in homogenized and non-homogenized pivotal expected values are identical.

Figure 1: Test for Common Values, Baseline Specification

(a) With Unobserved Heterogeneity



(b) Without Unobserved Heterogeneity



Note: Figures show estimated cumulative distributions of homogenized pivotal expected values.

Table 8: Test p-values
Baseline Specification

Test for Common Values

	With UH	No UH
{2,3} vs. {4,5}	0.021	0.299
{4,5} vs. {6,7}	0.274	0.754
Max (coarse binning)	0.022	0.590
Max (fine binning)	0.019	0.541

Specification Test

	With UH	No UH
{2,3} vs. {4,5}	0.906	0.066
{4,5} vs. {6,7}	0.901	0.076
Max (coarse binning)	0.985	0.081
Max (fine binning)	0.922	0.239

Contrast these results with those obtained in the model without unobserved heterogeneity ("No UH"). Here, the conclusions are essentially reversed. Not only is there no evidence of common values, but the specification tests suggest misspecification, with all three coarse binning tests yielding p-values below 0.10. Thus, at least in our data, failing to account for unobserved heterogeneity tends to hide the presence of common values and can erroneously suggest non-equilibrium bidding.

This is intuitive, but not a necessary implication of ignoring unobserved heterogeneity. On one hand, when we ignore unobserved heterogeneity and endogenous entry, an "endogenous treatment" bias works against the winner's curse effect we are seeking to detect: auctions with more bidders may have a larger winner's curse, but they also have more favorable unobservables. This suggests that the true effects of n on pivotal expected values could be masked or even reversed. But this intuition is incomplete. The model is misspecified when unobserved heterogeneity is present but ignored in the first-order conditions used to interpret the data. The cumulative distributions recovered in that case are not those of bidders' pivotal expected values. Moreover, the direction of the misspecification bias is unclear, and this bias may vary with n. Nonetheless, the results indicate that ignoring unobserved heterogeneity obscures the presence of common values in our sample.

6.3 Results under Alternative Specifications

We have explored the same testing strategies under several alternative specifications. In particular, we have examined:

(a) an extension of the bidding model allowing for a random reserve price, representing the Mineral Management Service's option to reject all bids;

(b) replacement of the ordered probit specification of the entry model with a semi-nonparametric estimator, following Gallant and Nychka (1987);

(c) dropping the year fixed effects from the index $\gamma(x_t, u_t)$, forcing unmeasured time-varying factors affecting the auctions into the unobservable; and

(d) dropping all drainage tracts from the sample.

Detailed discussion and results are presented in Appendix D. These alternative specifications lead to very similar patterns in the estimated distributions of pivotal expected values and to the same broad conclusions from the formal tests.

7 Affiliation, Common Values, and Seller Revenue

The presence of affiliation and common values can have important implications for auction design and for the way competition affects outcomes. Our estimates allow us to quantify some of these implications in the context of the OCS auctions.

7.1 Competition, Market Power, and Revenue

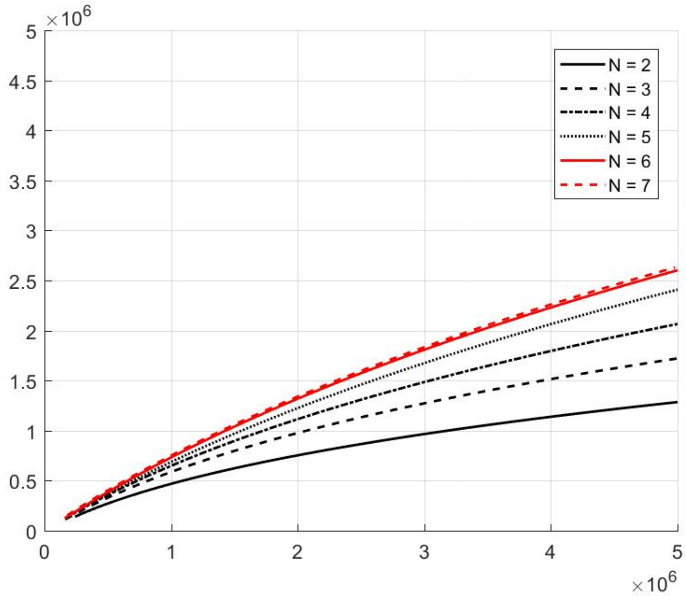
A well known feature of common value auctions is that added bidder competition can lead, counterintuitively, to less aggressive bidding and even to reduced seller revenue.[47] The winner's curse is a key force behind this possibility, although Pinkse and Tan (2005) demonstrated that bids can decline in the level of competition due to affiliation of signals alone. Here we use our estimates to examine the equilibrium effects of bidder competition on bid shading, the level of bids, and seller revenue.

In Figure 2 we plot, for different values of n, bidders' homogenized pivotal expected values against the associated homogenized bids implied by the first-order condition (9). Recall that pivotal expected values are strictly increasing in bidder types (signals). We know from theory that all types above the lowest shade their bids below their pivotal expected values, and that the degree of bid shading is increasing in type. Here we see that the estimated magnitude of this bid shading (reflecting bidders' market power) is substantial—all curves lie well below the 45 degree line. However, the gap shrinks as the level of competition rises from $n = 2$ to $n = 7$.

In Figure 3 we plot the estimated homogenized equilibrium bidding strategies $\beta^0(\cdot; n)$, where we normalize bidder signals to lie on $[0, 1]$ without loss. These estimated bid functions are strictly increasing, as implied by the model (but not imposed). They are also generally increasing with n—i.e., the effect of more intense competition generally dominates the effect of a more severe winner's curse. Using the same one-sided testing strategy used above, we reject (with a 10% significance threshold) the null hypothesis of bid functions that are constant with respect to n in favor of strategies that increase with n. Our point estimates suggest that this monotonicity reverses at $n = 7$. Such a reversal is consistent with the folk wisdom that equilibrium bids "eventually" decrease in the number of competitors in a common values auction

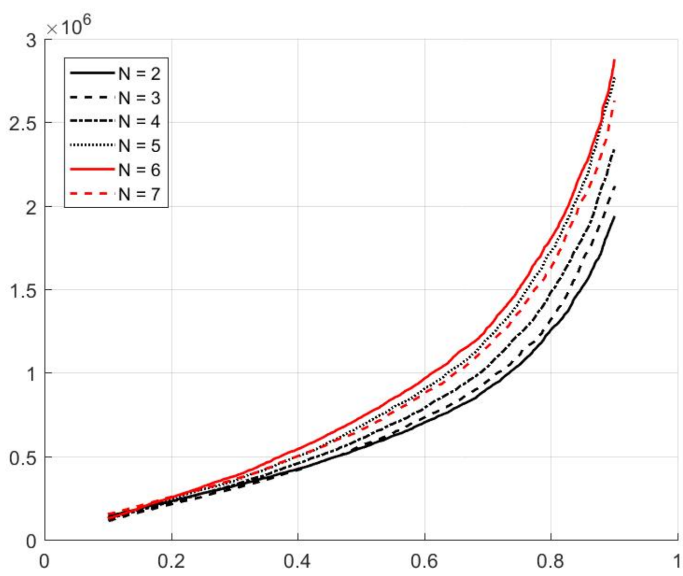
[47] See, e.g., Laffont (1997) and Hong and Shum (2002).

Figure 2: Competition and Bid Shading



Estimated pivotal expected values on horizontal axis with the associated homogenized equilibrium bids on the vertical axis, both in 1982 dollars.

Figure 3: Competition and Equilibrium Bidding Strategies



Bidder signals (normalized to $[0,1]$ on the horizontal axis with the associated homogenized equilibrium bids (in 1982 dollars) on the vertical axis.

(see, e.g., Laffont (1997)). However, even when comparing $n = 6$ to $n = 7$ in isolation, formal tests fail to reject the null that bids are weakly increasing with n at standard significance levels.

Finally, the second column (labeled "First-Price") of Table 9 shows, at each value of n, the implications of our estimates for a seller's expected revenue. To put revenues in a natural scale, we use the median value of the estimated index $\gamma(X_t, U_t)$ in our sample. Notably, even though our point estimates (Figure 3) suggested a non-monotonicity of bidding with respect to n, this effect is overcome by the fact that when n is larger the winning bid is the maximum among a larger number of bids. Thus, even our point estimates give no indication that a seller would profit from restricting entry to reduce the severity of the winner's curse faced by bidders.

Table 9: Revenue Gains through the Linkage Principle

	Expected Revenue			
n	First-Price	Second-Price	% gain	std. err.
2	7.02	7.05	0.48%	1.58%
3	9.21	9.31	1.06%	2.05%
4	11.48	11.79	2.73%	1.64%
5	14.95	15.64	4.62%	1.19%
6	16.91	17.51	3.57%	1.41%
7	16.94	17.43	2.89%	1.44%
overall	15.17	15.62	3.04%	0.49%

Simulated expected revenues in millions of 1982 dollars based on 1 million simulated auctions of each size n, scaled by the estimated median value of the index $\gamma(X_t, U_t)$ in our sample. Bootstrap standard errors based on 800 replications.

7.2 The Linkage Principle and Revenue Rankings

Milgrom and Weber (1982) and Milgrom (1987) identified the "linkage principle" as a key force determining a seller's preference among standard auction formats and information revelation policies in an affiliated values setting. Loosely, the linkage principle states that the information rents obtained by bidders can be limited (and the surplus extracted by the seller enhanced) by linking the price the winning bidder pays to realizations of random variables that are outside the winner's control but affiliated with the winner's private information. Moving from a first-price sealed bid auction to a second-price sealed bid auction, for example, enhances expected revenue by letting the second-highest bid determine the price: this bid is a function of the second-highest signal, which is affiliated with the winner's signal.[48] However despite the central theoretical role of the linkage principle and implied "revenue ranking" results, the existing literature provides few opportunities to quantify this effect empirically.[49] We know of no such evaluation in the case of OCS auctions.

The third column of Table 9 presents simulated expected revenues for second-price sealed bids auctions implied by our estimates.[50] We report results at each value of n from 2 to 7, as well as an overall figure obtained by mixing over the full range of N_t in our sample according to its empirical distribution. Although the results exhibit the revenue ranking implied by the theory, the estimated revenue gains from switching to a second-price auction are modest, with overall gains of 3% (although still around $450,000 per auction in 1982 dollars). This is consistent with our finding that correlation among bidders' private signals, while statistically significant, is relatively weak.

[48]Similarly, the Mineral Management Service's use of royalties, which link the total price paid to the realized value of the tract, can be justified by the linkage principle—at least in the absence of *ex post* moral hazard (see, e.g., Riley (1988) and DeMarzo, Kremer, and Skrzypacz (2005)).

[49]Shneyerov (2010) has done so for a sample of municipal bond auctions.

[50]Equilibrium revenue in a second-price auction equals the second-highest pivotal expected value.

As with the theoretical analysis of Milgrom and Weber (1982) and Milgrom (1987), an important caveat is that we have held bidder entry behavior fixed in our simulations. In reality, changes in auction design that reduce bidder surplus at a given level of competition should discourage entry, working against the anticipated gains in revenue. Quantifying the equilibrium effect would require estimation of a structural model of bidder entry (see, e.g., the extension discussed in section 8.1). However, because we find that even the gains considered by Milgrom and Weber (1982) and Milgrom (1987) would be fairly small, our results are quite informative. At least in the case of these auctions, any gains from moving to a second-price auction to better exploit the linkage principle appear to be limited.

8 Caveats, Extensions, and Challenges

Although our empirical approach offers several advantages for our study of OCS auctions, it relies on some significant assumptions that will not be suitable for all applications or questions. This is both an important caveat concerning our method and a call for further work. Here we briefly discuss some initial extensions and possible directions for future work.

8.1 Identifying a Structural Entry Model

We focused on identification and estimation of key features of the auction model. However, many interesting policy questions concerning auctions and procurement involve responses on the entry margin. Such responses generally cannot be evaluated using a only a reduced form for entry outcomes. However, the identification results we obtained above can make it possible to also identify the primitives of a structural entry model consistent with our assumed reduced form.

We demonstrate this in Appendix F for the model in Example 1. In the case of affiliated private values, our identification results above imply identification of the *ex ante* expected profit (gross of entry costs) of entering the auction as a function of the number of bidders. We show that this implies point identification of the structural entry model—i.e., of the entry cost functions for neighbors and non-neighbors. In the case of common values, because our identification results for the auction model deliver upper and lower bounds on the *ex ante* expected profit of entry, one obtains identification of bounds on these entry costs.

A full treatment of this extension, including development of appropriate estimation and inference procedures, is beyond the scope of the present paper. However, due to the substantial importance of entry in determining auction outcomes and the implications of auction policy, we view this as an important direction for further work.

8.2 Asymmetric Bidders

Our model assumed symmetry among bidders at the auction stage. Although we test and fail to reject symmetry in our data, other applications may demand allowing *ex ante* bidder heterogeneity. Symmetry is not essential to our method, although allowing asymmetry introduces two challenges. One concerns the potential multiplicity of equilibria in the auction. This issue arises regardless of whether one considers unobserved heterogeneity or endogenous entry. As a result, prior empirical work has relied on assumptions of suitable equilibrium selection rules, an assumption of independent private values (where equilibrium uniqueness can be guaranteed), or numerical verification of uniqueness conditional on the estimates.[51] An extension of our method to asymmetric bidders will face the same challenge, and we will focus here on the case of asymmetric independent private values for simplicity.

The second challenge is more tightly tied to our contribution and concerns whether

[51]See, e.g., Campo, Perrigne, and Vuong (2003), Krasnokutskaya (2011), Athey, Levin, and Seira (2011), Krasnokutskaya and Seim (2011), and Somaini (2015).

natural asymmetric games of entry followed by bidding lead to a deterministic reduced form entry equation like (3). Such an equation is essential to our approach, and asymmetries between potential entrants are known to create the potential for multiplicity in entry games. However, by focusing on the number of entrants within a given class of bidders, a deterministic relationship can arise naturally.

We demonstrate this in Appendix G. There we consider a fully specified asymmetric game similar to Example 1 in which the auction stage involves two classes of bidders (here, neighbors and non-neighbors), as is common in applications permitting bidder asymmetry. We show that equilibrium behavior in this example leads to a reduced form for neighbor entry taking the same form as (3). This allows straightforward adaptation of our key identification insights, although we leave open important questions concerning appropriate estimators.[52]

8.3 Binding Public Reserve Prices?

Our baseline model assumed the absence of a binding public reserve price at the auction. Although we developed an extension allowing a binding secret reserve price, these two cases will not cover all applications. One significant challenge to allowing a binding public reserve price within our framework would be the need to observe the number of firms entering (i.e., acquiring signals), as it is this entry outcome one might hope to exploit using a reduced form entry equation. In simpler models, firms endowed with signals are often called "potential bidders," and one often requires observation of the set (or number) of potential bidders for identification of auction models. However, when endogenenous signal acquisition is contemplated, it often becomes more difficult to justify an assumption that this set is correctly observed by the econometrician.

[52]Testing for common values in an asymmetric setting would require adjustments to the testing procedure as well, as one must focus on exogenous changes in participation that unambiguously raise or lower the severity of the winner's curse.

In some settings, the set of "planholders" (e.g., Krasnokutskaya and Seim (2011)) may both be observable and correctly define the set of firms acquiring signals. The potential adaptation of our ideas to such a setting is a topic we must leave to future research.

8.4 Overidentifying Restrictions and Ex Post Values

In our application we examined several testable implications of our model, some of which take the form of overidentifying restrictions:

(i) bidding strategies are strictly increasing;

(ii) pivotal expected values are weakly stochastically decreasing in n;

(iii) bid distributions respond to variation in n;

(iv) bid distributions for neighbors and non-neighbors are identical.

In several cases, our tests of these restrictions relied on other maintained hypotheses—e.g., our index structure—and therefore provide some assurance about these assumptions as well. In our discussion of identification we also noted several other sources of overidentifying restrictions.[53] However, we have left to future work the development of formal testing procedures. It would also be interesting to explore whether the sources of overidentification in our setting could allow identification in a more general model.

A closely related question is the potential value of additional observables. In the case of OCS auctions, an observable we have not exploited is (an estimate of) the ultimate value of each tract based on extracted volumes of oil and gas, realized

[53] A referee points out that additional restrictions arise from the fact in that in our model Z_t affects bids only through N_t but can exhibit more variation than N_t. For example, given any x, following the arguments used in Lemma 1 we have $\inf \{\ln B_{it}|N_t = \underline{n}(x,z), X_t = x, Z_t = z\} = \beta^0(s; \underline{n}(x,z)) + \gamma(x,0))$, which must be invariant to z within any set $\mathcal{Z}$ such that $\underline{n}(x,z)$ is the same for all $z \in \mathcal{Z}$.

market prices, and estimates of exploration and extraction costs.[54] Major challenges raised by the prospect of using these measures include those of measurement error, sample selection (*ex post* values are observed only for drilled tracts), and the fact that the observed values reflect just one realization of the time-series process whose *ex ante* distribution is relevant at an auction. If one abstracts from these issue and, in addition, assumes a pure common values model with no unobserved heterogeneity, HPP showed that *ex post* values can be used along with bids to obtain identification. Adapting their result to settings with unobserved heterogeneity raises another major challenge, however: the fact that in settings with unobserved heterogeneity one typically can identify the distribution of pivotal expected values, but not their individual realizations.[55] This contrasts with auction models without unobserved heterogeneity, and the distinction raises the question of whether/how one might combine *ex post* values with bid data to learn about the *ex ante* joint distribution of bidder signals and tract values.

9 Conclusion

We have proposed an empirical approach to first-price sealed bid auctions with affiliated values, unobserved auction-level heterogeneity, and endogenous bidder entry. Applying our method to OCS auction data leads us to reject the private values model in favor common values, a conclusion that is robust across a variety of specifications. We found that ignoring unobserved heterogeneity can hide the presence of common values, and that our specification tests allow us to reject specifications that ignore unobserved heterogeneity. Despite the presence of affiliated signals and common values, however, we found that the Mineral Management Service would not benefit from

[54] See, e.g., Hendricks and Porter (1988) or HPP.

[55] This is true of our method as well as approaches to unobserved heterogeneity based on mixtures or measurement error models.

limiting competition to soften the winner's curse, and that any revenue gains from moving to a second-price sealed bid auction to better exploit the linkage principle would likely be small.

While our empirical results confirm the conventional wisdom that oil lease auctions should be viewed as common values auctions, they also point to methodological challenges. We obtained a useful form of partial identification for a common values model, but the full set of primitives in common values models—even without unobserved heterogeneity—generally is not point identified without assumptions and data beyond those we required. Thus it will be important to continue pursuit of approaches to identification that exploit the features of particular settings (see, e.g., HPP or Somaini (2015)) and to explore extensions permitting unobserved heterogeneity in those frameworks. It may also prove productive to pursue other forms of partial identification that can be used to address positive and normative questions. The recent work of Syrgkanis, Tamer, and Ziani (2018) provides one such approach.

Appendices

A Equilibrium Entry in a Model of OCS Auctions

Here we consider a particular extensive form game of entry and bidding that is motivated by our application and yields the reduced form (3) for the entry outcome presented in the text, including the assumed weak monotonicity conditions.[56] This example also demonstrates how our model can accommodate auction-specific unobservables that are of arbitrary dimension and correlated with auction-specific observables, despite the apparent contradiction to our assumption that U_t is a scalar and independent of X_t. Accommodation of such correlation requires that we allow the interpretation of U_t to vary with the vector X_t. This precludes identification of (causal) effects of covariates on the auction; but in typical auction applications auction-level observables are primarily confounding factors to be controlled for rather than factors whose effects are of direct interest. This section also motivates the instrument used in our application. Finally, we discuss here the selection on unobservables that could be implied by considering only auctions attracting at least one bid, as necessitated by our data. We demonstrate that such selection introduces only an additional way in which the interpretation of U_t varies with X_t.

A.1 Model

Consider a game of entry and bidding for the lease of a tract t. Let $\mathcal{I}$ denote the set of all potential bidders ("firms"), and let $I = |\mathcal{I}|$. The set $\mathcal{I}$ can be partitioned into the set $\mathcal{Z}_t$ of "neighbor firms"—holders of active leases on adjacent ("neighbor") tracts—and all other firms, $\mathcal{I} \backslash \mathcal{Z}_t$. Denote the number of neighbor firms by $Z_t = |\mathcal{Z}_t|$.

[56] Example 1 in the text provided a sketch.

Let V_{it} denote the value of the lease to firm i (i's "valuation"). Let X_t and E_t denote, respectively, observed and unobserved (to us) characteristics of lease t that affect bidders' valuations. Let X_t include (among other relevant characteristics) the number of active leases on neighboring tracts and the set of bidders for each of those leases.[57] We make no restriction on the dimension of E_t and do not require independence between X_t and E_t.

The game consists of two stages. In the second stage, lease t is offered by first-price auction to the N_t bidders who enter in the first stage. We assume there is no binding reserve price in the auction.[58] In the first stage, bidders simultaneously choose whether to incur an entry cost in order to acquire a signal and participate in the auction.[59] Let $c_i(x_t)$ denote the entry cost for firm i. Neighbors have lower entry costs. In particular, $c_i(x_t) = c(x_t)$ for a neighbor firm, whereas non-neighbor firms have entry costs $c_i(x_t) = c(x_t) + \delta(x_t)$, with $\delta(x_t) > 0$.[60]

Firms acquiring signals become "bidders" and learn the number of competitors they face. Let N_t denote the number of bidders. Let S_{it} denote the signal received by bidder i. Given $N_t = n$, let $S_t = (S_{1t}, \ldots, S_{nt})$ and $V_t = (V_{1t}, \ldots, V_{nt},)$, where without loss we re-label bidders as firms $i = 1, \ldots n$. For any conditioning set $\Omega \subseteq (X_t, Z_t, E_t)$, let $F_{SV}(S_t, V_t | N_t, \Omega)$ denote the conditional distribution of bidders' signals and valuations. We assume $F_{SV}(S_t, V_t | N_t, X_t, Z_t, E_t)$ satisfies standard smoothness, symmetry, affiliation, and nondegeneracy conditions (see Assumption 2 in the text). We assume

[57] In practice we represent the set of bidders for neighboring tracts more parsimoniously with the number of such bidders.

[58] In our application we consider an extension allowing a random reserve price.

[59] As is standard in the literature, we assume that only bidders incurring the entry cost can submit a bid (see, e.g., Levin and Smith (1994), Li and Zheng (2009), Athey, Levin, and Seira (2011), Krasnokutskaya and Seim (2011), Gentry and Li (2014), or Bhattacharya, Roberts, and Sweeting (2014)).

[60] More generally, one can allow entry costs to depend on the instruments, writing $c(x_t, z_t)$ and $\delta(x_t, z_t)$. For example, in other applications one might have measures of signal acquisition costs that vary across time or location. However, our discussion of sample selection exploits the exclusion of Z_t from entry costs.

that Z_t alters the joint distribution of signals and valuations only through its effect on N_t, i.e.,

$$F_{SV}(S_t, V_t | N_t, Z_t, X_t, E_t) = F_{SV}(S_t, V_t | N_t, X_t, E_t),$$

and that Z_t is independent of E_t conditional on X_t. We discuss the justification for this conditional independence assumption below. We assume

$$V_{it} = V_{it}^0 \, \lambda(X_t, E_t), \tag{A.1}$$

where the function λ is positive and the random variables $\left(V_{1t}^0, \ldots, V_{n_t t}^0, S_{1t}, \ldots, S_{n_t t}\right)$ are independent of (X_t, E_t, Z_t) conditional on N_t. We assume that, for all $x \in \mathbb{X}$, $\lambda(x, E_t)$ is continuously distributed with convex bounded support.

Note that we have not restricted the dimension of E_t, imposed any monotonicity condition on λ, or required independence between X_t and E_t.[61] Nonetheless, we can obtain the model of unobserved heterogeneity in the text by representing the random variable $\lambda(X_t, E_t)$ in terms of its quantiles conditional on X_t. In particular, given $X_t = x$, let $F_\lambda(\cdot|x)$ denote the CDF of the random variable $\lambda(x, E_t)$, and let

$$U_t = F_\lambda(\lambda(x, E_t)|x). \tag{A.2}$$

For $u \in [0,1]$ define $F_\lambda^{-1}(u|x) = \inf\{\lambda : F_\lambda(\lambda|x) \geq u\}$ and let

$$\Gamma(x, u) = F_\lambda^{-1}(u|x). \tag{A.3}$$

Combining (A.2) and (A.3), for each x we have $F_\lambda(\Gamma(x, U_t)|x) = U_t = F_\lambda(\lambda(x, E_t)|x)$,

[61] This may be important, as the nomination process by which tracts were offered for lease in our sample period suggests that a tract with "undesirable" value of X_t may have been unlikely to be offered unless the value of E_t made the tract desirable. See, e.g., Hendricks, Porter, and Boudreau (1987) for a discussion of the nomination process.

i.e.,

$$\Gamma(x, U_t) = \lambda(x, E_t).$$

By construction, Γ is strictly increasing (and continuous) in its second argument, and U_t is uniform on $[0,1]$ conditional on X_t. And because U_t is a measurable function of E_t conditional on X_t, U_t is independent of Z_t conditional on X_t.

Note that in this new representation of the model, the distribution of U_t does not vary with X_t, but its interpretation generally will. Because $\Gamma(x_t, u_t) = \lambda(x_t, e_t)$ for all t by construction, $\Gamma(X_t, U_t)$ fully characterizes the variation and dependence in valuations and bids that arises from the observables and unobservables. Likewise, controlling for the value of $\Gamma(X_t, U_t)$ fully controls for the effects of auction observables and unobservables (X_t, E_t) on valuations, bids, and equilibrium first-order conditions. However, Γ does not characterize the effect of a change in X_t holding unobservables fixed, since our U_t is redefined at every value of X_t.

A.2 Equilibrium

We henceforth use the representation of the model just derived. The set of firms $\mathcal{I}$, the rules of the game, the values of (X_t, U_t, Z_t), and the distribution $F_{SV}\left(S_t, V_t | N_t, X_t, U_t\right)$ are common knowledge among firms. We consider perfect Bayesian equilibrium in pure strategies, with weakly increasing strategies in the auction stage.

The second stage of the game is identical to the first-price sealed bid auction with symmetric affiliated values studied by Milgrom and Weber (1982), who characterize the unique Bayes-Nash equilibrium in increasing bidding strategies. Bidder i's payoff in the auction stage can be written as a function of the commonly known (N_t, X_t, U_t) and the realized bidder signals S_t. As noted in the text, multiplicative separability of valuations is inherited by equilibrium bids. This implies that a bidder's *ex post* profit, denoted by $\pi\left(S_{it}, S_{-it}, N_t, X_t, U_t\right)$, is strictly increasing in the index $\Gamma(X_t, U_t)$

and, therefore, strictly increasing in U_t. Further, we assume the usual case in which the *ex ante* expected equilibrium payoff

$$\bar{\pi}\left(N_t, X_t, U_t\right) = E\left[\pi\left(S_{it}, S_{-it}, N_t, X_t, U_t\right)|N_t, X_t, U_t\right]$$

is strictly decreasing in N_t.[62]

In the entry stage, firms make decisions based on the cost of entry and expected profit from participating in the auction. Let $C_{it} = c_i\left(X_t\right)$. For firm i, entering when $n-1$ other firms will also enter implies expected profit

$$\bar{\pi}\left(n, X_t, U_t\right) - C_{it}.$$

Conditional on $\left(X_t, Z_t, U_t\right)$, and given equilibrium beliefs about payoffs in the auction stage, the entry stage is then equivalent to the entry game in Berry (1992). Berry showed that a pure strategy equilibrium exists and that with probability one all equilibria exhibit the same number of entrants, given by

$$\eta\left(X_t, Z_t, U_t\right) = \max_{0 \leq n \leq I}\left\{n : \bar{\pi}\left(n, X_t, U_t\right) - C_{it} \geq 0\right\}.$$

Recall that $\left(X_t, Z_t\right)$ determine the values of $\left\{C_{it}\right\}_{i \in \mathcal{I}}$. Thus, in any pure strategy perfect Bayesian equilibrium (with weakly increasing bidding) we have

$$N_t = \eta\left(X_t, Z_t, U_t\right).$$

Because $\bar{\pi}\left(N_t, X_t, U_t\right)$ is strictly increasing in U_t, η is weakly increasing in U_t.

[62] We know of no counterexample to strict monotonicity in N_t under the assumption that U_t is the only latent source of dependence between the entry and auction stages—i.e., that Assumption 12 holds. Nonmonotonicity (within the relevant range of N_t) could lead to existence of multiple equilibria with different numbers of bidders.

A.3 The Instrument

Our instrument for bidder entry Z_t is the number of neighbor firms. First consider the exclusion requirement (Assumption 5). We have assumed directly that Z_t is independent of (S_t, V_t^0) conditional on N_t, i.e., that X_t are the only observables directly affecting bidder valuations.[63] However we must verify that Z_t is also independent of U_t conditional on X_t. A tract with three neighbor leases, for example, may have one, two, or three neighbor firms, depending on which bidders for the neighboring leases won those auctions. Given the number of neighbor leases and the bidders for each neighbor tract (i.e., conditional on X_t), the number of distinct winners reflects only random variation in bidders' signals at prior auctions. Recall that signals are assumed independent of tract-specific unobserved heterogeneity and independent across tracts. Thus, even in the case of spatially correlated tract-level unobservables E_t, the conditional independence requirement $Z_t \perp\!\!\!\perp U_t | X_t$ will hold.

Regarding the "relevance" requirement for the instrument Z_t,[64] observe that changes in the number of neighbor firms affects entry because for some combinations of (X_t, Z_t, U_t) the market will accommodate the $n+1$st entrant only if there is a potential bidder with low signal acquisition cost. For example, we will sometimes have two entrants because the market would support entry by a third (low cost) neighbor, but not by a third firm that is a (high cost) non-neighbor. Thus, larger values of Z_t will lead, all else equal, to weakly larger numbers of entrants.

[63] This could fail here if the number of neighbor firms had a direct effect on tract value (given X_t), e.g., by driving up costs of negotiating production from common pools.

[64] This discussion is informal. We state formal "relevance" conditions for nonparametric identification in section 3.

A.4 Truncation

In the OCS data we observe no information about (even existence of) leases offered for sale but attracting no bids. Given the tract nomination process in place during the sample period, this may not have been a frequent phenomenon. But existence of such leases could imply a form of selection on unobservables: leases attracting no bids would be those with relatively undesirable unobservables. Here we demonstrate that such selection is accommodated by interpreting the unobservable in our model as that conditioned on the event that the auction attracts at least one bid.

In our model, offered leases attracting no bids are those for which

$$\pi\left(1, x_t, u_t\right) \leq c_{it} \quad \forall i \in \mathcal{I}. \tag{A.4}$$

Letting $\pi^{-1}\left(c; 1, x\right) = \sup\left\{u : \pi\left(1, x, u\right) < c\right\}$, we can rewrite (A.4) as[65]

$$u_t \leq \pi^{-1}\left(\min_{i \in \mathcal{I}} c_i\left(x_t\right); 1, x_t\right),$$

or, more simply given our definition of $c_i(x_t)$,

$$u_t \leq \underline{u}\left(x_t\right).$$

Recalling that the definition of U_t already changes with each value of X_t, we obtain the original model by redefining U_t to denote the value of the unobservable conditional on truncation at $\underline{u}(X_t)$.[66]

[65] Note that the threshold for attracting a single bidder varies depending on the presence of at least one neighbor tract (firm), but not with the number of neighbor firms Z_t.

[66] More formally, going back to the original formulation of the unobserved heterogeneity in terms of E_t, let $\tilde{\pi}\left(N_t, X_t, E_t\right)$ denote the expected second-stage payoff for a bidder facing $N_t - 1$ opponents given (X_t, E_t). Under the separable structure (A.1), we have $\tilde{\pi}\left(N_t, X_t . E_t\right) = \tilde{\pi}^0\left(N_t\right) \lambda\left(X_t, E_t\right)$. Given $X_t = x$, we have a zero-bidder auction when $\lambda\left(x, E_t\right) \leq \frac{\min_i c_i(x_t)}{\tilde{\pi}^0(1)}$. Thus, we modify the construction of U_t by letting $F_\lambda\left(\cdot|x\right)$ denote the CDF of $\lambda\left(x, E_t\right)$ conditional on $\lambda\left(x, E_t\right) > \frac{\min_i c_i(x_t)}{\tilde{\pi}^0(1)}$.

B Proofs Omitted from the Text

Corollary 1. Under Assumptions 1–6, the distribution of $U_t|(X_t, N_t)$ is identified.[67]

Proof. We can express $\Pr(U_t \leq u|X_t = x, N_t = n)$ as

$$F_{U|XN}(u|x,n) = \int F_{U|XZN}(u|x,z,n) \; d\zeta(z|x,n) \tag{B.1}$$

where $F_{U|XZN}$ is the distribution of $U_t|(X_t, Z_t, N_t)$ and ζ is the distribution of $Z_t|(X_t, N_t)$. Conditional on $N_t = n, Z_t = z$, and $X_t = x$, U_t is uniform on $[\tau_{n-1}(x,z), \tau_n(x,z)]$, and by Theorem 1 the endpoints $\tau_{n-1}(x,z)$ and $\tau_n(x,z)$ are identified. So $F_{U|XZN}$ is known. Since ζ is directly observed, the result follows from (B.1). □

Lemma 2. Under Assumptions 1–8, for all $n \geq \underline{n}$, all $(x,z) \in \mathbb{Y}(n)$, and all $(x',z') \in \mathbb{Y}(n)$, $\gamma(x, \tau_n(x,z)) - \gamma(x', \tau_{n-1}(x',z'))$ is identified.

Proof. For n^* as defined in Assumption 8, take $n \leq n^*$ and let $x(n)$, $z(n)$, and $\hat{z}(n)$ be as in part (i) of Assumption 8 so that

$$\begin{aligned} \underline{n}(x(n), z(n)) &= n \\ \underline{n}(x(n), \hat{z}(n)) &= n+1. \end{aligned} \tag{B.2}$$

Since $(x(n), z(n)) \in \mathbb{Y}(n)$ and $(x(n), \hat{z}(n)) \in \mathbb{Y}(n+1)$, Lemma 1 implies identification of

$$\gamma(x', \tau_{n-1}(x',z')) - \gamma(x(n), \tau_{n-1}(x(n), z(n))) \tag{B.3}$$

and

$$\gamma(x'', \tau_n(x'',z'')) - \gamma(x(n), \tau_n(x(n), \hat{z}(n))) \tag{B.4}$$

[67] For convenience we restate the results being proved in this appendix.

for all $(x', z') \in \mathbb{Y}(n)$ and $(x'', z'') \in \mathbb{Y}(n+1)$. By (4) and (B.2),

$$\tau_{n-1}(x(n), z(n)) = 0 = \tau_n(x(n), \hat{z}(n)),$$

so subtracting (B.3) from (B.4) yields identification of

$$\gamma(x'', \tau_n(x'', z'')) - \gamma(x', \tau_{n-1}(x', z')) \tag{B.5}$$

for all $(x'', z'') \in \mathbb{Y}(n+1)$ and $(x', z') \in \mathbb{Y}(n)$. By Assumption 7, there exists some (x'', z'') that is in both $\mathbb{Y}(n+1)$ and $\mathbb{Y}(n)$. The claim then follows from Lemma 1. A symmetric argument applies for $n > n^*$.[68] □

Lemma 3. Let Assumptions 1–8 hold. Then for all $n \geq \underline{n}$ and all $(x, z) \in \mathbb{Y}(n)$, the values of $\gamma(x, \tau_{n-1}(x, z))$ and $\gamma(x, \tau_n(x, z))$ are identified.

Proof. We proceed by induction, starting with $n = \underline{n}$. By the normalization (1), $\gamma(x^0, 0) = 0$, where for some z we have $(x^0, z) \in \mathbb{Y}(\underline{n})$. Lemma 1 then implies identification of $\gamma(x, \tau_{\underline{n}-1}(x, z))$ for all $(x, z) \in \mathbb{Y}(\underline{n})$. Lemma 2 then implies identification of $\gamma(x, \tau_{\underline{n}}(x, z))$ for all $(x, z) \in \mathbb{Y}(\underline{n})$. Now take any $n > \underline{n}$ and suppose that $\gamma(x, \tau_{n-1}(x, z))$ is known for all $(x, z) \in \mathbb{Y}(n-1)$. By Assumption 7 there exists a point $(\tilde{x}, \tilde{z})$ in $\mathbb{Y}(n-1) \cap \mathbb{Y}(n)$. Since we have already identified $\gamma(\tilde{x}, \tau_{n-1}(\tilde{x}, \tilde{z}))$, by Lemma 1 we also know the value of $\gamma(x, \tau_{n-1}(x, z))$ for all (x, z) in $\mathbb{Y}(n)$. By Lemma 2, this implies identification of $\gamma(x, \tau_n(x, z))$ for all (x, z) in $\mathbb{Y}(n)$. □

[68] Note that the arguments used to show Lemmas 1 and 2 will often imply several forms of overidentification. For example, Lemma 1 implies overidentification of $\gamma(x', \tau_m(x', z')) - \gamma(x, \tau_m(x, z))$ for any m which is both smaller than $\min\{\overline{n}(x', z'), \overline{n}(x, z)\}$ and larger than $\max\{\underline{n}(x', z'), \underline{n}(x, z)\}$. And while Assumption 7 ensures only that there exist one $(\tilde{x}, \tilde{z}) \in \mathbb{Y}(n)$ that is also in $\mathbb{Y}(n-1)$, when there is more than one such pair the proof of Lemma 2 will provide multiple ways of constructing the same value of a given difference $\gamma(x'', \tau_n(x'', z'')) - \gamma(x', \tau_{n-1}(x', z'))$. Finally, in practice there may often be more than one value of n^* satisfying Assumption 8, resulting in some duplication in the differences identified in the two halves of the proof of Lemma 2.

Lemma 4. Under Assumptions 1–9, $\tau_{n-1}(X_t, Z_t)$ is continuous in Z_t on the pre-image of $(0,1)$.

Proof. Fix $n, x,$ and z such that $\tau_{n-1}(x,z) \in (0,1)$. Let $\tau = \tau_{n-1}(x,z)$ and let $\nu > 0$ be sufficiently small that $\tau + \nu < 1$ and $\tau - \nu > 0$. We show that for any such ν there exists $\epsilon > 0$ such that for every z' satisfying $\|z' - z\| < \epsilon$ we have $\tau_{n-1}(x, z') \in (\tau - \nu, \tau + \nu)$. Let $\delta = \nu/2$. By the definition of $\tau_{n-1}(x,z)$ and weak monotonicity of η in U_t, $\eta(x, z, \tau - \delta) < n$. So by Assumption 9 there exists $\epsilon_1 > 0$ such that for any z' satisfying $\|z' - z\| < \epsilon_1$, $\eta(x, z', \tau') < n$ for some $\tau' \in (\tau - 2\delta, \tau)$. Similarly, because $\eta(x, z, \tau + \delta) \geq n$, Assumption 9 ensures that there exists $\epsilon_2 > 0$ such that for any z' satisfying $\|z' - z\| < \epsilon_2$, $\eta(x, z', \tau'') \geq n$ for some $\tau'' \in (\tau, \tau + 2\delta)$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we have shown that for any z' satisfying $\|z' - z\| < \epsilon$, $\eta(x, z', \tau') < n$ for some $\tau' \in (\tau - \nu, \tau)$ while $\eta(x, z', \tau'') \geq n$ for some $\tau'' \in (\tau, \tau + \nu)$. At such z', $\tau_{n-1}(x, z')$ must lie in $[\tau', \tau'']$. □

Lemma 5. Under Assumptions 1–11, conditional on any $N_t = n$, the joint density of $(B^0_{1t}, \ldots, B^0_{nt})$ is identified.

Proof. Fix $N_t = n$ and $X_t = x$, where x is as in Assumption 11. Let $\psi_{\tilde{B}}$ denote the characteristic function of the log bids $(\tilde{B}_{1t}, \ldots, \tilde{B}_{nt})$ conditional on $X_t = x$ and $N_t = n$. Let ψ_γ denote the characteristic function of $\gamma(x, U_t)$ conditional on $N_t = n$. By (10), for $(r_1, \ldots, r_n) \in \mathbb{R}^n$ we have

$$\psi_{\tilde{B}}(r_1, \ldots, r_n) = \psi_{\tilde{B}^0}(r_1, \ldots, r_n)\, \psi_\gamma(r_1 + \cdots + r_n),$$

where $\psi_{\tilde{B}^0}$ is the characteristic function of the log homogenized bids $\left(\tilde{B}^0_{1t}, \ldots, \tilde{B}^0_{nt}\right)$ conditional on $N_t = n$. Since the distribution of $U_t|(X_t, N_t)$ is known (Corollary 1) and γ is a known function (Theorem 2), ψ_γ is known. So under Assumption 11 the equation

$$\psi_{\tilde{B}^0}(r_1, \ldots, r_n) = \frac{\psi_{\tilde{B}}(r_1, \ldots, r_n)}{\psi_\gamma(r_1 + \cdots + r_n)}$$

uniquely determines $\psi_{\tilde{B}^0}(r_1, \ldots, r_n)$ for almost all $(r_1, \ldots, r_n)$. By continuity of characteristic functions this yields identification of $\psi_{\tilde{B}^0}$, implying identification of the joint density of $\left(\tilde{B}^0_{1t}, \ldots, \tilde{B}^0_{nt}\right)$.[69] The result then follows. □

C Tests of Bidder Symmetry

Here we examine the bid data to assess the assumption of symmetry between neighbor firms and non-neighbors in the auction stage—in particular, the implication that equilibrium bids of neighbors and non-neighbors in any given auction are drawn from the same marginal distribution. The key challenges arise from the need to account for (a) auction heterogeneity through (X_t, U_t); (b) dependence across auctions due to spatial correlation of U_t; (c) dependence among bids in the same auction due to affiliation of bidder signals; and (d) the variation in equilibrium bidding strategies with the number of bidders.

We consider tests based on within-auction differences in log bids, exploiting our separability assumption.[70] Recall that equilibrium log bids are given by

$$\tilde{B}_{it} = \tilde{B}^0_{it} + \gamma(X_t, U_t),$$

so that

$$\tilde{B}_{it} - \tilde{B}_{jt} = \tilde{B}^0_{it} - \tilde{B}^0_{jt} \tag{C.1}$$

for any bids i, j in auction t.

Fix a value of $n \geq 2$ and consider two distinct n-bidder auctions, t and t', such that

[69] Because the same argument holds at all $x \in \mathbb{X}$ such that the characteristic function of $\gamma(x, U_t)$ is nonvanishing a.s., the argument demonstrating Lemma 5 may often yield overidentification.

[70] Our reliance on the separability assumption implies that our tests below are in fact joint tests of symmetry and separability. For other applications where such a joint test rejects, an interesting question would be how one might evaluate the two hypotheses separately.

there is at least one neighbor bidder in auction t and one non-neighbor in auction t'. Without loss, let bidder 1 be a neighbor in auction t and bidder 2 be a non-neighbor in auction t'. For each of the two auctions, let bidder 3 denote a randomly selected (neighbor or non-neighbor) bidder in that auction.[71] Then, by (C.1),

$$\tilde{B}_{1t} - \tilde{B}_{3t} \stackrel{d}{=} \tilde{B}_{2t'} - \tilde{B}_{3t'}. \tag{C.2}$$

We test the null (C.2) using standard two-sample tests of equal distributions. By relying on within-auction differences we eliminate the effects of (X_t, U_t). And by using distinct auctions t and of t' to construct the two samples, we also avoid dependence across samples. Thus, this approach overcomes challenges (a)–(c) above. This leaves the question of how to combine data across values of n in a way that overcomes challenge (d).[72] This requires that we avoid systematic differences across the two samples in the mixing over n. We consider two ways of doing this.

In the first, we limit attention to auctions with at least one neighbor bidder and one non-neighbor bidder. We begin by randomly partitioning these auctions into two samples. For each auction in the first (second) sample, we take the difference between the log-bid of a randomly selected neighbor (non-neighbor) and that of a randomly selected opponent; these differences are then placed in the "neighbor sample" ("non-neighbor sample").[73] We test the null of equal distributions using a both a Kolmogorov-Smirnov test and a Cramér-von Mises test. Because the random components of this procedure imply random variation in test results, we repeat the procedure

[71] Despite our indexing convention here, we do not require more than 2 bidders per auction. In the case of a 2-bidder auction, "bidder 3" is simply the "other" bidder at this auction.

[72] As elsewhere in the paper, we abuse notation slightly and let "n" refer to the number of bidders when association with a particular auction is not essential.

[73] In the neighbor (non-neighbor) sample, the neighbor's (non-neighbor's) log bid comes first in the associated difference in (C.2), ensuring that the two distributions differ under the alternative even when $n = 2$.

many times and report average p-values.[74] Because we randomly assign auctions to the two samples, there is no *ex ante* difference in the mixing over n in the two samples. And because we average over many replications of the random procedure, differences in the mixing over n that do arise in each replication will "average out" across replications. Indeed, if we instead stratify on n when randomly partitioning the sample of auctions, the average p-values obtained from each test are virtually identical.

This first approach has the appeal of comparing differenced bids of neighbors and non-neighbors from the same population of auctions. However, because a relatively small share of auctions has a bidder of each type, this limits the size of the samples compared (to 578 total auctions, or 239 for each sample). Our second approach instead uses all auctions with at least two bidders, including those with only neighbors or only non-neighbors as bidders. This enables us to more than double the sample size, although slightly more care must be taken to ensure equal mixing over n in the two distributions. In particular, we assign auctions to the two samples using a procedure that stratifies by n. For the "neighbor sample" we consider all auctions with a neighbor bidder. Let $NB(n)$ denote the number of n-bidder auctions in this sample. For each auction in this sample we difference the log bids of a randomly selected neighbor and a randomly selected bidder among its opponents. We construct the "non-neighbor sample" using the remaining auctions, which have only non-neighbor bidders. In our data the number of such auctions exceeds $NB(n)$ for all n. So for each n we select $NB(n)$ auctions at random and construct the difference in log bids between two randomly selected bidders. As with the first testing approach, we iterate this procedure many times and report average p-values.

Table 10 shows the average p-values resulting from both approaches, using 800 replications. None of the four tests rejects at standard significance levels.

[74] The sources of randomness involve the initial partition of auctions, the selection of which neighbor (non-neighbor) is chosen from each auction, and which opposing bid is used to construct the difference.

Table 10: Two-Sample Tests of Bidder Symmetry

	observations per sample	average p-values K-S Test	average p-values CvM Test
Approach 1	239	0.3047	0.7215
Approach 2	521	0.6764	0.3523

Average p-values over 800 replications.

D Tests for Common Values: Alternative Specifications

Here we consider our tests for common values (and our specification test) under several alternative specifications.

D.1 Random Reserve Price

Our baseline model ignores the fact that the MMS retained (and sometimes exercised) a right to reject all bids. A bidder who anticipates this rejection possibility would have an incentive for more aggressive bidding than is implied by the standard equilibrium first-order condition. In our sample this effect is likely to be small in auctions attracting at least three bidders, where well over 99 percent of all winning bids were accepted.[75] However, the threat of rejection could be an important factor in auctions attracting only one or two bidders.

To incorporate this feature, we follow Hendricks, Porter, and Spady (1989) and

[75]Rejection of the high bid occurred more often at auctions dropped from our sample due either to missing values or to their being auctions of leases on partial-block tracts.

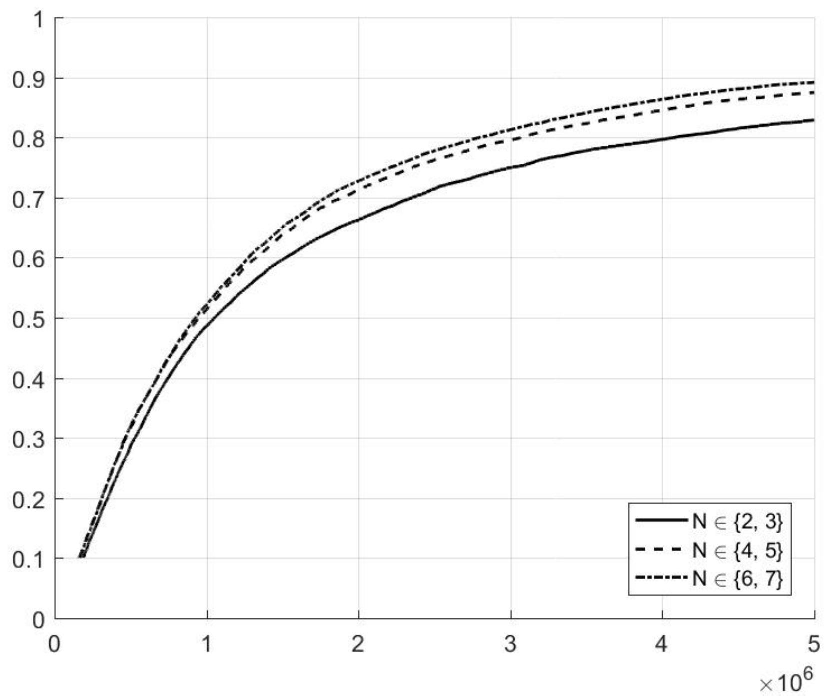
Hendricks, Porter, and Wilson (1994)) by modeling bid rejection with a random reserve price whose value is unknown to bidders. From a bidder's perspective, the random reserve price is effectively an additional bid (submitted by the auctioneer), leading to a first-order condition similar to that in the baseline case. We follow Bajari and Hortaçsu (2003) by modeling the reserve price as drawn from the same marginal distribution as a bid, although we re-scale the seller's "bid" to fit the observed MMS rejection decisions. We provide additional detail and discussion in Appendix E.

Figures 4a and 4b show the estimated marginal distributions of pivotal expected values obtained with this alternative model. These are very similar to the estimates obtained in the baseline model. In Figure 4a, where we allow unobserved heterogeneity, we see the pattern of first-order stochastic dominance predicted by common values, with the magnitude of the estimated shifts in the distributions declining with n. Figure 4b again shows that, if unobserved heterogeneity is ignored, we obtain estimated distributions ordered in the direction opposite that implied by common values. As shown in Table 11, the formal tests are also very similar to those in the baseline case. In particular, when we account for unobserved heterogeneity, the test for common values leads us to reject private values, with p-values between 0.02 and 0.03. However, we se no evidence of common values when we ignore unobserved heterogeneity.[76]

[76] As discussed in Appendix E, with the random reserve price, violations of weak stochastic ordering need not imply rejection of the model. Thus, although we display the results of the specification tests, apparent rejections observed when we ignore unobserved heterogeneity should be viewed only as suggestive in this case.

Figure 4: Random Reserve

(a) With UH



(b) No UH

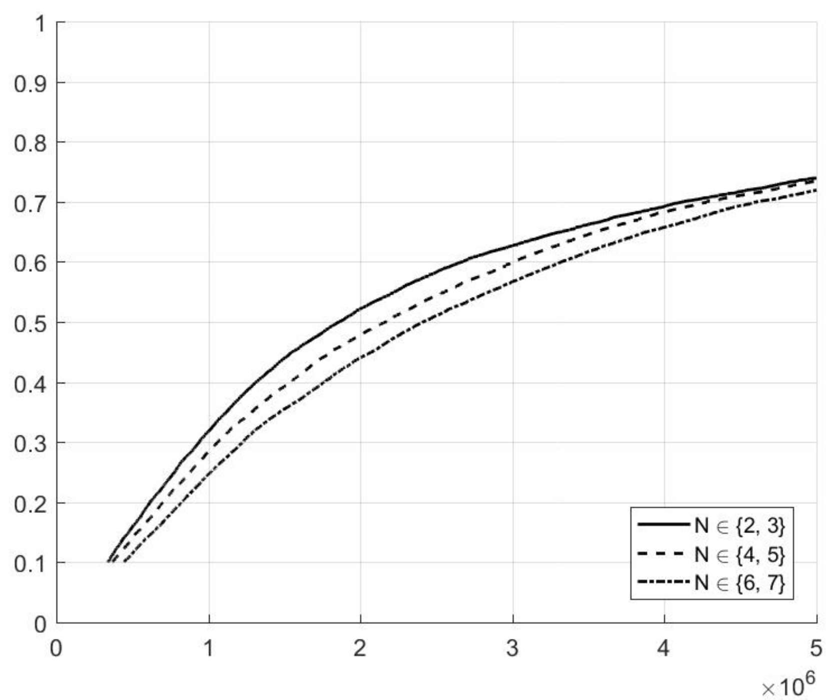


Table 11: Test p-values
Random Reserve Specification

Test for Common Values

	With UH	No UH
{2,3} vs. {4,5}	0.023	0.306
{4,5} vs. {6,7}	0.278	0.759
Max (coarse binning)	0.027	0.605
Max (fine binning)	0.028	0.664

Specification Test*

	With UH	No UH
{2,3} vs. {4,5}	0.847	0.036
{4,5} vs. {6,7}	0.839	0.071
Max (coarse binning)	0.960	0.075
Max (fine binning)	0.910	0.245

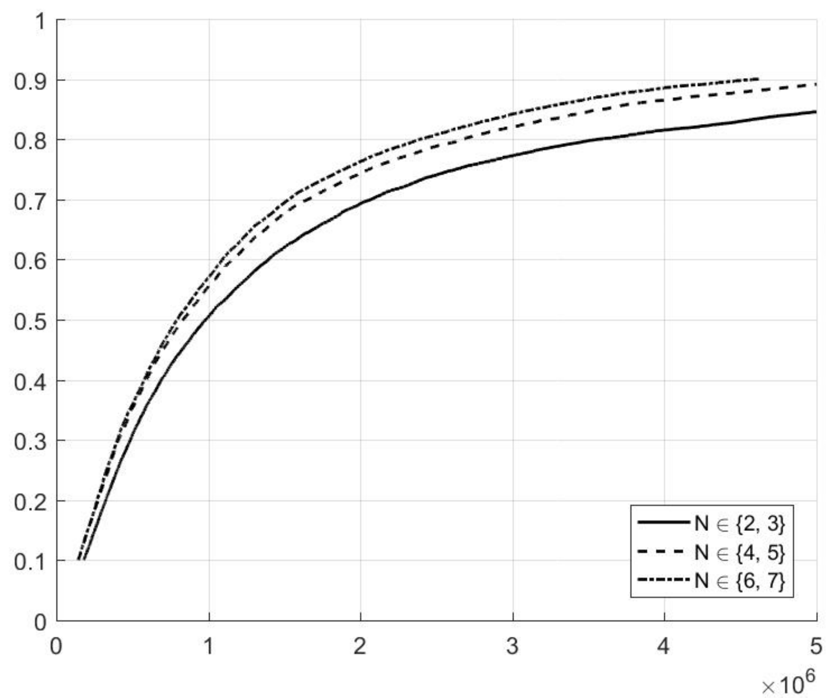
* See footnote 76 and Appendix E.

D.2 Semi-nonparametric Entry Model

In the results presented so far, the entry model was estimated as an ordered probit: in the notation of section 4.1.1, the distribution $H(\cdot)$ was specified as the standard normal. Although H could be chosen arbitrarily if we specified each $\alpha_n(x,z)$ nonparametrically, with our parametric specification of the thresholds $\alpha_n(x,z)$ the choice of H could matter. Thus, we consider here an alternative semi-nonparametric specification in which $H(\cdot)$ is approximated flexibly with Hermite polynomials.

Following Gallant and Nychka (1987), H is specified as the continuous distribution function with density $h(\epsilon) = \frac{1}{\theta_h}\left(\sum_{k=0}^{\tilde{K}} \sigma_k \epsilon^k\right)^2 \phi(\epsilon)$, where $\tilde{K}$ is the order of the Hermite polynomial approximation, σ_k are parameters, ϕ is the standard normal density, and $\theta_h = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\tilde{K}} \sigma_k \epsilon^k\right)^2 \phi(\epsilon) d\epsilon$. We set $\tilde{K} = 3$. Because the entry model affects only the way that we control for unobserved heterogeneity, we examine here only the results obtained from the model allowing unobserved heterogeneity.

Figure 5: SNP Entry Model
With UH



In Figure 5 we plot the estimated distributions of homogenized pivotal expected values obtained when we use this alternative specification of the entry model. These distributions exhibit patterns very similar to those from the baseline model. Table 12 confirms that the formal test results are also very similar. Indeed, the statistical evidence of common values is slightly stronger (smaller p-values) with this more flexible specification.

Table 12: Test p-values
With SNP Entry Model

Test for Common Values

	With UH
{2,3} vs. {4,5}	0.014
{4,5} vs. {6,7}	0.198
Max (coarse binning)	0.015
Max (fine binning)	0.004

Specification Test

	With UH
{2,3} vs. {4,5}	0.923
{4,5} vs. {6,7}	0.948
Max (coarse binning)	0.995
Max (fine binning)	0.933

D.3 No Year Fixed Effects

Our previous specifications included year fixed effects. These provide flexible control for a number of common-knowledge time-varying factors such as macro shocks, variation in oil and gas prices, changes in industry structure, regulatory changes, etc., which vary substantially over the three decades of our sample. We saw the importance of this temporal variation in Table 6. However, in some applications one might not have sufficient sample size to allow such flexibility. Without the fixed effects, time varying factors would be an additional source of unmeasured heterogeneity, and it is interesting to explore what happens in such cases. We therefore consider a specification that drops the year fixed effects.

The estimated distributions of pivotal expected values are shown in Figures 6a and 6b. When we account for unobserved heterogeneity and endogenous entry we obtain estimated distributions of pivotal expected values that are again ordered as we would expect under common values. The specification ignoring unobserved heterogeneity yields estimated distributions that are ordered—now more sharply than in the baseline specification—in the direction indicating misspecification of the model.

When we examine the formal test results in Table 13, the statistical evidence aligns with the impression given by the figures. For the model allowing unobserved heterogeneity, the rejections of private values in favor of common values are at slightly larger significance levels than in the baseline specification, particularly in the case of fine binning. However, the more striking difference is the stronger rejection of the model that assumes no unobserved heterogeneity. This is as we would expect: as more auction-level factors are forced into the unobservable, the misspecification implied by ignoring unobserved heterogeneity is likely to be more severe.

Table 13: Test p-values
No Year Fixed Effects

Test for Common Values

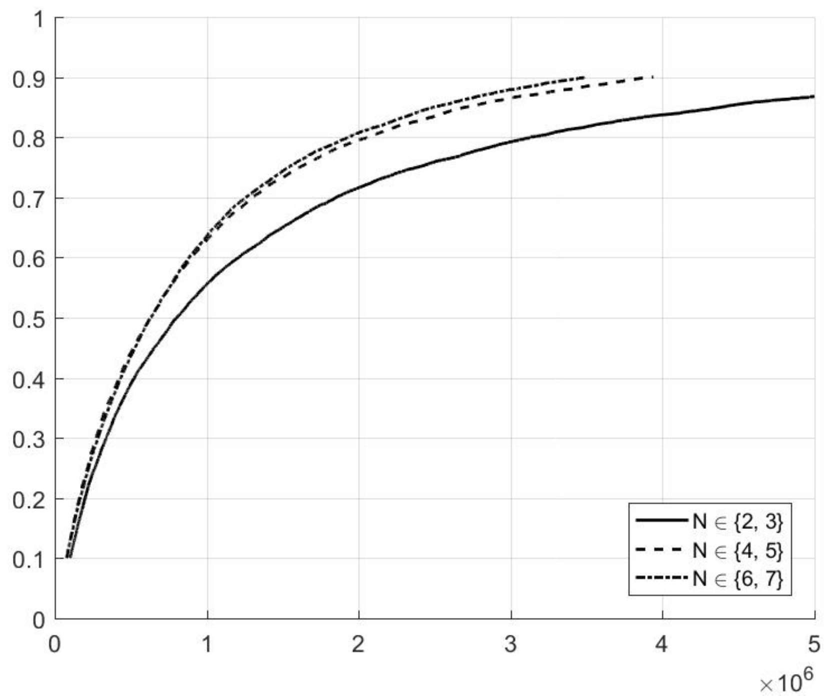
	With UH	No UH
{2,3} vs. {4,5}	0.024	0.406
{4,5} vs. {6,7}	0.319	0.841
Max (coarse binning)	0.028	0.748
Max (fine binning)	0.095	0.985

Specification Test

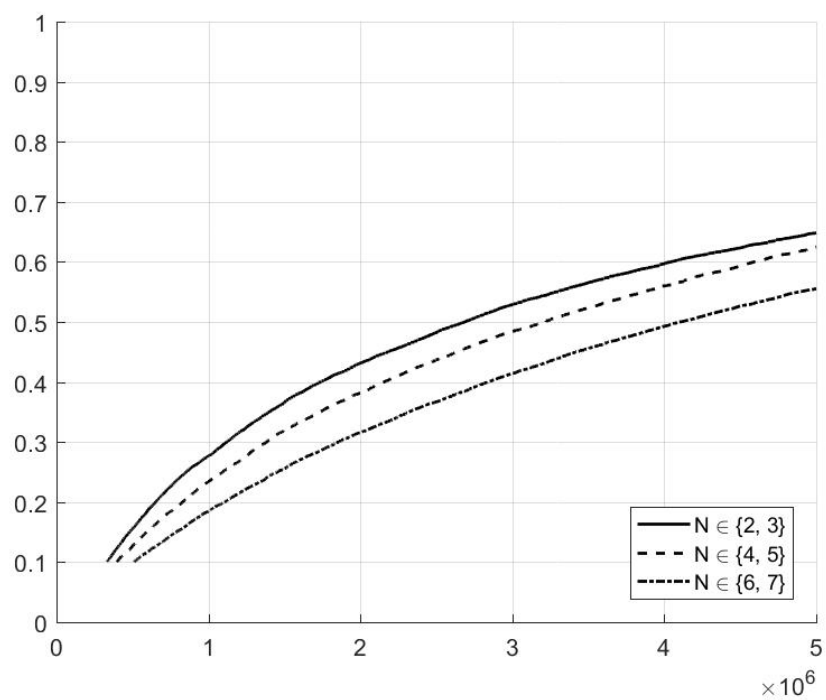
	With UH	No UH
{2,3} vs. {4,5}	0.976	0.069
{4,5} vs. {6,7}	0.744	0.009
Max (coarse binning)	0.937	0.009
Max (fine binning)	0.942	0.625

Figure 6: No Year Fixed Effects

(a) With UH



(b) No UH



D.4 No Drainage Tracts

"Drainage" tracts are those adjacent to productive leases. It is natural to imagine that owners of neighbor leases have an advantage in assessing the value of adjacent tracts. The path-breaking paper of Hendricks and Porter (1988) focused on drainage tracts and assumed that only the neighbor firm had access to a private signal of the tract value, whereas non-neighbors relied solely on common knowledge information. Our model of entry in OCS auctions (see Example 1) can be viewed as relaxing this structure by allowing less extreme forms of asymmetries in signal acquisition costs and permitting competition between multiple neighbors.[77] However, our model is one in which bidders choosing to acquire signals have private information of (*ex ante*) equal precision: bidders are symmetric in the auction stage. Formal tests of this assumption (see Appendix C) give no indication that this assumption is violated. But as a robustness check we can repeat our entire analysis dropping all drainage tracts.

Unsurprisingly, restricting the sample in this way eliminates much of the variation in our instrument. This is an important limitation. Recalling the discussion in section 3, it means that the results in this case will likely be more heavily reliant on the functional form of the index function γ, and are likely to be less precisely estimated. The estimated distributions in this case, shown in Figures 7a and 7b, are in fact very similar to those obtained from the full sample. However, the formal test results show that there is indeed a loss of precision. When we allow for unobserved heterogeneity, the coarse binning test for common values comparing "low" and "medium" competition yields a p-value of 0.162, with a p-value of 0.168 for the max test. With fine binning, the max test yields a p-value of 0.075. The fine binning comparison between $n = 2$

[77] Hendricks and Porter (1988) assume that if there are multiple neighbors, only one submits a serious bid. This leads to the result that non-neighbors bid in equilibrium despite having no information, using mixed strategies. Such behavior is possible in equilibrium when there is no cost of bidder entry.

and $n = 3$, where we expect the largest effect of the winner's curse, yields a p-value of 0.030. Specification tests applied to the model ignoring unobserved heterogeneity yield p-values between 0.13 and 0.19 from the coarse binning tests.

While we can still reject private values in favor of common values at the 10% level (or at the 5% level using the fine binning test for $n = 2$ vs. $n = 3$), the statistical evidence is less definitive. These results indicate that, while the patterns in the estimated distributions obtained from the full sample are robust to exclusion of the drainage tracts, the variation in the instrument these tracts provide is important for the precision of the results.

Table 14: Test p-values
No-drainage Specification

Test for Common Values

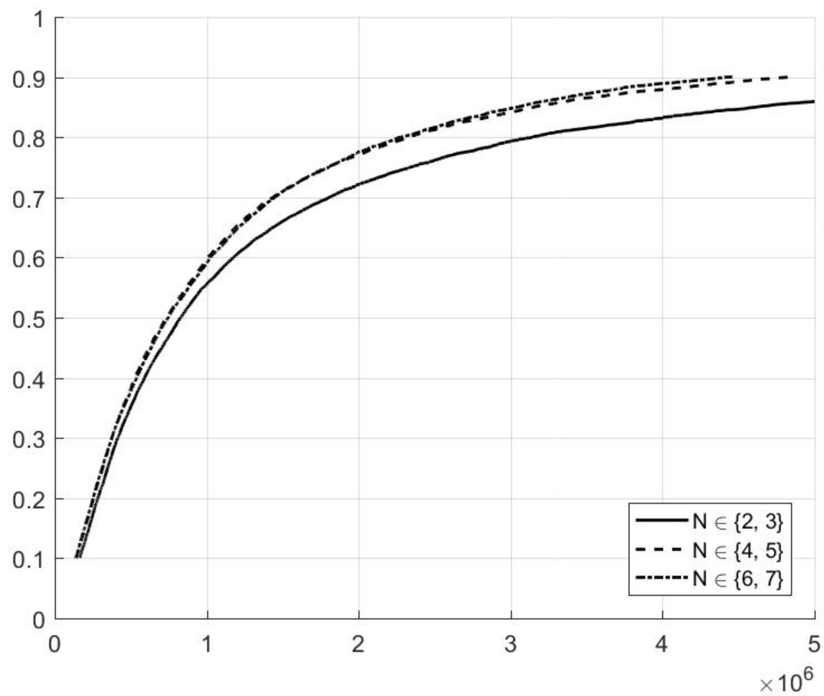
	With UH	No UH
{2,3} vs. {4,5}	0.162	0.344
{4,5} vs. {6,7}	0.485	0.651
Max (coarse binning)	0.168	0.463
Max (fine binning)	0.075	0.508

Specification Test

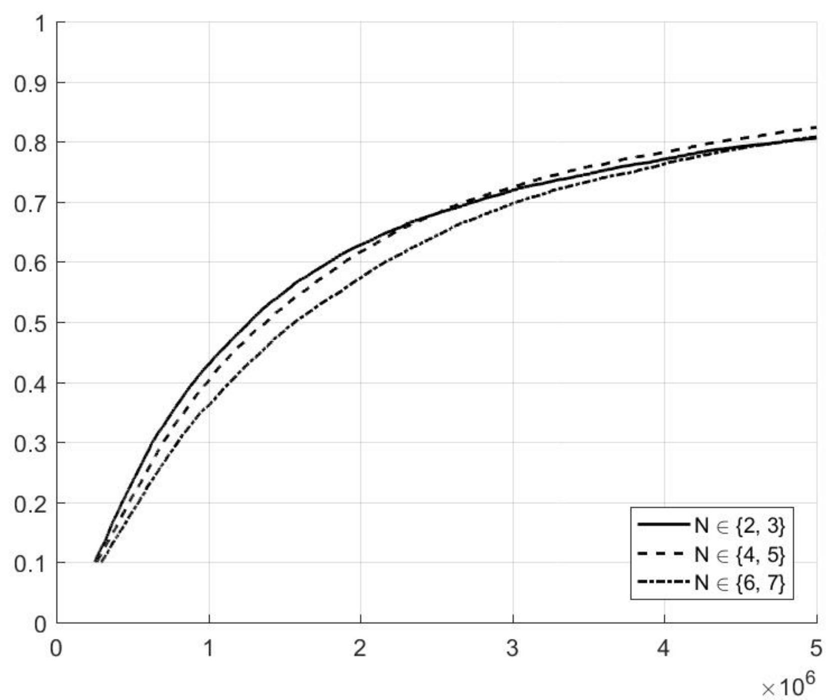
	With UH	No UH
{2,3} vs. {4,5}	0.856	0.188
{4,5} vs. {6,7}	0.742	0.134
Max (coarse binning)	0.863	0.149
Max (fine binning)	0.724	0.246

Figure 7: No Drainage Tracts

(a) With UH



(b) No UH



E Additional Detail: Random Reserve Price

Here we provide additional detail regarding the random reserve price model discussed in Appendix D.1. As discussed there, the MMS occasionally exercised its right to reject all bids, typically when the number or level of bids received was low. Following Hendricks, Porter, and Spady (1989) and Hendricks, Porter, and Wilson (1994), we model this rejection policy by assuming that at each auction t the MMS used a random reserve price R_t whose realization was unknown to bidders.

We let the distribution of R_t vary with the number of bidders and with the auction characteristics (X_t, U_t). The dependence on (X_t, U_t) is assumed to mirror that of tract valuations. Thus, we assume

$$R_t = R_t^0 \times \Gamma(X_t, U_t),$$

where R_t^0 is independent of (X_t, U_t, S_t, V_t) conditional on N_t. In this formulation, a homogenized winning bid $M_t^0 = \max_i\{B_{it}^0\}$ is accepted if and only if $M_t^0 \geq R_t^0$. Let $H_R(\cdot|N_t)$ denote the distribution of R_t^0 given N_t.

Following Bajari and Hortaçsu (2003), for each value of n we model the distribution $H_R(\cdot|n)$ using the marginal distribution of homogenized bids in an n-bidder auction. However, to fit the data, we allow the MMS "bid" to be less aggressive than a real bid by introducing a scaling factor σ_n.[78] In particular, we assume

$$H_R(r|n) = \min\left\{1, G_{B^0}\left(\frac{r}{\sigma_n}; n\right)\right\}.$$

Thus, the random reserve price has the same distribution as a re-scaled bid.

Motivated by the observed rejection frequencies (see Table 15 below), we estimate

[78] Very similar results are obtained if we instead fit a lognormal distribution for R_t^0 or allow the homogenized reserve price to be correlated with the homogenized bids.

separate scaling factors for $n = 1$, $n \in \{2,3\}$ and $n \geq 4$. Taking the estimates of $\theta = (\theta_\tau, \theta_\gamma, \theta_b)$ as given, we fit these scaling parameters σ to the observed bid acceptance decisions

$$Y_{it} = 1\{M_{it} \geq R_t\}$$

using the quasi-likelihood function

$$\mathcal{L}(y;\sigma,\hat{\theta}) =$$
$$\prod_{t=1}^{T} \frac{1}{\tau_{n_t}(x_t, z_t; \hat{\theta}_\tau) - \tau_{n_t-1}(x_t, z_t; \hat{\theta}_\tau)} \int_{\tau_{n_t-1}(x_t,z_t;\hat{\theta}_\tau)}^{\tau_{n_t}(x_t,z_t;\hat{\theta}_\tau)} H_R\left(m_t - \gamma(x_t, u; \hat{\theta}_\gamma)|n_t; \hat{\theta}_b, \sigma\right)^{y_t} \times$$
$$\left(1 - H_R\left(m_t - \gamma(x_t, u; \hat{\theta}_\gamma)|n_t; \hat{\theta}_b, \sigma\right)\right)^{1-y_t} du. \quad \text{(E.1)}$$

Table 15 shows the actual and fitted bid acceptance rates for each value of n.

Table 15: Random Reserve Model Fit

	acceptance rate	
n	actual	fitted
1	0.921	0.917
2	0.978	0.981
3	0.997	0.991
4	0.991	0.997
5	1.000	0.995
6	1.000	0.998
7	1.000	0.998
8	1.000	0.998
9	1.000	0.997
10	1.000	0.999
11–18	1.000	0.999

Introducing the random reserve price requires only a minor change in the inversion of bidder first-order conditions. Under the null hypothesis of private values, bidders' homogenized pivotal expected values are identified using a revised first-order condition

$$w^0(s_{it};n_t) = b_{it}^0 + \frac{G_{M^0|B^0}(b_{it}^0;b_{it}^0,n_t)H_R(b_{it}^0;n_t)}{G_{M^0|B^0}(b_{it}^0;b_{it}^0,n_t)h_R(b_{it}^0;n_t) + g_{M^0|B^0}(b_{it}^0;b_{it}^0,n_t)H_R(b_{it}^0;n_t)}. \quad \text{(E.2)}$$

This allows testing of the private values null as in the baseline model.

An nuance here is that under the alternative of common values, the right-hand side of (E.2) is not equal to $w^0(s_{it}, s_{it}; n_t)$ but to a weighted average[79]

$$\varphi(s_{it}, n_t)\underline{w}^0(s_{it}; n_t) + [1 - \varphi(s_{it}, n_t)]\, w^0(s_{it}; n_t), \quad \text{(E.3)}$$

where

$$\underline{w}^0(s_{it}; n_t) = E\left[V_{it}^0 \middle| S_{it} = s_{it}, \max_{j\neq i} S_{jt} \leq s_{it}, N_t = n_t\right] \quad \text{(E.4)}$$

denotes a bidder's expected (homogenized) valuation conditional on his signal and on winning the auction. This expectation is slightly different from the homogenized pivotal expected value. And although both $\underline{w}^0(s_{it}; n_t)$ and $w^0(s_{it}; n_t)$ are decreasing in n_t in a common values model, the weights $\varphi(s_{it}, n_t)$ also vary with n_t, leaving an ambiguous prediction regarding how the distribution of the weighted average varies with n_t. This implies that violations of the private values null might fail to reveal themselves, and further that there may be no relationship between the estimated distributions obtained from the first-order condition that can be explained only by violation of the model's maintained hypotheses.[80]

Given the stochastic ordering we do find, this caveat implies that the evidence we

[79] The weight $\varphi(s_{it}, n_t)$ is equal to

$$h(b_{it}^0, n_t)G_{M|B}(b_{it}^0; b_{it}^0, n_t)/\left[h(b_{it}^0, n_t)G_{M|B}(b_{it}^0; b_{it}^0, n_t) + H(b_{it}^0, n_t)g_{M|B}(b_{it}^0; b_{it}^0, n_t)\right].$$

Observe that all weight is placed on the pivotal expected value when the rejection probability $1 - H(b_{it}^0, n_t)$ is zero.

[80] A referee has noted that this issue will not arise if the weights $\varphi(s, n)$ are weakly increasing in n, which is itself a verifiable condition.

obtain in favor of common values from this specification is conditioned on a maintained assumption that our model is correctly specified. Such conditioning is typical in hypothesis testing but unlike the baseline specification, where we could partition the set of all possible outcomes to those consistent with private values, those consistent with common values, and those inconsistent with the model.

F Identification of a Structural Entry Model

This appendix provides identification results, referenced in section 8.1, for the fully specified model of entry in Example 1.[81]

F.1 Affiliated Private Values

We begin with the case in which bidders have affiliated private values in the auction stage. Our identification results in the text immediately imply identification of the *ex ante* gross profit of entering auction t, defined as $\bar{\pi}(N_t, X_t, U_t)$ in Appendix A. Recall that this function is strictly increasing in U_t. The conditional distribution of $U_t|X_t, Z_t$ is also known (it is uniform $[0,1]$). Thus, we need only show identification of the entry costs $c(X_t, Z_t)$ and $c(X_t, Z_t) + \delta(X_t, Z_t)$ for neighbors and non-neighbors, respectively.

Consider arbitrary values, x and z, of X_t and Z_t. For $n \leq z$, our equilibrium entry conditions imply (see also Berry (1992))

$$\begin{aligned}
\Pr(N_t < n|X_t = x, Z_t = z) &= \Pr\left(\bar{\pi}(n, x, U_t) < c(x,z)|X_t = x, Z_t = z\right) \\
&= \Pr\left(U_t < \bar{\pi}^{-1}(c(x,z); n, x)\,|X_t = x, Z_t = z\right) \\
&= \bar{\pi}^{-1}(c(x,z); n, x)\,.
\end{aligned}$$

[81] Here we work with the more general version of the model in which firms' entry costs are permitted to vary with Z_t.

Applying the function $\bar{\pi}(n, x, \cdot)$ to both sides, we obtain

$$\bar{\pi}(n, x, \Pr(N_t < n | X_t = x, Z_t = z)) = c(x, z).$$

Because the function $\bar{\pi}$ is known and $\Pr(N_t < n | X_t = x, Z_t = z)$ is observed, identification of $c(x, z)$ follows.

For $n > z$, in equilibrium we obtain $N_t = n$ only when at least one non-neighbor enters. Thus, the entry equilibrium conditions imply

$$\begin{aligned} \Pr(N_t < n | X_t = x, Z_t = z) &= \Pr(\bar{\pi}(n, X_t, U_t) < c(x, z) + \delta(x, z) | X_t = x, Z_t = z) \\ &= \bar{\pi}^{-1}(c(x, z) + \delta(x, z); n, x). \end{aligned}$$

We now obtain identification of $\delta(x, z)$ from

$$\bar{\pi}(n, x, \Pr(N_t < n | X_t = x, Z_t = z)) = c(x, z) + \delta(x, z).$$

F.2 Common Values

Now consider the case of common values, where we will obtain partial identification of the entry cost functions. The *ex ante* gross expected profit from entering the auction is given by

$$\bar{\pi}(n, x, u) = \int_{\underline{s}}^{\overline{s}} \int_{\underline{s}}^{s} [w(s, y; x, u, n) - \beta(s; x, u, n)] \, dF_{sy}(s, y)$$

where $Y_i = y$ denotes the realized maximum signal among a bidder i's opponents and $F_{sy}(s, y)$ denotes the joint distribution of (S_i, Y_i).[82] Our identification results in the

[82] Strict monotonicity of equilibrium bidding strategies implies that the joint distribution of signals (appropriately normalized, e.g., to have uniform marginals) is identified from the joint distribution of homogenized bids.

text for the auction model allow us to bound this function using the fact that for $y < s$,

$$w(y, y; x, u, n) \leq w(s, y; x, u, n) \leq w(s, s; x, u, n),$$

where our results imply that the outer values are identified. Let $\bar{\pi}_H(n, x, u)$ and $\bar{\pi}_L(n, x, u)$ denote the resulting bounds. Note that these functions will also be strictly increasing in U_t.

To demonstrate the partial identification of entry costs, again fix X_t and Z_t at arbitrary values x and z. For $n \leq z$, we now obtain

$$\begin{aligned}
\Pr(N_t < n | X_t = x, Z_t = z) &\leq \Pr\left(\bar{\pi}_L(n, x, U_t) < c(x, z) | X_t = x, Z_t = z\right) \\
&= \Pr\left(U_t < \bar{\pi}_L^{-1}\left(c(x, z); n, x\right) | X_t = x, Z_t = z\right) \\
&= \bar{\pi}_L^{-1}\left(c(x, z); n, x\right).
\end{aligned}$$

Applying the strictly increasing function $\bar{\pi}_L(n, x, \cdot)$ to both sides, we obtain

$$\bar{\pi}_L\left(n, x, \Pr(N_t < n | X_t = x, Z_t = z)\right) \leq c(x, z).$$

A similar argument applies using the upper bound $\bar{\pi}_H(n, x, u)$ to construct an upper bound on $c(x, z)$, and to construct upper and lower bounds on $\delta(x, z)$.

G Asymmetric Bidders

Here we sketch a variation of Example 1 permitting asymmetry between neighbors and non-neighbors in the auction. As discussed in section 8.2, we show here that equilibrium in this example leads to a reduced form entry equation for neighbor firms taking the same form as (3). Adaptation of the identification results in section 3 would then be straightforward: identification of the entry model and index function would follow the arguments in section 3.1 (focusing on neighbor entry alone), and identification of the auction model would follow standard arguments, using equilibrium first-order conditions.

To be clear, allowing asymmetry in the auction introduces challenges that are not special to our approach. Most significant is a lack of results ensuring a unique equilibrium in the auction itself. Due to this challenge, the empirical literature on auctions with asymmetry has typically either assumed independent private values (e.g., Athey, Levin, and Seira (2011), Krasnokutskaya (2011), Krasnokutskaya and Seim (2011)), or relied on an assumption of unique equilibrium selection (e.g., Campo, Perrigne, and Vuong (2003), Somaini (2015)). If one is willing to address equilibrium multiplicity in one of these ways, our approach can still offer a solution to the challenges of unobserved heterogeneity and endogenous bidder entry. Given our purpose, here we avoid discussion of equilibrium selection in the auction stage by limiting attention to the asymmetric IPV case.

We retain most elements of Example 1. However, we drop the assumption of exchangeable bidders, instead assuming exchangeability only within each of the two groups. We assume firm information at the entry stage is asymmetric as well. Neighbor firms observe (Z_t, X_t, U_t) before their entry decisions. Non-neighbor firms observe (Z_t, X_t) but learn U_t (along with their private signals) only upon entry. Neighbors have commonly known entry cost c. Non-neighbors draw private entry costs C_{it} independently from a smooth distribution F_C.[83] This private cost may reflect, for example, the extent to which firm i has already collected the seismic data it would need to analyze in order to learn the signal S_{it}.

To ease notation, we will denote neighbor firms as "group 1" and non-neighbor firms as "group 2." For clarity, we let K_{1t} and K_{2t} denote the number of firms in each group; however, we again assume $K_{1t} = Z_t$ and $K_{2t} = K - K_{1t}$, where K is

[83]For simplicity we suppress any dependence of c or the distribution F_C on (X_t, Z_t).

the total number of firms in the industry.[84] Common knowledge among all firms are $(K_{1t}, K_{2t}, c, F_C, Z_t, X_t)$.

We consider a two-stage game of simultaneous entry, followed by a first-price auction in which the unobserved heterogeneity u_t as well as entry outcomes (n_{1t}, n_{2t}) are common knowledge. Following, e.g., Campo, Perrigne, and Vuong (2003), Athey, Levin, and Seira (2011), Krasnokutskaya (2011), and Krasnokutskaya and Seim (2011), we consider group-symmetric pure strategies for the auction continuation game. Such strategies take the form $\beta_k(\cdot; X_t, U_t, N_{1t}, N_{2t})$ for each group $k = 1, 2$, where N_k denotes the number of bidders from group k. As usual, equilibrium bidding strategies inherit the multiplicative separability of valuations, so that

$$\beta_k(\cdot; X_t, U_t, N_{1t}, N_{2t}) = \beta_k^0(\cdot; N_{1t}, N_{2t})\,\Gamma(X_t, U_t). \tag{G.1}$$

The *ex ante* expected profits to a bidder in the auction, given $(X_t, U_t, N_{1t}, N_{2t})$, then inherit the multiplicative structure as well and can be written $\pi_k^0(N_{1t}, N_{2t})\,\Gamma(X_t, U_t)$. Here, $\pi_k^0(N_{1t}, N_{2t})$ is strictly decreasing in both arguments. Recall from Appendix A that Γ is strictly increasing and continuous in U_t.

We consider group-symmetric pure strategies in the entry stage as well. Such strategies take the form of degenerate entry probabilities

$$e_1(X_t, Z_t, U_t) \in \{0, 1\}$$
$$e_2(X_t, Z_t, C_{it}) \in \{0, 1\}$$

for groups 1 and 2, respectively. We further assume a unique selection of e_2 conditional on the auction-specific factors (X_t, Z_t) that are common knowledge at the

[84] More generally it is sufficient that (X_t, Z_t) (with Z_t potentially comprising multiple instruments) determine K_{1t} and K_{2t}.

entry stage.[85] Denote by

$$\varrho\left(X_t, Z_t\right) = \Pr\left(e_2\left(X_t, Z_t, C_{it}\right) = 1 | X_t, Z_t\right)$$

the implied *ex ante* (before knowledge of C_{it}) probability of entry by a given non-neighbor i.

We henceforth condition on a given value of (X_t, Z_t) and suppress these arguments in the notation. Given any strategy e_2, a firm of type 1 then has expected gross profit from entering conditional on n_1 (i.e., the total number of type-1 entrants, including this firm) given by

$$E_{N_{2t}}\left[\pi_1^0\left(n_1, N_{2t}\right)\Gamma\left(u\right)|\varrho\right] = \sum_{n_2=0}^{K_{2t}} \binom{K_{2t}}{n_2} \varrho^{n_2}\left(1-\varrho\right)^{K_{2t}-n_2} \pi_1^0\left(n_1, n_2\right)\Gamma\left(u\right). \tag{G.2}$$

Because this expected profit is strictly decreasing in n_1, by the arguments in Berry (1992) there will be a unique number of neighbor entrants consistent with mutual best-responses to the entry behavior characterized by ϱ; i.e., the best response of neighbors is summarized by the reduced form

$$N_{1t} = \eta\left(U_t; \varrho\right). \tag{G.3}$$

Because (G.2) is strictly increasing in u, η is weakly increasing in U_t.

Similar to the entry model in the text, the entry of neighbors can then be charac-

[85] This assumption is weaker than the common requirement of unique equilibrium selection conditional on common knowledge observables, as we allow multiplicity in the strategies used by group 1. As usual, this equilibrium multiplicity will turn out to be limited to the form of variation in which firms enter, not the number of such entrants. Other work on asymmetric auctions with entry (e.g., Athey, Levin, and Seira (2011), Krasnokutskaya and Seim (2011)) rely on numerical confirmation of a unique entry equilibrium.

terized by thresholds $\tau_\ell(\varrho) \in [0,1]$ where

$$\tau_\ell(\varrho) = 0 \text{ for } \ell < \underline{n}(\varrho),$$
$$\tau_\ell(\varrho) = 1 \text{ for } \ell \geq \overline{n}(\varrho),$$

and, for $\ell = \underline{n}(\varrho), \ldots, \overline{n}(\varrho) - 1$, $\tau_\ell(\varrho)$ is defined by

$$E_{N_{2t}}\left[\pi_1^0(\ell+1, N_{2t})\,\Gamma(\tau_\ell(\varrho))\,|\varrho\right] = c.$$

Thus, given ϱ, up to zero-probability events we have $N_{1t} = n$ if and only if $U_t \in [\tau_{n-1}(\varrho), \tau_n(\varrho)]$.

Lemma 6. *For all $\ell \in \{0,1,\ldots\}$, $\tau_\ell(\varrho)$ is continuous in ϱ.*

Proof. Define an extension of the function Γ to the domain $\mathbb{R}$ as follows:

$$\bar{\Gamma}(u) = \begin{cases} \Gamma(u) & u \in [0,1] \\ \Gamma(1)\,u & u > 1 \\ \Gamma(0) + u & u < 0. \end{cases}$$

Observe that $\bar{\Gamma}$ has range $\mathbb{R}$, is continuous, and is strictly increasing. Thus, given any ϱ and $\ell \in \{0,1,\ldots\}$, there exists a unique $\bar{\tau}_\ell(\varrho) \in \mathbb{R}$ defined by

$$E_{N_{2t}}\left[\pi_1^0(\ell+1, N_{2t})\,\bar{\Gamma}(\bar{\tau}_\ell(\varrho))\,|\varrho\right] = c.$$

For all τ the expectation $E_{N_{2t}}\left[\pi_1^0(\ell+1, N_{2t})\,\bar{\Gamma}(\tau)\,|\varrho\right]$ is continuous with respect to ϱ, so $\bar{\tau}_\ell(\varrho)$ is continuous with respect to ϱ. Now observe that for any $\ell \in \{0,1,\ldots\}$, $\tau_\ell(\varrho) = \max\{0, \min\{1, \bar{\tau}_\ell(\varrho)\}\}$. Because the min and max operators are continuous, the result follows. $\square$

Now consider the entry decision of a non-neighbor firm i, taking the aggregate entry of neighbors as given by $\eta(\cdot;\varrho)$. Firm i's expected payoff from entering the auction is $\bar{\pi}_2(\varrho) - C_{it}$, where

$$\bar{\pi}_2(\varrho) = \sum_{\hat{n}=0}^{K_2-1} \binom{K_2-1}{\hat{n}} \varrho^{\hat{n}} (1-\varrho)^{K_2-1-\hat{n}} \int_0^1 \pi_2^0 \left(\eta(u;\varrho), \hat{n}+1\right) \Gamma(u)\, du. \tag{G.4}$$

Firm i's best response is to enter when $\bar{\pi}_2(\varrho) - C_{it}$ is nonnegative. This best response behavior implies

$$\varrho = F_C\left(\bar{\pi}_2(\varrho)\right). \tag{G.5}$$

To show that these strategies form a Bayes Nash equilibrium, we need to show that (G.5) has a fixed point in $\varrho \in [0,1]$. This will follow from Brouwer's fixed point theorem if $\bar{\pi}_2(\varrho)$ is continuous. From (G.4) it is clear that it is sufficient that

$$\int_0^1 \pi_2^0 \left(\eta(u;\varrho), \hat{n}+1\right) \Gamma(u)\, du$$

be continuous in ϱ. Observe that

$$\int_0^1 \pi_2^0 \left(\eta(u;\varrho), \hat{n}+1\right) \Gamma(u)\, du = \sum_{n_1=\underline{n}(\varrho)}^{\overline{n}(\varrho)} \int_{\tau_{n_1-1}(\varrho)}^{\tau_{n_1}(\varrho)} \pi_2^0 \left(n_1, \hat{n}+1\right) \Gamma(u)\, du. \tag{G.6}$$

To see that the right-hand-side is continuous in ϱ, observe that a change in ϱ can alter this expression in two ways. One is through the limits of integration associated with a given value of n_1. In this case the effects of ϱ are continuous, since each threshold $\tau_\ell(\varrho)$ is continuous in ϱ. The second is through a change in $\underline{n}(\varrho)$ or $\overline{n}(\varrho)$. The effects in this case are also continuous. For example, consider the case in which an increase in ϱ from ϱ^0 to any $\varrho' \in (\varrho^0, \varrho^0+\epsilon)$ results in $\underline{n}(\varrho') < \underline{n}(\varrho^0)$.[86] Continuity of

$$E_{N_{2t}}\left[\pi_1^0\left(n, N_{2t}\right)\Gamma(0)\,|\varrho\right]$$

[86] An increase in ϱ can only reduce $\underline{n}(\varrho)$.

with respect to ϱ (recall (G.2)) implies that for sufficiently small ϵ, $\underline{n}(\varrho') = \underline{n}(\varrho^0) - 1$. Any such change from ϱ^0 to ϱ' will therefore add the term

$$\int_0^{\tau_{\underline{n}(\varrho')}(\varrho')} \pi_2^0 (n_1, \hat{n}+1) \Gamma(u) \; du \tag{G.7}$$

to the beginning of the sum (G.6). Letting $\ell = \underline{n}(\varrho') = \underline{n}(\varrho^0) - 1$, note that $\tau_\ell(\varrho^0) = 0$. Thus, by Lemma 6, by setting $|\varrho' - \varrho^0|$ sufficiently small, $\tau_\ell(\varrho')$ will be made arbitrarily close to zero, implying an arbitrarily small value of (G.7). A similar argument applies to all other cases in which a change in ϱ causes a change in the limits of the summation in (G.6).

Thus, equilibrium exists. Recalling that we conditioned on (X_t, Z_t), (G.3) provides a reduced form

$$N_{1t} = \eta(X_t, Z_t, U_t)$$

for neighbor entry, with weak monotonicity in U_t.

H Additional Computational Detail

Here we discuss additional computational aspects of our estimation procedure. A useful feature of the Bernstein polynomial specification is the ease of imposing otherwise complex functional restrictions through linear restrictions on the vector of Bernstein coefficients. For instance, a necessary and sufficient condition for the function to integrate to one is that $\sum_{j=0}^{m} \theta_{b,n}^{(j)} = m+1$. Additionally, the Bernstein polynomials allow easy transformation between the density and associated cumulative distribution. In particular,

$$\tilde{G}_{B_i^0}(\tilde{b}^0; \tilde{\theta}_b, n) = \sum_{j=0}^{m+1} \tilde{\theta}_{b,n}^{(j)} q_{j,m+1}\left(\Phi\left(\tilde{b}^0\right)\right), \tag{H.1}$$

where $\tilde{\theta}_{b,n} = (\tilde{\theta}_{b,n}^{(0)}, \ldots, \tilde{\theta}_{b,n}^{(m+1)})' = M\theta_{b,n}$ for a known matrix M, and $\tilde{\theta}_b = \{\tilde{\theta}_{b,n}\}_{n=\underline{n}}^{\overline{n}}$. This is useful because we require the CDF of the bid marginal when applying the copula $\chi(\cdot;\rho_n)$ to compute the joint density in (13). In contrast to numerical integration, the transformation from density to CDF with Bernstein polynomials involves only a linear transformation of parameters and is exact.

One key computational issue is how one can derive tractable expressions for the conditional distributions that appear in the bid first-order condition (15). Below we show how the Gaussian copula can be leveraged to write $\tilde{G}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0, n)$ and $\tilde{g}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0, n)$ using integrals of known normal distributions, which are easily computed using readily available methods. We first derive an expression for $\tilde{G}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0, n)$. An expression for $\tilde{g}_{M|B}(\tilde{b}_{it}^0|\tilde{b}_{it}^0, n)$ follows naturally.

First, note that

$$\begin{aligned}\tilde{G}_{M|B}(b|b,n) &= \Pr\left(\tilde{B}_{2t}^0 \le b, \ldots, \tilde{B}_{nt}^0 \le b | \tilde{B}_{1t}^0 = b\right) \\ &= \Pr\left(\Phi^{-1}\left(\tilde{G}_{B_i^0}\left(\tilde{B}_{2t}^0\right)\right) \le b^\star, \ldots, \Phi^{-1}\left(\tilde{G}_{B_i^0}\left(\tilde{B}_{nt}^0\right)\right) \le b^\star \mid \Phi^{-1}\left(\tilde{G}_{B_i^0}\left(\tilde{B}_{1t}^0\right)\right) = b^\star\right),\end{aligned} \tag{H.2}$$

where $b^\star = \Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)$ and we drop the dependence of $\tilde{G}_{B_i^0}$ on $\tilde{\theta}_b$ and n for notational convenience. The second equality holds for every $b \in \mathbb{R}$, since the estimated $\tilde{G}_{B_i^0}$ has always full support.

Second, note that the Gaussian copula parametrization of the joint distribution of homogenized bids implies

$$\left(\Phi^{-1}\left(\tilde{G}_{B_i^0}\left(\tilde{B}_{1t}^0\right)\right), \ldots, \Phi^{-1}\left(\tilde{G}_{B_i^0}\left(\tilde{B}_{nt}^0\right)\right)\right) \sim N(0, \Sigma_{\rho_n}),$$

where Σ_{ρ_n} is the covariance matrix with constant pairwise correlation ρ_n.

Finally, the conditional normal distribution is also normal, so we can re-write the conditional probability (H.2) as

$$\tilde{G}_{M|B}(b|b,n) = \tilde{\Phi}\left(\Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right), \dots, \Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)\right),$$

where $\tilde{\Phi}(\cdot)$ is the multivariate Normal CDF with mean $1_{(n-1)\times 1}\, \rho_n\, \Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)$ and variance $\Sigma_{\rho_n} - {\rho_n}^2$. Therefore computation of $\tilde{G}_{M|B}(b|b,n)$ requires only the evaluation of a known normal CDF.

This procedure naturally suggests one way of computing $\tilde{g}_{M|B}(b|b,n)$ as a multivariate integral. By definition,

$$\tilde{g}_{M|B}(b|b,n) = \frac{d\tilde{\Phi}\left(\Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right), \dots, \Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)\right)}{db},$$

By symmetry and using the Leibniz integral rule,

$$\tilde{g}_{M|B}(b|b,n) =$$

$$(n-1)\frac{\tilde{g}_{B_i^0}(b)}{\phi(\tilde{G}_{B_i^0}(b))}\int_{-\infty}^{\Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)}\int_{-\infty}^{\Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right)} \tilde{\phi}\left(\Phi^{-1}\left(\tilde{G}_{B_i^0}(b)\right), b_3, \dots, b_n\right) db_3 \dots b_n.$$

where we drop the dependence of $\tilde{g}_{B_i^0}$ on θ_b and n. The integral above is over a known multivariate normal density. It can be computed quickly and reliably using the algorithm suggested by Genz (1992).

I Additional Tables

Table 16: Test p-values
Baseline Specification, Fine Binning

Test for Common Values

	With UH	No UH
2 vs. 3	0.001	0.046
3 vs. 4	0.212	0.703
4 vs. 5	0.662	0.869
5 vs. 6	0.625	0.733
6 vs. 7	0.073	0.270
Max	0.019	0.541

Specification Test

	With UH	No UH
2 vs. 3	1.000	1.000
3 vs. 4	0.730	0.229
4 vs. 5	0.567	0.086
5 vs. 6	0.627	0.249
6 vs. 7	0.562	0.440
Max	0.922	0.239

Table 17: Entry Model Estimates
Alternative Specifications

		No Year FE			No Drainage			SNP		
		Est.	SE	SE (BB)	Est.	SE	SE (BB)	Est.	SE	SE (BB)
# active leases	X	-0.102	0.026	0.031	-0.036	0.037	0.044	-0.094		0.042
isolated lease		0.402	0.086	0.096	0.434	0.129	0.141	0.557		0.141
# firms that bid for neighbors		0.014	0.011	0.011	0.052	0.020	0.021	0.022		0.015
reoffered tract		-0.293	0.062	0.069	-0.108	0.093	0.106	-0.242		0.093
neighbor expired		-0.053	0.068	0.074	0.017	0.092	0.108	-0.022		0.104
# neighbors drilled		-0.011	0.027	0.029	-0.090	0.041	0.042	-0.053		0.040
# neighbor hits		0.055	0.028	0.027	-0.353	0.253	0.254	0.077		0.038
depth		-0.111	0.162	0.360	-0.704	0.198	0.309	-0.612		0.336
depth squared		-0.060	0.065	0.234	0.174	0.073	0.133	0.121		0.153
royalty rate		0.024	0.015	0.020	-0.012	0.023	0.031	-0.004		0.026
time controls		Sale year dummies			Sale year dummies			Sale year dummies		
# neighbor firms	Z	0.151	0.040	0.041	0.046	0.070	0.073	0.182		0.059

Table 18: Index Function Estimates
Alternative Specifications

		SNP			No time FE			No Drainage		
		Est.	SE	SE (BB)	Est.	SE	SE (BB)	Est.	SE	SE (BB)
# active leases	X	0.013	0.018	0.022	-0.006	0.027	0.027	0.006	0.023	0.030
isolated lease		0.060	0.060	0.091	0.027	0.079	0.111	0.048	0.085	0.123
# firms that bid for neighbors		0.032	0.008	0.011	0.052	0.010	0.013	0.036	0.013	0.019
reoffered tract		-0.182	0.048	0.070	-0.245	0.078	0.091	-0.102	0.064	0.093
neighbor expired		-0.277	0.047	0.068	-0.119	0.062	0.083	-0.203	0.060	0.087
# neighbors drilled		0.047	0.020	0.025	0.122	0.027	0.028	0.058	0.029	0.042
# neighbor hits		-0.021	0.020	0.026	-0.047	0.028	0.031	-0.096	0.177	0.219
depth		-0.248	0.137	0.274	1.203	0.146	0.365	-0.442	0.160	0.297
depth squared		0.094	0.065	0.120	-0.307	0.064	0.186	0.161	0.071	0.124
royalty rate		-0.007	0.010	0.017	0.029	0.015	0.015	0.015	0.013	0.023
time controls		Sale year dummies			Sale year dummies			Sale year dummies		
unobserved heterogeneity	U	1.319	0.291	0.405	1.989	0.584	0.646	0.941	0.358	0.513

Table 19: Copula Correlation Estimates
Alternative Specifications

	SNP			No time FE			No Drainage		
n	Est.	SE	SE (BB)	Est.	SE	SE (BB)	Est.	SE	SE (BB)
2	0.043	0.041	0.050	0.197	0.049	0.051	0.024	0.056	0.060
3	0.031	0.030	0.040	0.212	0.040	0.042	0.007	0.033	0.047
4	0.090	0.031	0.032	0.275	0.038	0.038	0.053	0.034	0.036
5	0.136	0.030	0.041	0.301	0.031	0.042	0.111	0.035	0.042
6	0.112	0.035	0.037	0.259	0.032	0.042	0.083	0.034	0.036
7	0.111	0.044	0.043	0.342	0.061	0.041	0.090	0.043	0.040
8	0.133	0.039	0.040	0.261	0.039	0.052	0.160	0.052	0.046
9	0.295	0.071	0.067	0.400	0.076	0.073	0.290	0.071	0.082
10	0.158	0.051	0.068	0.307	0.055	0.055	0.155	0.053	0.063
11- 18	0.108	0.025	0.020	0.246	0.029	0.031	0.098	0.026	0.026

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