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# Unobserved Heterogeneity in Auctions

By

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# Unobserved Heterogeneity in Auctions\*

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## Abstract

A common concern in the empirical study of auctions is the likely presence of auction-specific factors that are common knowledge among bidders but unobserved to the econometrician. Such unobserved heterogeneity confounds attempts to uncover the underlying structure of demand and information, typically a primary feature of interest in an auction market. Unobserved heterogeneity presents a particular challenge in first-price auctions, where identification arguments rely on the econometrician's ability to reconstruct from observables the conditional probabilities that entered each bidder's equilibrium optimization problem. When bidders condition on unobservables, it is not obvious that this is possible. Here we discuss several approaches to identification developed in recent work on first-price auctions with unobserved heterogeneity. Despite the special challenges of this setting, all of the approaches build on insights developed in other areas of econometrics, including those on control functions, measurement error, and mixture models. Because each strategy relies on different combinations of model restrictions, technical assumptions, and data requirements, their relative attractiveness will vary with the application. However, this varied menu of results suggests both a type of robustness of identifiability and the potential for expanding the frontier with additional work.

**Keywords:** nonparametric identification, control function, measurement error, finite mixture, quasi-control function

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# 1 Introduction

The econometrics of auctions has been an active area of research over the last thirty years. Auctions often provide applications in which an economic model can be tightly matched to actual market institutions, and where the equilibrium relationships obtained from a rich theoretical literature can often be “inverted” to allow identification and estimation of model primitives. This makes auctions attractive for applied work aimed at combining theory and data to produce quantitative answers to questions about procurement practices, market design, and the roles of strategic behavior and asymmetric information in determining market outcomes.<sup>1</sup>

In this review we examine recent developments taking on an important challenge in this literature: unobserved heterogeneity. In many applications one suspects that there is auction-specific information commonly known among bidders but unavailable to researchers. The presence such unobserved heterogeneity can be important. The underlying information structure is at the heart of many questions concerning bidder behavior, auction design, the division of surplus, etc., so the distinction between private information and information that is merely omitted from the analysis is essential. Accounting for unobserved heterogeneity is particularly important and challenging in first-price sealed bid auctions (henceforth “first-price auctions”) because standard identification results rely on the econometrician’s ability to reconstruct from observables the conditional probabilities entering bidders’ first-order conditions.<sup>2</sup> This strategy is threatened if bidders condition on common knowledge information unavailable to the researcher.

Further complicating matters is the problem of endogenous bidder entry. The effect of competition on bids and revenues can be theoretically ambiguous and is, therefore, itself an important empirical question. More broadly, a variety identification and testing approaches

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<sup>1</sup>Surveys of the literature can be found in, e.g., Hendricks and Paarsch (1995), Athey and Haile (2006, 2007), Hendricks and Porter (2007), and Hickman, Hubbard, and Saglam (2012). See also the monograph of Paarsch and Hong (2006).

<sup>2</sup>See Guerre, Perrigne, and Vuong (2000) and the extensive literature that follows.

in the auction literature exploit exogenous variation the level of competition.<sup>3</sup> However, once the potential for unobserved auction-level heterogeneity is acknowledged, one naturally suspects that such unobservables will affect bidders' participation decisions as well. Thus, for example, high levels of bidder participation may reflect latent auction-level factors that also alter bidders' valuations or information structure. In the case of first-price auctions this means that ignoring unobserved heterogeneity can lead to double trouble: misspecification of bidders' first-order conditions and endogeneity of a key covariate.

Some of the errors that will result from ignoring unobserved heterogeneity in auctions are intuitive. For example, one will infer from bids too much within-auction correlation in bidders' private information, and too much cross-auction variation in this information. Recent work has demonstrated that ignoring unobserved heterogeneity can also lead to quantitatively important distortions of other less transparent forms, including (a) erroneous estimates of market power and information rents to bidders (e.g., Krasnokutskaya (2011), Krasnokutskaya and Seim (2011), Athey, Levin, and Seira (2011)); (b) incorrect conclusions about optimal auction design (e.g., Krasnokutskaya (2011), Roberts (2013)); and (c) wrong conclusions about the presence/significance of the winner's curse (e.g., Haile, Hong, and Shum (2003), Compiani, Haile, and Sant'Anna (2018)). Thus, as the literature has matured, increasing attention has been given to the need for strategies allowing identification even in the presence of unobserved heterogeneity.

Standard methods for handling unobserved heterogeneity in econometrics typically are not directly applicable in auction settings. For example, nonlinearity rules out reliance on first differences, and the (typically) small number of bids per auction rules out a fixed effects approach.<sup>4</sup> More fundamental is the fact that the observation-specific components of the "error terms" in an auction model are equilibrium transformations of bidders' private information, i.e., of key primitives of interest. Assumptions (e.g., standard IV conditions)

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<sup>3</sup>See, e.g., Gilley and Karels (1981), Athey and Haile (2002), Haile, Hong, and Shum (2003), Guerre, Perrigne, and Vuong (2009), Campo, Guerre, Perrigne, and Vuong (2011), and Gillen (2010).

<sup>4</sup>An exception is the case of certain large multi-unit auctions—e.g., Treasury auctions—where it is possible to treat each auction in isolation. See, e.g., Cassola, Hortaçsu, and Kastl (2013) and Hortaçsu, Kastl, and Zhang (2018).

that merely restore independence (or conditional independence) of the error terms, therefore, generally do not enable identification of the relevant model primitives. Nonetheless, we will see below that the existing identification strategies have connections to methods developed in other areas of econometrics, including control function methods, measurement error methods, and methods for mixture models. Some of these connections are more direct than others, and in some cases the new insights discussed here may prove useful in other types of structural econometric models as well.

Because the primary threats of unobserved heterogeneity concern identification, we focus below exclusively on different approaches to nonparametric identification in the presence of unobserved heterogeneity. And because unobserved heterogeneity is particularly challenging in first-price auctions, we focus exclusively on this case. Each of the approaches we discuss requires assumptions beyond those of a standard baseline model.<sup>5</sup> Some rule out correlation between different bidders' information; some require an auxiliary equation and exclusion restriction; some restrict the support of the unobservable; some restrict the way that unobservables enter the auction model; and some rule out endogenous bidder entry. As a result, none of these approaches dominates another, and the most relevant result in practice will depend on the details of the application, the questions of interest, and the data available. Our goal here is to describe a range of alternative strategies currently available, focusing on the key insights permitting identification without reliance on parametric assumptions.

In the following section we set up an affiliated values auction model with unobserved heterogeneity and point out the key challenge that arises from the presence of unobservables in the equilibrium first-order conditions. In section 3 we discuss identification obtained using a control function strategy. There, in essence, an auxiliary outcome (e.g., the number of bidders entering the auction) is used to allow one to indirectly condition on the unobservable. Section 4 then discusses identification obtained by adapting results from the literature on measurement error. Here, one typically requires a model with independent bidder types so

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<sup>5</sup>We emphasize, however, that we will not attempt to state the most general versions of the results possible. We focus on a version of each approach meant to convey the main insights most clearly.

that observed correlation among bids can be attributed to the unobserved heterogeneity. Bids can then serve as multiple noisy measures of the unobserved state, allowing application of classic or modern results. In section 5 we discuss a “quasi-control function” approach that avoids the strict monotonicity requirement of the control function strategy, instead exploiting the way that bounds on the unobservable implied by entry outcomes shift with auction-level observables. Finally, in section 6 we discuss identification obtained using a nonlinear finite mixture representation of the data generating process. This approach allows a softening of the index structure required by several other strategies, while assuming that the unobserved heterogeneity reflects a finite set of latent states.

## 2 Model

### 2.1 Setup

Our baseline model is the standard symmetric affiliated values model of first-price sealed bid auctions (Milgrom and Weber (1982)).<sup>6</sup> For simplicity we focus on the case with no binding reserve price. Bidders in auction  $t$  are risk neutral and indexed by  $i = 1, \dots, N_t$ . In addition to the number of bidders  $N_t$ , auction  $t$  is associated with characteristics

$$C_t = (X_t, Z_t, U_t).$$

Among the three components of  $C_t$  are two important distinctions. First,  $(X_t, Z_t)$  are observable to econometrician, whereas  $U_t \in \mathbb{R}$  is unobserved. Second, the variables  $(X_t, U_t)$  may affect bidder valuations, but  $Z_t$  does not—an exclusion restriction we will formalize below when we discuss results that require it.

The realization of  $(N_t, C_t)$  is common knowledge among bidders at auction  $t$ . Each bidder

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<sup>6</sup>Milgrom and Weber (1982) considered bidding in a single auction, and by using this model we follow most of the literature in assuming that one observes data from independent auctions. Spatial dependence is discussed by, e.g., Hendricks, Pinkse, and Porter (2003) and Compiani, Haile, and Sant’Anna (2018). Work on the econometrics of dynamic auction models includes Jofre-Bonet and Pesendorfer (2003) and Balat (2011).

$i$  has valuation for the good for sale given by a random variable  $V_{it} \in \mathbb{R}$ . However, beyond  $(N_t, C_t)$ , each bidder  $i$  observes only a private signal  $S_{it} \in \mathbb{R}$ . Thus, a bidder may not know her own valuation. A bidder’s expectation of her valuation given her information set is  $E[V_{it}|S_{it}, C_t, N_t]$ . More relevant for what follows, however, is a bidder’s expected valuation conditional on  $(S_{it}, C_t, N_t)$  and an additional assumption that her equilibrium bid is pivotal (see, e.g., Milgrom and Weber (1982)):

$$w(s; n, c) \equiv E \left[ V_{it} \mid S_{it} = \max_{j \neq i} S_{jt} = s, N_t = n, C_t = c \right].$$

Following Compiani, Haile, and Sant’Anna (2018), we refer to  $w(S_{it}; N_t, C_t)$  as bidder  $i$ ’s “pivotal expected value” in auction  $t$ . We let  $V_t = (V_{1t}, \dots, V_{n_t t})$  and  $S_t = (S_{1t}, \dots, S_{n_t t})$ .

The affiliated values model incorporates several important special cases often considered in applications. By specifying  $S_{it} = V_{it}$  one obtains a private values model, and in that case  $w(S_{it}; N_t, C_t) = V_{it}$ . A further requirement of mutual independence among  $(V_{1t}, \dots, V_{n_t t})$  conditional on  $C_t$  will yield the “independent private values” (IPV) model.<sup>7</sup> When  $E[V_{it}|S_t]$  has nontrivial dependence on  $S_{-it}$ , one has a “common values” model, also known as a model with “interdependent values.” This is a broad class of models distinguished by the presence of a winner’s curse. A special case, referred to as “pure common values,” arises when the value of the good is identical for all bidders.

Given  $C_t = c$  and  $N_t = n$ , let  $F_{SV}(S_t, V_t|n, c)$  denote the joint distribution of bidders’ signals and valuations in auction  $t$ . We assume this distribution is affiliated, exchangeable in the bidder indices  $i$ , admits  $\mathcal{C}^1$  density that is positive on  $(\underline{s}, \bar{s})^n \times (\underline{v}, \bar{v})^n$ , and is such that the conditional expectation  $E[V_{it}|S_{it}, S_{-it}, N_t, X_t, U_t]$  exists and is strictly increasing in  $S_{it}$ .

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<sup>7</sup>A relaxation of the IPV model specifies valuations as independent only after conditioning on a latent variable  $\omega_t$  as well as the common knowledge auction level information  $C_t$ . This is a special case of correlated private values known in the literature as the “conditionally independent private values” model (see, e.g., Li, Perrigne, and Vuong (2000)). An important distinction between that model and models with unobserved heterogeneity is that in the latter  $U_t$  is observed by bidders, while in the former  $\omega_t$  is unknown to bidders. These are different models with sometimes very different implications. For example, revenue equivalence holds under independence conditional on  $C_t$ , but not in the conditionally independent private values model. Thus, distinguishing between these information structures is an important motivation for exploring models that permit both dependent private information and unobserved heterogeneity.



We assume that observed bids reflect a symmetric Bayes Nash equilibrium in pure, strictly increasing, differentiable strategies.<sup>8</sup> Let

$$\beta(\cdot; N_t, C_t) : [\underline{s}, \bar{s}] \rightarrow \mathbb{R}$$

denote the equilibrium bidding strategy. Bidder  $i$ 's equilibrium action (bid) in auction  $t$  can then be represented by the random variable

$$B_{it} = \beta(S_{it}; N_t, C_t).$$

Let  $B_t = (B_{1t}, \dots, B_{nt})$ . Let  $M_{it} = \max_{j \neq i} B_{jt}$  denote the maximum bid among  $i$ 's competitors at the auction. In effect, bidder  $i$  competes only against  $M_{it}$ : given  $\{S_{it} = s, N_t = n, C_t = c\}$  her equilibrium bid solves

$$\max_b E[(V_{it} - b) 1\{M_{it} < b\} | S_{it} = s, N_t = n, C_t = c]. \quad (1)$$

## 2.2 The Identification Challenge

Our discussion of identification focuses primarily on one primitive of interest: the joint distribution  $F_w(\cdot | n, c)$  of bidders' pivotal expected values  $(w(S_{1t}; n, c), \dots, w(S_{nt}; n, c))$  given  $N_t = n, C_t = c$ . In private values models this is identical to the joint distribution of valuations conditional on  $N_t = n, C_t = c$ . In a common values setting, identification of  $F_w(\cdot | n, c)$  is a form of partial identification sufficient to address some important questions.<sup>9</sup> Of course, because  $C_t$  is not fully observed, one may also be interested in the distribution of  $U_t$  conditional on  $X_t, Z_t, N_t$ . Thus, we also discuss identification of this conditional distribution.

From the perspective of the econometrician, we assume that the observables consist of

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<sup>8</sup>See Athey and Haile (2007) for a review of results on existence and uniqueness.

<sup>9</sup>As discussed by, e.g., Laffont and Vuong (1996) and Athey and Haile (2007), in a common values model (even without unobserved heterogeneity) one is typically forced either to settle for such partial identification or to rely on a combination of additional restrictions and observables beyond bids and auction-level covariates. Hendricks, Pinkse, and Porter (2003) and Somaini (2018) offer two such approaches.

$N_t$ ,  $X_t$ ,  $B_t$ , and in some cases excluded variables  $Z_t$ . Importantly,  $U_t$  is unobserved. To see why this poses a potential problem for identification, it is useful to review the pathbreaking insight of Guerre, Perrigne, and Vuong (2000).<sup>10</sup> For simplicity, take the case of private values, where bidder  $i$  observes  $V_{it} = v$ ,  $N_t = n$ ,  $C_t = c$ . Let

$$G_{M|B}(m|b, n, c) = \Pr(M_{it} \leq m | \beta(S_{it}; n, c) = b, N_t = n, C_t = c),$$

and let

$$g_{M|B}(m|b, n, c) = \frac{\partial}{\partial m} G_{M|B}(m|b, n, c).$$

Bidder  $i$ 's equilibrium bid  $b$  must then solve<sup>11</sup>

$$\max_{\tilde{b}} (v - \tilde{b}) G_{M|B}(\tilde{b}|b, n, c). \quad (2)$$

This problem has first-order condition

$$v = b + \frac{G_{M|B}(b|b, n, c)}{g_{M|B}(b|b, n, c)}. \quad (3)$$

If there is no unobserved heterogeneity ( $U_t$  is degenerate), all terms on the right-hand side of (3) are observable. We then immediately have nonparametric identification of each bidder's valuation and, therefore, of the joint distribution  $F_V(V_{1t}, \dots, V_{nt}|n, c)$  (Li, Perrigne, and Vuong (2002)). With unobserved heterogeneity, however, bidders condition on information unavailable to the econometrician:  $C_t$  includes  $U_t$ . As a result, the right-hand side of (3) is no longer observed. Indeed, without additional information or structure, the functions  $G_{M|B}$  and  $g_{M|B}$  on the right-hand side are not identified.

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<sup>10</sup>See also Laffont and Vuong (1993), Li, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), Athey and Haile (2002), Hendricks, Pinkse, and Porter (2003), Haile, Hong, and Shum (2003), Campo, Perrigne, and Vuong (2003), Hortaçsu and McAdams (2010), and Kastl (2011), among many other papers building on this insight.

<sup>11</sup>Observe that because equilibrium bids are strictly increasing in signals, conditioning on the value of a bidder's equilibrium bid is equivalent to conditioning on his signal. Hence, given the private values assumption, (1) and (2) are equivalent.

This challenge extends to the more general affiliated values model, where the first-order condition takes the form (see, e.g., Haile, Hong, and Shum (2003) or Athey and Haile (2007))

$$w(s_{it}; n_t, c_t) = b_{it} + \frac{G_{M|B}(b_{it}|b_{it}, n_t, c_t)}{g_{M|B}(b_{it}|b_{it}, n_t, c_t)}. \quad (4)$$

Absent unobserved heterogeneity, this equation implies identification of each  $w(s_{it}; n_t, c_t)$  and, therefore,  $F_w(\cdot|n_t, c_t)$ . But when the econometrician is unable to observe all elements of  $c_t$ , this strategy breaks down. Additional data or structure will be needed to obtain identification of our primitives of interest.

### 2.3 Separability and “Homogenization”

One form of additional structure relied on by several of the approaches discussed below involves a separability assumption regarding the way the auction characteristics  $C_t$  affect bidder valuations. Suppose for example that

$$V_{it} = \Gamma(X_t, U_t) V_{it}^0. \quad (5)$$

where, conditional on  $N_t$ ,

$$(V_{1t}^0, \dots, V_{N_t t}^0, S_{1t}, \dots, S_{N_t t}) \perp\!\!\!\perp (X_t, U_t). \quad (6)$$

Alone, (5) has no content; however, when combined with (6) it requires that, conditional on  $N_t$ ,  $(X_t, U_t)$  affect the auction only through the multiplicative index  $\Gamma(X_t, U_t)$ . Note that under (5) and (6) we may take an arbitrary point  $x^0$  and let

$$\Gamma(x^0, 0) = 1 \quad (7)$$

without loss of generality.

This type of structure proves useful because separability is preserved by equilibrium

bidding. In particular, it is easily confirmed that<sup>12</sup>

$$\beta(S_{it}; n_t, x_t, u_t) = \Gamma(x_t, u_t) \beta^0(S_{it}; n_t), \quad (8)$$

where  $\beta^0$  denotes the symmetric Bayes Nash equilibrium bidding strategy for a standardized auction  $t$  at which  $\Gamma(X_t, U_t) = \Gamma(x^0, 0) = 1$ . Letting  $B_{it}^0 = \beta^0(S_{it}; n_t)$ , we can rewrite (8) as

$$B_{it} = \Gamma(x_t, u_t) B_{it}^0. \quad (9)$$

Following Haile, Hong, and Shum (2003), we refer to the random variable  $B_{it}^0$  as bidder  $i$ 's "homogenized" bid at auction  $t$ . Likewise we refer to  $V_{it}^0$ , as  $i$ 's "homogenized valuation" and to  $w^0(S_{it}; N_t) \equiv w(S_{it}; N_t)/\Gamma(X_t, U_t)$  as  $i$ 's "homogenized pivotal expected value." One can easily confirm that homogenized pivotal expected values must satisfy the first-order condition

$$w^0(s_{it}; n_t) = b_{it}^0 + \frac{G_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t, c_t)}{g_{M^0|B^0}(b_{it}^0|b_{it}^0, n_t, c_t)}, \quad (10)$$

where

$$b_{it}^0 = b_{it}/\Gamma(x_t, u_t)$$

$$M_{it}^0 = M_{it}/\Gamma(x_t, u_t)$$

$$G_{M^0|B^0}(m^0|b^0, n) = \Pr(M_{it}^0 \leq m | B_{it}^0 = b^0, N_t = n)$$

$$g_{M^0|B^0}(m^0|b^0, n) = \frac{\partial}{\partial m^0} G_{M^0|B^0}(m^0|b^0, n).$$

Equations (9) and (10) imply that, after rescaling bids appropriately, one can proceed as if the data reflected a sample of homogeneous auctions to recover estimates of the homogenized pivotal expected values  $w^0(s_{it}; n_t)$ . Of course, to do so one must first recover the scaling factors  $\Gamma(x_t, u_t)$ . Observe that knowledge of each  $w^0(s_{it}; n_t)$  and  $\Gamma(x_t, u_t)$  will imply

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<sup>12</sup>See Haile, Hong, and Shum (2003), Athey and Haile (2007), or Krasnokutskaya (2011). We focus here on the case of multiplicative separability although an analogous result holds by essentially the same argument under additive separability.

identification of  $F_w(\cdot|n, c)$ .

### 3 Control Function Approaches

The earliest and simplest approaches to unobserved heterogeneity in first-price auctions rely on a control function. The essence of the approach is to exploit an auxiliary observable outcome that indirectly fixes the realization of the unobservable  $U_t$ , enabling the econometrician to condition on it.<sup>13</sup> This approach was first proposed by Campo, Perrigne, and Vuong (2003) and further developed by Haile, Hong, and Shum (2003), both using the number of bidders  $N_t$  as the auxiliary outcome (see also Guerre, Perrigne, and Vuong (2009)). This strategy turns the “problem” of endogenous bidder entry into an economic relationship to be exploited.

Suppose that the number of bidders at each auction  $t$  satisfies

$$N_t = \eta(X_t, Z_t, U_t),$$

where the function  $\eta$  is strictly increasing in its final argument. In addition suppose that  $Z_t$  is independent of  $(S_t, V_t)$  conditional on  $X_t, U_t, N_t$ . As Campo, Perrigne, and Vuong (2003) point out, existence of a functional relationship between  $(X_t, Z_t, U_t)$  and  $N_t$  arises naturally when the meaningful decision to “enter” an auction takes place before a bidder learns her signal (thus, no “selective entry”) and entry outcomes do not reflect randomization (e.g., mixed strategies). The monotonicity requirement captures the natural idea that unobservables can be ordered in a way such that auctions with “better” unobservables attract (weakly) more

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<sup>13</sup>This is more than typically required of a control function in a regression setting, where conditioning on a control variable need only deliver independence between regressors and the structural error in the outcome equation (e.g., Chesher (2003), Imbens and Newey (2009)). Identification of the auction model is equivalent to identification of the bidding equation  $B_{it} = \beta(S_{it}; N_t, X_t, U_t)$  ((4) is the inverse of this equation), where both  $S_{it}$  and  $U_t$  are structural errors. Knowledge of objects like average effects or the “quantile structural function” (the function defining the quantiles of  $B_{it}$  conditional on  $N_t, X_t$ ) generally are not sufficient to address the economic questions of interest in an auction setting—indeed, the latter is directly observable. This reflects the fact that the problem in the auction setting is not merely the endogeneity of a covariate, but the fact that equilibrium bidding strategies condition on the realization of  $U_t$ . Olley and Pakes (1996) provide an early example of this kind of approach in a very different economic setting.

bidders. However, it is not necessary to assume that this order (i.e., notion of “better”) also reflects the way that unobservables affect valuations.

The value of strict monotonicity for identification comes from the fact that conditioning on the observables  $(X_t, Z_t, N_t)$  indirectly fixes  $U_t$ . And when combined with the assumed conditional independence of  $Z_t$ , this implies that the joint distribution of bids  $(B_{1t}, \dots, B_{nt})$  conditional on  $(X_t, Z_t, N_t = n)$  is identical to that conditional on  $(X_t, U_t, N_t)$ . Thus we can rewrite the first-order condition (4) as

$$w(s_{it}; n_t, c_t) = b_{it} + \frac{G_{M|B}(b_{it}|b_{it}, n_t, x_t, z_t)}{g_{M|B}(b_{it}|b_{it}, n_t, x_t, z_t)}.$$

Now all terms on the right-hand side are observable, yielding identification of the realizations of  $w(s_{it}; n_t, c_t)$  for all  $i$  and  $t$ , even though the value of  $c_t$  itself is not identified.

This alone is sufficient to address many questions of interest. For example, in a private values setting, where  $w(s_{it}; n_t, c_t) = v_{it}$ , it is sufficient to determine the joint distribution of bidder valuations holding  $N_t$  and  $C_t$  fixed (the latter at an unknown value of  $U_t$ , however), or to determine the division of surplus between the seller and bidders. However, this does not enable one to assess the effects of  $N_t, U_t$ , or  $X_t$  on bidders’ information/valuations. Thus, an additional assumption will sometimes be needed—e.g., to assess the effect of competition on bid levels or to test for common values.

Suppose, for example, that  $N_t$  is the only endogenous component of  $(N_t, X_t, Z_t)$ , i.e.,

$$(X_t, Z_t) \perp\!\!\!\perp U_t. \tag{11}$$

This assumption immediately implies identification of the function  $\eta$ , implying identification of the realization of each  $U_t$  as well. Since  $U_t$  is the only unobservable component of  $C_t$ , by (4) we have identification of each realization  $w(S_{it}; n, c)$  given any  $(n, c)$  in the support of  $(N_t, C_t)$ . This implies identification of the function  $w$ , the conditional distribution  $F_w(\cdot|C_t)$ , and the distribution of  $U_t$  conditional on any subvector of  $(X_t, Z_t, N_t)$ .

An additional benefit of the independence assumption (11) is provision of a source of

exogenous variation in the level of competition  $N_t$ . This has been exploited by Haile, Hong, and Shum (2003) and Guerre, Perrigne, and Vuong (2009) to provide an approach for detecting the presence of common values and to identify a model with risk averse bidders, respectively, while allowing for unobserved heterogeneity.

An attractive aspect of this identification approach is that it requires no restriction on functional form or on the information structure of the baseline auction model. This distinguishes it from the approaches discussed below. But it has limitations as well. One is the requirement of an instrument that allows  $N_t$  to vary while  $(X_t, U_t)$  are held fixed. Natural instruments may be measures of the number of potential bidders (e.g., Haile (2001), Hendricks, Pinkse, and Porter (2003), Haile, Hong, and Shum (2003)), or of bidder entry costs (e.g., Kong (2017a), Compiani, Haile, and Sant’Anna (2018)). However, instruments will not be available in all applications. Another limitation is that, because  $N_t$  is discrete, strict monotonicity requires that  $U_t$  be discrete, with support limited by that of  $N_t$ . Finally, reliance on a reduced form for bidder entry implies that some types of counterfactuals (those that would alter the reduced form) will not be identified, at least without additional work.<sup>14</sup>

One may be able to avoid these limitations if there is an alternative continuous auxiliary outcome measure that can replace the number of bidders. Roberts (2013) points out that in many applications there is a natural candidate: the seller’s reserve price.<sup>15</sup> If the seller observes the unobserved heterogeneity, it may be natural to assume that the reserve price is strictly increasing in  $U_t$  conditional on other observables. Focusing on the case of independent private values, Roberts (2013) shows that, given a standard regularity condition, this strict monotonicity follows if the seller observes  $U_t$  and sets the profit-maximizing reserve price.<sup>16</sup>

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<sup>14</sup>As discussed by Compiani, Haile, and Sant’Anna (2018), once the auction model primitives are identified using the reduced form entry model, it may become possible to identify the preferred structural model of entry consistent with the reduced form.

<sup>15</sup>Adapting bidder first-order conditions to the case of a binding reserve price is straightforward. See, e.g., Athey and Haile (2002).

<sup>16</sup>Following Matzkin (2003), Roberts (2013) shows that the assumption of independence between  $Z_t$  and  $V_t$  may also be avoided under certain additional functional form restrictions.

## 4 Measurement Error Approaches

The control function approach discussed above lets the researcher to pin down, or fix, the value of  $U_t$  through an auxiliary observable. In effect, such a strategy makes the unobservable observable. This is not the only possible approach. In fact there is a close parallelism between nonlinear measurement error models and econometric models with unobserved heterogeneity. Results from the measurement error literature can therefore offer useful identification strategies for auction models with unobserved heterogeneity. With these approaches one forgoes pinning down the unobserved heterogeneity for each observation and instead relies on a decomposition of observable distributions into the components reflecting model primitives and those reflecting unobserved heterogeneity.

A pioneering application of results from the measurement error literature to auctions with unobserved heterogeneity is due to Krasnokutskaya (2011). Her result relies on the celebrated lemma of Kotlarski (1967). Kotlarski's Lemma is concerned with nonparametric deconvolution with repeated measurements subject to independent separable measurement error. Consider a triplet of mutually independent random variables  $(Y^*, \eta_1, \eta_2)$  and let

$$\begin{aligned} Y_1 &= Y^* + \eta_1 \\ Y_2 &= Y^* + \eta_2. \end{aligned}$$

For normalization purposes let  $E[\eta_1] = 0$ . Assuming that the characteristic functions of  $Y_1$  and  $Y_2$  are non-vanishing,<sup>17</sup> Kotlarski (1967) shows that the distribution  $F(Y^*, \eta_1, \eta_2) = F(Y^*)F(\eta_1)F(\eta_2)$  is identified from the joint distribution of  $(Y_1, Y_2)$ . This result is constructive, which can be useful for nonparametric estimation.

Kotlarski's Lemma has wide applications in econometrics: it applies to measurement error problems, panel data, and as Krasnokutskaya (2011) shows, auction models with unobserved heterogeneity.<sup>18</sup> Krasnokutskaya (2011) considers an IPV model in which the unobserved

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<sup>17</sup>Evdokimov and White (2012) provide some results relaxing this assumption.

<sup>18</sup>A related model in the auction literature is considered by Li, Perrigne, and Vuong (2000).



heterogeneity enters multiplicatively, i.e.,

$$V_{it} = U_t V_{it}^0,$$

where

$$U_t \perp\!\!\!\perp V_{1t}^0 \perp\!\!\!\perp V_{2t}^0 \dots \perp\!\!\!\perp V_{N_t}^0.$$

Here  $V_{it}^0$  can be regarded as  $i$ 's homogenized valuation as before.<sup>19</sup> By the preservation of separability under equilibrium bidding, we then have

$$B_{it} = U_t B_{it}^0,$$

with

$$U_t \perp\!\!\!\perp B_{1t}^0 \perp\!\!\!\perp B_{2t}^0.$$

By considering (any) two bidders at auction  $t$ , we have

$$\begin{aligned} \ln B_{1t} &= \ln U_t + \ln B_{1t}^0 \\ \ln B_{2t} &= \ln U_t + \ln B_{2t}^0. \end{aligned}$$

It is then immediate from Kotlarski's Lemma that the joint distribution of  $(U_t, B_{1t}^0, B_{2t}^0)$  is nonparametrically identified from the observable joint distribution of  $(B_{1t}, B_{2t})$ . Using the first-order condition (10), one can then recover the marginal distributions of valuations for bidders 1 and 2. In the case of more than 2 bidders, a similar argument yields the distribution of valuations for other bidders. In the case of symmetric bidders, this result implies a form of overidentification, allowing falsification of the model.

Note that the data requirement for Krasnokutskaya's identification results is modest, and it does not require an instrument. Also, whereas some of the approaches we discuss require that the unobserved heterogeneity be discrete, here such an assumption is unnecessary. And

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<sup>19</sup>Here we implicitly condition on any observable auction characteristics  $X_t$ .

although we have assumed symmetry for simplicity, here this is not required. A critical requirement, however, is the independence of bidder types—here, of the homogenized valuations  $V_{1t}^0, \dots, V_{nt}^0$ . In essence, therefore, this kind of approach is limited to IPV settings.<sup>20</sup> A separable structure and statistical independence between the unobserved heterogeneity and valuations are also critical; it is this structure that creates the equivalence to the to the classical measurement error setting.

An alternative approach that is also based on a result from the measurement error literature has been proposed by Hu, McAdams, and Shum (2013). They build upon an identification result for nonlinear measurement error models due to Hu (2008). Consider a random vector  $(Y, W^*, W, Z)$ , where  $Y$  denotes the dependent variable,  $W^*$  the unobserved independent variable,  $W$  a mismeasured indicator of  $W^*$  and  $Z$  an instrument. Here  $W^*$  is assumed to be supported by a finite set, and  $W$  and  $Z$  are assumed to have the same finite support. The goal is to identify the joint CDF  $F_{Y, W^*, W, Z}$  of  $(Y, W^*, W, Z)$ . Suppose

$$Y \perp\!\!\!\perp W \perp\!\!\!\perp Z | W^*,$$

which, among other things, allows for nonclassical measurement error. Also assume a full rank condition for the conditional distribution function of  $W^*$  given  $Z$ , which is essentially an instrument relevance condition. Then with additional conditions that guarantee uniqueness of the eigenvector-eigenvalue decomposition for an observable matrix, Hu (2008) shows that  $F_{Y, W^*, W, Z}$  is identified.

Hu, McAdams, and Shum (2013) show that the above result can be used to establish identification in auction models with unobserved heterogeneity when there are at least three bidders. Like that of Krasnokutskaya (2011), the identification strategy in Hu, McAdams, and Shum (2013) is essentially limited to the IPV model, although here one can dispense

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<sup>20</sup>Common values models with independent types exist, but are not easily motivated in applications.

with any separability requirement. In particular, Hu, McAdams, and Shum (2013) assume

$$V_{1t} \perp\!\!\!\perp V_{2t} \perp\!\!\!\perp \dots \perp\!\!\!\perp V_{N_t t} | U_t,$$

and that  $U_t$  is finitely supported, as in Hu (2008). Pick an arbitrary three bidders (wlog, the first three of the  $N_t$  bidders). The independence above implies

$$B_{1t} \perp\!\!\!\perp B_{2t} \perp\!\!\!\perp B_{3t} | U_t.$$

Hu, McAdams, and Shum (2013) treat  $B_{1t}$ ,  $B_{2t}$ ,  $B_{3t}$  and  $U_t$ , respectively, as the dependent variable  $Y$ , the mismeasured indicator  $W$ , the instrument  $Z$ , and the true measurement  $W^*$  in Hu’s measurement error model described above. To do this, they discretize  $B_{2t}$  and  $B_{3t}$ . Moreover, Hu, McAdams, and Shum (2013) show that if  $V_{it} | U_t$  is increasing in  $U_t$  in terms of first-order stochastic dominance (FOSD), then the rank condition and the uniqueness conditions for eigenvector-eigenvalue decomposition in Hu (2008) is satisfied.<sup>21</sup> Thus, by the result above, the joint distribution of  $(B_{1t}, B_{2t}, B_{3t}, U_t)$  is identified (up to the discretization). This in turn allows one to recover the joint distribution of valuations and the unobserved heterogeneity through the first-order condition (4).

There have been at least two important extensions of this result in the recent literature. Gentry and Li (2014) have showed that it can be used to obtain identification in an IPV model with endogenous bidder entry under the additional assumption that higher values of the unobservable lead to higher entry probabilities. They allow for a binding reserve price and selective entry, the latter being ruled out by the control function and quasi-control function approach of Campo, Perrigne, and Vuong (2003), Haile, Hong, and Shum (2003), or Compiani, Haile, and Sant’Anna (2018). Their result does not require an instrument for entry, but can allow it when a source of exogenous variation is required. Balat (2011) has shown that one need not rely on bids to provide the analogs of the “measurements” and instrument

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<sup>21</sup>This FOSD condition is implied by multiplicative (or additive) separability. Luo (2018) shows that the FOSD requirement can be weakened.

in Hu (2008). He instead exploits the availability multiple measures of participation—that at sequential stages (prequalification and bidding) or among multiple subgroups (e.g., large firms and small firms), each of which is assumed to respond (stochastically) to the auction-level unobservable. When such data are available, an advantage of this approach is that it can allow one to drop the assumption of independent bidder types.

## 5 A “Quasi-Control Function” Approach

Compiani, Haile, and Sant’Anna (2018) have recently considered an approach that shares many features of the control function strategy (section 3) while avoiding its strict monotonicity requirement and requirement of discrete unobserved heterogeneity. Like Haile, Hong, and Shum (2003), Compiani, Haile, and Sant’Anna (2018) assume that the number of bidders participating in auction  $t$  can be represented by a reduced-form relation

$$N_t = \eta(X_t, Z_t, U_t),$$

where

$$U_t \perp\!\!\!\perp (X_t, Z_t) \tag{12}$$

and

$$F_{SV}(S_t, V_t | N_t, Z_t, X_t, U_t) = F_{SV}(S_t, V_t | N_t, X_t, U_t).$$

However, Compiani, Haile, and Sant’Anna (2018) require only weak monotonicity of  $\eta$  in  $U_t$ .<sup>22</sup>

Compiani, Haile, and Sant’Anna (2018) combine this structure with the separability assumptions (5) and (6), assuming in addition that the index function  $\Gamma(X_t, U_t)$  scaling bidder valuations is also weakly increasing in  $U_t$ . Thus, Compiani, Haile, and Sant’Anna (2018) assume that unobservables making the good for sale (weakly) more valuable also lead

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<sup>22</sup>With this structure,  $U_t$  may be assumed uniform on  $[0, 1]$  without loss.

to (weakly) higher levels of participation.<sup>23</sup>

With weak monotonicity of  $\eta$ , the observed values of  $(N_t, X_t, Z_t)$  does not determine the realizations of  $U_t$ , although they do imply bounds. Compiani, Haile, and Sant'Anna (2018) show that point identification of the model can be obtained by exploiting the way that these bounds alter the support of equilibrium bids.

Their argument proceeds in three steps. First consider the bounds on each realized  $u_t$ . Fix  $(X_t, Z_t) = (x, z)$  and let the (conditional) support of  $N_t$  be  $\{\underline{n}(x, z), \underline{n}(x, z) + 1, \dots, \bar{n}(x, z)\}$ . The function  $\eta(x, z, \cdot)$  is then characterized by a set of thresholds:  $N_t = n$  if and only if  $U_t \in [\tau_{n-1}(x, z), \tau_n(x, z)]$ . Observed conditional entry probabilities therefore satisfy

$$\Pr(N_t = n | X_t = x, Z_t = z) = \tau_n(x, z) - \tau_{n-1}(x, z) \quad n = \underline{n}(x, z), \dots, \bar{n}(x, z). \quad (13)$$

Of course,  $\tau_{\underline{n}(x, z)-1}(x, z) = 0$  and  $\tau_{\bar{n}(x, z)}(x, z) = 1$ , giving an initial value from which to solve (13) for all thresholds  $\tau_n(x, z)$ . By (13), these known thresholds bound the realization of each  $U_t$ ; indeed, conditional on  $(X_t, Z_t, N_t) = (x, z, n)$ ,  $U_t$  is uniform on  $[\tau_{n-1}(x, z), \tau_n(x, z)]$ .

Next, to pin down the function  $\Gamma$ , take logs of (9) to obtain

$$\ln B_{it} = \gamma(X_t, U_t) + \ln \beta^0(S_{it}, N_t), \quad (14)$$

where we have defined  $\gamma(X_t, U_t) = \Gamma(X_t, U_t)$ . Now observe that

$$\sup \{ \ln B_{it} | N_t = n, X_t = x, Z_t = z \} = \gamma(x, \tau_n(x, z)) + \ln \beta^0(\bar{s}; n)$$

while

$$\sup \{ \ln B_{it} | N_t = n, X_t = \hat{x}, Z_t = \hat{z} \} = \gamma(\hat{x}, \tau_n(\hat{x}, \hat{z})) + \ln \beta^0(\bar{s}; n).$$

By differencing these equations, one learns all first differences of the form  $\gamma(\hat{x}, \tau_n(\hat{x}, \hat{z})) -$

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<sup>23</sup>Compiani, Haile, and Sant'Anna (2018) motivate this structure with an example of a fully specified two-stage game of entry and bidding, where entering the auction involves costly acquisition of a signal  $S_{it}$ . In that example, they show how the restriction to a scalar unobservable, the independence between  $X_t$  and  $U_t$ , and the required weak monotonicity conditions can be obtained as results rather than assumptions.

$\gamma(x, \tau_n(x, z))$ . Additional first differences can be obtained by varying the value of  $N_t$  conditioned on and differencing first differences with common terms. More first difference are obtained by applying similar argument using the infimum rather than supremum. Given “enough” of these first-differences, the fact that we know the value of the function at one point (recall (7)) will determine  $\gamma$  over its entire domain. Compiani, Haile, and Sant’Anna (2018) provide sufficient conditions—roughly, that  $U_t$  and  $Z_t$  act as continuous substitutes in the “production” of bidder entry, and that  $Z_t$  have variation sufficient to offset certain discrete variation in  $U_t$ . They also discuss the partial identification of  $\gamma$  obtained when the instrument induces more limited variation (even no variation) in bidder entry.

With  $\gamma$  known, identification of  $F_w(\cdot|n, c)$  follows easily. Fix  $X_t = x, N_t = n$  and recall (14). The random variable  $\gamma(x, U_t)$  is independent of  $(\ln B_{1t}^0, \dots, \ln B_{nt}^0)$  and has (now) a known distribution. Thus, because the joint distribution of  $(\ln B_{1t}, \dots, \ln B_{nt})$  is observed, a standard deconvolution result implies identification of the joint distribution of the log homogenized bids  $(\ln B_{1t}^0, \dots, \ln B_{nt}^0)$ . Since homogenized pivotal expected values must satisfy the first-order condition (10), their joint distribution is identified. Because we know the function  $\gamma$ , this also implies identification of the joint distribution  $F_w(\cdot|n, c)$  for all  $n$  and  $c$ .

The approach of Compiani, Haile, and Sant’Anna (2018) shares with the control function approach of Haile, Hong, and Shum (2003) the potential disadvantage of relying on a reduced form for bidder entry outcomes. And like the strategy of Krasnokutskaya (2011), it relies on a separability requirement limiting the way auction characteristics alter the environment. On the other hand, the approach deals explicitly with the endogeneity of bidder entry, requires no further restriction on the baseline auction model, and provides a strategy for isolating the exogenous variation in competition induced by the instrument.

## 6 A Mixture Model Approach

Finite mixtures have been widely used in order to incorporate unobserved heterogeneity in econometric models.<sup>24</sup> Nonparametric identifiability of finite mixture models is a challenging subject, although some approaches have been suggested in the recent literature. For example, Kitamura and Laage (2017) show that a finite mixture regression model can be identified nonparametrically under reasonable assumptions. This result can be useful for auction models with unobserved heterogeneity, as we demonstrate below.

Kitamura and Laage (2017) consider a  $J$ -components mixture of nonparametric regressions

$$Y = \gamma(X, U) + \epsilon_U, \quad \Pr\{U = u\} = \lambda_u, u = 1, \dots, J$$

where the econometrician observes the outcome variable  $Y$  and the covariate  $X$ , whereas  $U$  and  $\epsilon_U$  remain unobserved. The functions  $\{\gamma(\cdot, u)\}_{u=1}^J$ , the probability weights  $\{\lambda_u\}_{u=1}^J$ , and the distributions of  $\{\epsilon_u\}_{u=1}^J$  are unknown. The key assumptions for the identification results in Kitamura and Laage (2017) are: (i)  $\epsilon_u \perp\!\!\!\perp X$  for every  $u \in \{1, \dots, J\}$  and (ii) there exists a segment (in  $x$ ) where  $\gamma(x, u), u \in \{1, \dots, J\}$  are non-parallel. Then with a regularity condition in terms of the characteristic functions and the moment generating functions of  $\epsilon_u, u \in \{1, \dots, J\}$ , Kitamura and Laage (2017) show that  $\{\gamma(\cdot, u)\}_{u=1}^J, \{\lambda_u\}_{u=1}^J$  and the distribution functions  $F_{\epsilon_u}, u = 1, \dots, J$  are all nonparametrically identified.

To apply this result to an auction model with unobserved heterogeneity, the multiplicative structure (5) for valuations is once again maintained, although this time we impose the following independence condition: conditional on  $N_t$  and  $U_t$ ,

$$(V_{1t}^0, \dots, V_{N_t t}^0, S_{1t}, \dots, S_{N_t t}) \perp\!\!\!\perp X_t. \tag{15}$$

This assumption relaxes condition (6) by avoiding the requirement of independence between homogenized valuations/signals and  $U_t$ . Thus, for example, bidders' private information may

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<sup>24</sup>See Compiani and Kitamura (2016) for a recent review.

interact with the unobservable, not just through an index of auction characteristics  $(X_t, U_t)$ . These structures imply

$$w(S_{it}; N_t, X_t, U_t) = \Gamma(X_t, U_t)w^0(S_{it}; N_t, U_t), \quad (16)$$

where  $w^0(s; n, u) = E[V_{it}^0 | S_{it} = \max_{j \neq i} S_{jt} = s, N_t = n, U_t = u]$ . As before,  $w^0(s; n, u)$  is a homogenized pivotal expected value, although here it is allowed to depend on  $U_t$ . On the other hand, we assume that the support of  $U_t$  is  $\{1, \dots, J\}$ , so that the finite mixture identification result in Kitamura and Laage (2017) applies.

Just like the quasi-control function approach in Section 5 and the measurement error approach by Krasnokutskaya (2011), the mixture approach in this section imposes the multiplicative separability requirement, but note that it differs from the other two in that the separability is required only in terms of  $X_t$ , but not in terms of  $U_t$ . Also, the mixture approach, as in Hu, McAdams, and Shum (2013), assumes a discrete and finite support for unobserved heterogeneity. On the other hand, it applies to general affiliated values models (as the control function and quasi-control function approaches do), including common values models, while allowing for a general type of unobserved heterogeneity.

By the separability preserving property of equilibrium bidding, from (16) we obtain

$$B_{it} = \Gamma(X_t, U_t)B_{it}^{0,U_t},$$

where  $B_{it}^{0,U_t}$  is bidder  $i$ 's homogenized valuation (i.e.  $B_{it}^{0,U_t} = \beta^0(S_{it}; N_t, U_t)$ ). Here the more flexible index structure is also inherited by equilibrium bids:  $B_{it}^{0,U_t}$  can depend on  $U_t$  even after homogenization. Taking logs of both sides and letting  $b_{it} = \log B_{it}$ ,  $\gamma(X_t, U_t) = \log \Gamma(X_t, U_t)$  and  $b_{it}^{0,U_t} = \log B_{it}^{0,U_t}$ , we have

$$b_{it} = \gamma(X_t, U_t) + b_{it}^{0,U_t}.$$



Note that, conditional on  $N_t = n$ , our independence assumptions imply

$$(b_{1t}^{0,u}, \dots, b_{nt}^{0,u}) \perp\!\!\!\perp X_t.$$

Finally, we assume

$$U_t \perp\!\!\!\perp X_t$$

conditional on  $N_t$ . Although it may be possible to relax this assumption, this generally rules out entry outcomes  $N_t$  that depend on both  $X_t$  and  $U_t$ .

We are now in a position to apply the identification result of Kitamura and Laage (2017). For the remainder of this section we fix  $N_t$  at a arbitrary value  $n$  and suppress the index  $n$  except where it is necessary. Take any vector  $c = (c_1, \dots, c_n)'$  from  $\mathbb{R}^n$  and construct the linear combination of log bids

$$\sum_{i=1}^n c_i b_{it} = \left( \sum_{i=1}^n c_i \right) \gamma(X_t, U_t) + \sum_{i=1}^n c_i b_{it}^{0, U_t}.$$

Rewrite this as

$$\tilde{b}_t^c = C \gamma(X_t, U_t) + \tilde{b}_t^{c, 0, U_t}, \tag{17}$$

with  $\tilde{b}_t^c \equiv \sum_{i=1}^n c_i b_{it}$ ,  $C \equiv \sum_{i=1}^n c_i$  and  $\tilde{b}_t^{c, 0, U_t} \equiv \sum_{i=1}^n c_i b_{it}^{0, U_t}$ . Note that

$$\tilde{b}_t^{c, 0, u} \perp\!\!\!\perp X_t \tag{18}$$

holds for each  $u \in \{1, \dots, J\}$ . Define

$$\lambda_u = \Pr\{U_t = u\}, \quad u \in \{1, \dots, J\}. \tag{19}$$

Applying the result by Kitamura and Laage (2017) outlined above to (17), (18) and (19), we see that  $C \gamma(\cdot, u)$ ,  $\lambda_u$ , and the distribution of  $\tilde{b}_t^{c, 0, u}$  are all identified for every  $c \in \mathbb{R}^n$  and each  $u \in \{1, \dots, J\}$ . But  $C$  is known, so  $\gamma(\cdot, \cdot)$  is identified. As  $c \in \mathbb{R}^n$  is arbitrary,

the (marginal) distribution of every linear combination  $\tilde{b}_t^{c,0,u}$  of  $(\ln B_{1t}^{0,u}, \dots, \ln B_{1n}^{0,u})$  is identified. Thus, by the Cramér-Wold device, the joint distribution of  $(B_{1t}^{0,u}, \dots, B_{nt}^{0,u})$  is identified. Since  $\gamma(\cdot, \cdot)$  is already known, this implies the identification of the joint distribution of  $(B_{1t}, \dots, B_{nt})|n, x, u$ . From the first order condition for equilibrium bidding the joint distribution of  $(w(S_{1t}; n, x, u), \dots, w(S_{nt}; n, x, u))$  is then uniquely determined.

## 7 Conclusion

We have discussed several strategies for allowing unobserved heterogeneity in empirical models of first-price auctions without sacrificing nonparametric identifiability. Some of these methods are now well established in the empirical literature, while others have been developed only recently. In all cases, some combination of new structure or new data must be added to the starting point in which one observes only bids and covariates, and where no structure is placed on the model beyond those needed to characterize equilibrium behavior. The results we have discussed offer a range of alternatives, several of which have close analogs in other types of econometric models.

Each of the identification strategies we have discussed offers advantages and disadvantages relative to others. And, while we have focused exclusively on identification, nonparametric/semiparametric estimators based on these identification results introduce additional trade-offs, suggesting that the most suitable approach in practice will vary with the application. We view these nonparametric identification results as relevant for empirical work employing parametric models as well. Even when practical concerns dictate the use of parametric assumptions for estimation, it is valuable to understand whether such assumptions are essential maintained hypotheses or merely choices of finite sample approximation method; without this, we cannot be precise about the foundation on which we build knowledge from data. Here, the availability of several alternative sufficient conditions for nonparametric identification suggest a form of robust identification that should encourage the use of models incorporating unobserved heterogeneity in practice. Indeed, applications using estimators

based these identification strategies so far indicate that accounting for unobserved heterogeneity can be important for the policy conclusions one reaches.<sup>25</sup>

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<sup>25</sup>See, e.g., Haile, Hong, and Shum (2003), An, Hu, and Shum (2010), Krasnokutskaya (2011), Krasnokutskaya and Seim (2011), Roberts (2013), Athey, Levin, and Seira (2011), Balat (2011), Kong (2017b), and Compiani, Haile, and Sant’Anna (2018).

## References

- AN, Y., Y. HU, AND M. SHUM (2010): “Estimating First-Price Auctions with an Unknown Number of Bidders: A Misclassification Approach,” *Journal of Econometrics*, 157, 328–341.
- ATHEY, S., AND P. A. HAILE (2002): “Identification of Standard Auction Models,” *Econometrica*, 70, 2107–2140.
- (2006): “Empirical Models of Auctions,” in *Advances in Economics and Econometrics, Theory and Applications: Ninth World Congress of the Econometric Society*, ed. by R. Blundell, W. Newey, and T. Persson. Cambridge University Press, vol. 2.
- (2007): “Nonparametric Approaches to Auctions,” in *Handbook of Econometrics*, ed. by J. J. Heckman, and E. Leamer. Elsevier, vol. 6A, pp. 3847–3965.
- ATHEY, S., J. LEVIN, AND E. SEIRA (2011): “Comparing Open and Sealed Bid Auctions: Evidence from Timber Auctions,” *Quarterly Journal of Economics*, 126, 207–257.
- BALAT, J. (2011): “Highway Procurement and the Stimulus Package: Identification and Estimation of Dynamic Auctions with Unobserved Heterogeneity,” working paper, Yale University.
- CAMPO, S., E. GUERRE, I. M. PERRIGNE, AND Q. VUONG (2011): “Semiparametric Estimation of First-Price Auctions with Risk Averse Bidders,” *Review of Economic Studies*, 78, 112–147.
- CAMPO, S., I. M. PERRIGNE, AND Q. VUONG (2003): “Asymmetry in First-Price Auctions with Affiliated Private Values,” *Journal of Applied Econometrics*, 18, 197–207.
- CASSOLA, N., A. HORTAÇSU, AND J. KASTL (2013): “The 2007 Subprime Market Crisis Through the Lens of European Central Bank Auctions for Short-Term Funds,” *Econometrica*, 81, 1309–1345.

- CHESHER, A. (2003): “Identification in Nonseparable Models,” *Econometrica*, 71, 1405–1441.
- COMPIANI, G., P. HAILE, AND M. SANT’ANNA (2018): “Common Values, Unobserved Heterogeneity, and Endogenous Entry in U.S. Offshore Oil Lease Auctions,” Discussion paper, Yale University, Cowles Foundation CFDP No. 2137.
- COMPIANI, G., AND Y. KITAMURA (2016): “Using mixtures in econometric models: a brief review and some new results,” *The Econometrics Journal*, 19(3), C95–C127.
- EVDOKIMOV, K., AND H. WHITE (2012): “Some Extensions of a Lemma of Kotlarski,” *Econometric Theory*, 28(4), 925–932.
- GENTRY, M., AND T. LI (2014): “Identification in Auctions with Selective Entry,” *Econometrica*, 82, 314–344.
- GILLEN, B. (2010): “Identification and Estimation of Level- $k$  Auctions,” Discussion paper, Caltech.
- GILLEY, O. W., AND G. V. KARELS (1981): “The Competitive Effect in Bonus Bidding: New Evidence,” *Bell Journal of Economics*, 12, 637–648.
- GUERRE, E., I. M. PERRIGNE, AND Q. VUONG (2000): “Optimal Nonparametric Estimation of First-Price Auctions,” *Econometrica*, 68, 525–574.
- (2009): “Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions,” *Econometrica*, 77, 1193–1227.
- HAILE, P. A. (2001): “Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales,” *American Economic Review*, 91, 399–427.
- HAILE, P. A., H. HONG, AND M. SHUM (2003): “Nonparametric Tests for Common Values in First-Price Sealed-Bid Auctions,” Cowles Foundation Discussion Paper 1444.

- HENDRICKS, K., AND H. J. PAARSCH (1995): “A Survey of Recent Empirical Work Concerning Auctions,” *Canadian Journal of Economics*, 28, 403–426.
- HENDRICKS, K., J. PINKSE, AND R. H. PORTER (2003): “Empirical Implications of Equilibrium Bidding in First-Price, Symmetric, Common Value Auctions,” *Review of Economic Studies*, 70, 115–145.
- HENDRICKS, K., AND R. H. PORTER (2007): “An Empirical Perspective on Auctions,” in *Handbook of Industrial Organization*, ed. by M. Armstrong, and R. Porter. Elsevier, vol. 3, pp. 2073–2143.
- HICKMAN, B. R., T. P. HUBBARD, AND Y. SAGLAM (2012): “Structural Econometric Methods in Auctions: A Guide to the Literature,” *Journal of Econometric Methods*, 1, 67–106.
- HORTAÇSU, A., J. KASTL, AND A. ZHANG (2018): “Bid Shading and Bidder Surplus in the US Treasury Auction System,” *American Economic Review*, 108, 147–169.
- HORTAÇSU, A., AND D. MCADAMS (2010): “Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market,” *Journal of Political Economy*, 118, 833–865.
- HU, Y. (2008): “Identification and estimation of nonlinear models with misclassification error using instrumental variables: A general solution,” *Journal of Econometrics*, 144(1), 27–61.
- HU, Y., D. MCADAMS, AND M. SHUM (2013): “Identification of First-Price Auctions with Non-Separable Unobserved Heterogeneity,” *Journal of Econometrics*, 174, 186–193.
- IMBENS, G. W., AND W. K. NEWEY (2009): “Identification and Estimation of Triangular Simultaneous Equations Models without Additivity,” *Econometrica*, 77, 1481–1512.
- JOFRE-BONET, M., AND M. PESENDORFER (2003): “Estimation of a Dynamic Auction Game,” *Econometrica*, 71, 1443–1489.

- KASTL, J. (2011): “Discrete Bids and Empirical Inference in Divisible Good Auctions,” *Review of Economic Studies*, 78, 974–1014.
- KITAMURA, Y., AND L. LAAGE (2017): “Nonparametric Analysis of Finite Mixtures,” Discussion paper, Yale University.
- KONG, Y. (2017a): “Selective Entry in Auctions: Estimation and Evidence,” Discussion paper, Rice University.
- (2017b): “Sequential Auctions with Synergy and Affiliation Across Auctions,” Discussion paper, Rice University.
- KOTLARSKI, I. (1967): “On Characterizing the Gamma and the Normal Distribution,” *Pacific Journal of Mathematics*, 20(1), 69–76.
- KRASNOKUTSKAYA, E. (2011): “Identification and Estimation in Procurement Auctions under Unobserved Auction Heterogeneity,” *Review of Economic Studies*, 28, 293–327.
- KRASNOKUTSKAYA, E., AND K. SEIM (2011): “Bid Preference Programs and Participation in Highway Procurement Auctions,” *American Economic Review*, 101, 2653–2686.
- LAFFONT, J.-J., AND Q. VUONG (1993): “Structural Econometric Analysis of Descending Auctions,” *European Economic Review*, 37, 329–341.
- (1996): “Structural Analysis of Auction Data,” *American Economic Review, Papers and Proceedings*, 86, 414–420.
- LI, T., I. PERRIGNE, AND Q. VUONG (2000): “Conditionally Independent Private Information in OCS Wildcat Auctions,” *Journal of Econometrics*, 98, 129–161.
- (2002): “Structural Estimation of the Affiliated Private Value Auction Model,” *RAND Journal of Economics*, 33, 171–193.

- LUO, Y. (2018): “Unobserved Heterogeneity in Auctions under Restricted Stochastic Dominance,” Discussion paper, University of Toronto, Department of Economics, Working Paper No. 606.
- MATZKIN, R. L. (2003): “Nonparametric Estimation of Nonadditive Random Functions,” *Econometrica*, 71(5), 1339–1375.
- MILGROM, P. R., AND R. J. WEBER (1982): “A Theory of Auctions and Competitive Bidding,” *Econometrica*, 50, 1089–1122.
- OLLEY, G. S., AND A. PAKES (1996): “The Dynamics of Productivity in the Telecommunications Equipment Industry,” *Econometrica*, 64, 1263–1297.
- PAARSCH, H. J., AND H. HONG (2006): *An Introduction to the Structural Econometrics of Auction Data*. Cambridge, MA: MIT Press.
- ROBERTS, J. W. (2013): “Unobserved Heterogeneity and Reserve Prices in Auctions,” *RAND Journal of Economics*, pp. 712–732.
- SOMAINI, P. (2018): “Identification in Auction Models with Interdependent Costs,” Discussion paper, Stanford University.