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On the Galois Action on Motivic Fundamental Groups of Punctured Elliptic and Rational Curves

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Abstract

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The main motive of this thesis is to study the action of the motivic Galois group on the motivic fundamental group of an algebraic curve X punctured at a finite set of points S:

$$\mathrm{Gal}_{\mathrm{Mot}} \circlearrowleft \pi_1^{\mathrm{Mot}}(X - S, v_0). \tag{1}$$

The algebraic, geometric, and analytic aspects of this action are examined in two cases: for $X = \mathbb{P}^1$ and for X an elliptic curve.

To study this action, we rely on motivic correlators, canonical elements in the fundamental Lie coalgebra of the category of mixed motives over a number field. We trace three themes:

(1) The Lie coalgebra structure on the motivic correlators. Using combinatorial arguments with an injection of Hodge theory, we find general families of relations (*double shuffle relations*) on these elements.

(2) The Hodge realization of the structure in (1). The canonical real periods of the motivic correlators are the Hodge correlator functions, functions of several X-valued variables that are computed as Feynman integrals. We find new functional equations on the Hodge correlator integrals. The proofs of results related to themes (1) and (2) are closely intertwined.

(3) The geometry of modular manifolds. The phenomenon that the complex computing cohomology of some locally symmetric space can be mapped to the standard cochain complex of the motivic Lie coalgebra had been observed in several cases. Our work on Bianchi hyperbolic threefolds and the motivic fundamental groups of CM elliptic curves is a variation on this theme.

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Chapter 1

Introduction

The main motive of this thesis is to study the action of the motivic Galois group on the motivic fundamental group of an algebraic curve X punctured at a finite set of points S:

$$\mathrm{Gal}_{\mathrm{Mot}} \circlearrowright \pi_1^{\mathrm{Mot}}(X - S, v_0). \tag{1.1}$$

This thesis consists of two parts, in which the algebraic, geometric, and analytic aspects of this action are examined in two cases: for $X = \mathbb{P}^1$ (Chapter 2) and for X an elliptic curve (Chapter 3).

To study this action, we rely on *motivic correlators*, canonical elements in the fundamental Lie coalgebra of the category of mixed motives over a number field that were constructed by Goncharov in [G9]. We trace three themes:

(1) The Lie coalgebra structure on the motivic correlators. Using combinatorial arguments with an injection of Hodge theory, we find general families of relations (*double shuffle relations*) on these elements.

(2) The Hodge realization of the structure in (1). The canonical real periods of the motivic correlators are the Hodge correlator functions, functions of several X-valued variables that are computed as Feynman integrals. We find new functional equations on the Hodge correlator integrals. The proofs of results related to themes (1) and (2) are closely intertwined.

(3) The geometry of modular manifolds. The phenomenon that the complex computing cohomology of some locally symmetric space can be mapped to the standard cochain complex of the motivic Lie coalgebra had been observed in several cases ([G10, G8]). Our work on Bianchi hyperbolic threefolds and the motivic fundamental groups of CM elliptic curves is a variation on this theme.

The structure of the text is the following. The remainder of the introduction is an exposition of the common background to our results: mixed motives and motivic correlators for general curves (§1.1) and a special case, mixed Tate motives and motivic correlators for rational curves (§1.2). In Chapters 2 and 3 we present our results on the motivic fundamental groups of rational curves and CM elliptic curves, respectively.

1.1 Hodge and motivic correlators

1.1.1 Hodge correlator integrals

Let X be a complex curve (in this thesis, $X = \mathbb{P}^1(\mathbb{C})$ or an elliptic curve). The Hodge correlator functions, defined in [G9], are functions

$$\mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n),$$

where each x_i is either a point of X or a 1-form representing a class in $H^1(X; \mathbb{C})$. The *depth* of this expression is the number of points among the x_i minus one, if X is an elliptic curve, or the number of points distinct from 0 among the x_i minus one, if $X = \mathbb{P}^1$. The *weight* is n plus the depth.

The Hodge correlators depend on the choice of a *tangential base point* – a base point $s \in X$ and a tangent vector v_0 at s. If $n = 1$ and $x_0, x_1 \in X$, then $\mathrm{Cor}_{\mathcal{H}}(x_0, x_1)$ is a (normalized) Green's function with pole at s. In particular,

– If $X = \mathbb{P}^1$ and $s = \infty$, then

$$\mathrm{Cor}_{\mathcal{H}}(x_0, x_1) = G_\infty(x_0, x_1) = (2\pi i)^{-1} \log |x_0 - x_1| + C.$$

The constant C depends on the choice of tangent vector at ∞, but the correlator is independent of this constant in weight > 2, so we will ignore it when convenient. (When $v_0 = \frac{-1}{z^2}\frac{\partial}{\partial z} = \frac{\partial}{\partial w}$ with $w = z^{-1}$, we have $C = 0$.) The correlator for other tangential base points can be derived using the fact that it is invariant under automorphisms of $\mathbb{P}^1$ acting on the base point and the arguments.

– If X is the elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, where $\mathrm{Im}(\tau) > 0$, with coordinate z inherited from the complex plane, then

$$\mathrm{Cor}_{\mathcal{H}}(x_0, x_1) = G_s(x_0, x_1) = G_{\mathrm{Ar}}(x_0, x_1) - G_{\mathrm{Ar}}(x_0, s) - G_{\mathrm{Ar}}(s, x_1) + C.$$

Here G_{Ar} is the Arakelov Green's function, the unique (up to constant) solution to the elliptic partial differential equation $(2\pi i)^{-1}\partial\overline{\partial} G_{\mathrm{Ar}}(x) = \mathrm{vol}_E - \delta_0$. It has the Fourier expansion

$$G_{\mathrm{Ar}}(z) = \frac{2\mathrm{Im}(\tau)}{2\pi i} \sum_{\gamma \in (\mathbb{Z}+\mathbb{Z}\tau)\setminus\{0\}} \frac{\exp\left(2\pi i\, \mathrm{Im}(z\overline{\gamma})/\mathrm{Im}(\tau)\right)}{|\gamma|^2}. \tag{1.2}$$

The Arakelov Green's function has a logarithmic singularity at 0. Hence, the function $\mathrm{Cor}_{\mathcal{H}}(x_0, x_1)$ has singularities of the form $\log|z|$ at the divisors $x_0 = x_1$, $x_0 = s$, $x_1 = s$.

– **Remark:** The Green's function on $\mathbb{P}^1$ is a specialization of the one on E. To be exact, write G^{E_τ} for the Green's function on $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with base point 0. Then, taking z to be the coordinate on E_τ inherited from the complex plane, such that the section $z \in E_\tau$ approaches $e^{2\pi i z} \in \mathbb{P}^1$, with appropriate choice of tangential base points,

$$\lim_{\tau\to+i\infty} G^{E_\tau}(z_1, z_2) = G_1\left(e^{2\pi i z_1}, e^{2\pi i z_2}\right) = \log\left|\frac{1}{e^{2\pi i z_1}-1} - \frac{1}{e^{2\pi i z_2}-1}\right|.$$

(This can be shown by a residue computation or an application of the Kronecker limit formula. We will require this fact in §3.4.2.)

If $n \geq 2$, the Hodge correlators are defined as sums of integrals determined by plane trivalent trees. Picture the $x_0, \dots, x_n$ written counterclockwise along the boundary of a disc and consider a trivalent tree T embedded in the disc with leaves at the $n+1$ boundary points. The tree has $n-1$ interior vertices V° and $2n-1$ edges $E_0, \dots, E_{2n-2}$. The embedding into the plane gives a canonical orientation $\mathrm{Or}_T \in \{\pm 1\}$ (an ordering of the edges up to even permutation).

Assign to each interior vertex $v \in V^\circ$ a copy of X, called X_v, with coordinate x_v. Then assign to each edge E_i either a function f_i or a 1-form ω_i, as follows:

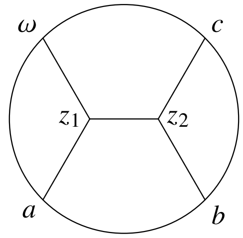
(1) If $E_i = (u, v)$ is an interior edge, let $f_i = G_s(x_u, x_v)$, a function on $X_u \times X_v$.

(2) If $E_i = (u, x_j)$ is a boundary edge with the leaf decorated by a point $x_j \in X$, let $f_i = G_s(x_u, x_j)$, a function on X_u.

(3) If $E_i = (u, x_j)$ is a boundary edge with $x_j = \omega$ a 1-form, let $\omega_i = \omega(x_u)$, a 1-form on X_u.

Without loss of generality, $E_0, \dots, E_k$ are the edges labeled by functions (i.e., not boundary edges decorated by 1-forms). Suppose also that each form is either purely holomorphic or purely antiholomorphic (which we may do because the Hodge correlators are linear in the forms); let there be p and q forms of these types, respectively. Then, setting $d^{\mathbb{C}} = \partial - \overline{\partial}$, we define

$$c_T(x_0, \dots, x_n) = (-2)^k \binom{k}{\frac{1}{2}(k+p-q)}^{-1} \mathrm{Or}_T \int_{X^{V^\circ}} f_0\, d^{\mathbb{C}} f_1 \wedge \cdots \wedge d^{\mathbb{C}} f_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{2n-2}. \tag{1.3}$$



$$\int_{z_1, z_2} G(z_1, a)\, d^{\mathbb{C}} G(z_1, z_2) \wedge d^{\mathbb{C}} G(z_2, b) \wedge d^{\mathbb{C}} G(z_2, c) \wedge \omega(z_1)$$

Figure 1.1. One of the trees contributing to $\mathrm{Cor}_{\mathcal{H}}(a, b, c, \omega)$.

The Hodge correlator is the sum of such expressions over all plane trivalent trees,

$$\mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n) = \sum_T c_T\,(x_0, \ldots, x_n). \tag{1.4}$$

The Hodge correlator is independent of the choice of ordering of edges. As a function of the arguments that are points on X, it is either purely real or purely imaginary.

Figure 1.1 shows the integral corresponding to one of the two trees contributing to $\mathrm{Cor}_{\mathcal{H}}(a, b, c, \omega)$.

The Hodge correlators satisfy a family of *(first) shuffle relations*. For $i, j > 0$, let $\Sigma_{i,j}$ be the set of (i, j)-shuffles, permutations $\sigma \in S_{i+j}$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+j)$. The (i, j)-shuffle relation states:

$$\sum_{\sigma \in \Sigma_{i,j}} \mathrm{Cor}_{\mathcal{H}}(x_0, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(i+j)}) = 0. \tag{1.5}$$

1.1.2 Correlator coalgebra and fundamental group

We review the construction of Hodge and motivic correlators from [G9]. The Hodge correlator functions $\mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n)$ of the previous section are upgraded to elements of the Tannakian Lie coalgebra $\mathrm{Lie}^{\vee}_{\mathrm{Hod}}$ of the category of $\mathbb{R}$-mixed Hodge structures:

$$\mathrm{Cor}_{\mathrm{Hod}}(z_0, \ldots, z_n) \in \mathrm{Lie}^{\vee}_{\mathrm{Hod}}. \tag{1.6}$$

The upgraded Hodge correlators (1.6) satisfy the first shuffle relations, and their coproduct in the coalgebra $\mathrm{Lie}^{\vee}_{\mathrm{Hod}}$ is given by a simple formula, which we state below.

Hodge-theoretic setup

Let $\mathrm{MHS}_{\mathbb{R}}$ of be the tensor category of $\mathbb{R}$-mixed Hodge structures and $\mathrm{HS}_{\mathbb{R}}$ the category of $\mathbb{R}$-pure Hodge structures. Every object of $\mathrm{MHS}_{\mathbb{R}}$ is filtered by weight, and $\mathrm{MHS}_{\mathbb{R}}$ is generated by the simple objects $\mathbb{R}(p,q)+\mathbb{R}(q,p)$ $(p,q\in\mathbb{Z})$. (We write $\mathbb{R}(p)$ for $\mathbb{R}(p,p)$.) By Deligne's theory [D2], the cohomology of a (possibly singular) complex variety is a mixed Hodge structure.

The *Galois Lie algebra* of the category of mixed Hodge structures, $\mathrm{Lie}_{\mathrm{Hod}}$, is the algebra of tensor derivations of the functor $\mathrm{gr}^W : \mathrm{MHS}_{\mathbb{R}} \to \mathrm{HS}_{\mathbb{R}}$. It is a graded Lie algebra in the category $\mathrm{HS}_{\mathbb{R}}$, and $\mathrm{MHS}_{\mathbb{R}}$ is equivalent to the category of graded $\mathrm{Lie}_{\mathrm{Hod}}$-modules in $\mathrm{HS}_{\mathbb{R}}$. Let $\mathrm{Lie}_{\mathrm{Hod}}^{\vee}$ be its graded dual. A canonical *period map*

$$p : \mathrm{Lie}_{\mathrm{Hod}}^{\vee} \to \mathbb{R}$$

was defined in [G9], §1.11.

Let X be a smooth curve, $S\subset X$ a finite set of punctures, $s\in S$ a distinguished puncture (called the base point), and v_0 a distinguished tangent vector at s. The pronilpotent completion $\pi_1^{\mathrm{nil}}(X\setminus(S\cup\{s\}),v_0)$ of the fundamental group $\pi_1(X\setminus S,s)$ carries a mixed Hodge structure, depending on v_0, and thus there is a map

$$\mathrm{Lie}_{\mathrm{Hod}} \to \mathrm{Der}\left(\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X\setminus S, v_0)\right).$$

Hodge correlator coalgebra

The *Hodge correlator coalgebra* is defined as a vector space by

$$\mathcal{CL}_{X,S,v_0}^{\vee} := \frac{T(\mathbb{C}\,[S]^{\vee}\oplus H^1(X;\mathbb{C}))}{\text{relations}}\otimes H_2(X).$$

Note that $H_2(X)\cong\mathbb{R}(1)$. If $[h]\in H_2(X)$ is the fundamental class, we write $x(1)$ for $x\otimes[h]$. This coalgebra is graded by weight. It is more finely graded by the Hodge bidegree, or *type*, where points in S have type $(1,1)$, holomorphic and antiholomorphic 1-forms have type $(1,0)$ or $(0,1)$, respectively, and $H_2(X)$ has type $(-1,-1)$, extended to be additive with respect to the tensor product. The weight of an element of type (p,q) is $p+q$. This algebra also carries a filtration by depth, defined analogously to the depth for the Hodge correlator functions introduced above.

The relations are the following:

(1) Cyclic symmetry: $x_0\otimes\cdots\otimes x_n = x_1\otimes\cdots\otimes x_n\otimes x_0$.

(2) (First) shuffle relations:

$$\sum_{\sigma\in\Sigma_{i,j}} x_0\otimes x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(i+j)}=0.$$

(3) The elements of nonpositive weight are set to 0.

An action of the graded dual Lie algebra $C\mathcal{L}_{X,S,v_0}$ by derivations on $\mathrm{gr}^W\pi_1^{\mathrm{nil}}(X\setminus S,v_0)\otimes\mathbb{C}$ was constructed in [G9]. This action is injective; its image consists of the *special derivations*

$$\mathrm{Der}^S\left(\mathrm{gr}^W\pi_1^{\mathrm{nil}}(X\setminus S,v_0)\otimes\mathbb{C}\right),$$

those which act by 0 on the loop around ∞ and preserve the conjugacy classes of all the loops $x\in S\setminus\{s\}$.

Dualizing this map composed with the action of $\mathrm{Lie}_{\mathrm{Hod}}$, we get the *Hodge correlator morphism* of Lie coalgebras:

$$\mathrm{Cor}_{\mathrm{Hod}}:C\mathcal{L}^\vee_{X,S,v_0}\to\mathrm{Lie}^\vee_{\mathrm{Hod}}.$$

Let $\mathrm{Lie}^\vee_{\mathrm{Hod}}(X,S,v_0)$ denote the image of this action, and let $\mathrm{Lie}^\vee_{\mathrm{Hod}}(X,S)$ denote the algebra generated by the $\mathrm{Lie}^\vee_{\mathrm{Hod}}(X,S,v_0)$ for all choices of base point. We will use the shorter notation $\mathrm{Cor}_{\mathrm{Hod}}(x_0,\ldots,x_n)$ for $\mathrm{Cor}_{\mathrm{Hod}}\left((x_0\otimes\cdots\otimes x_n)(1)\right)$, or $\mathrm{Cor}_s(\ldots)$, when we wish to specify the base point.

The Lie coalgebra structure on $C\mathcal{L}^\vee_{X,S,v_0}$ has a simple description on the generators. There are two terms in the coproduct, δ_{S} and δ_{Cas}, which are each sums over "cuts" of the element

$$C=(x_0\otimes\cdots\otimes x_n)\otimes[h],$$

which we picture as $x_0,\ldots,x_n$ written counterclockwise around a circle.

(1) Term δ_S: Consider a line inside the circle beginning at a point on the circle labeled by a puncture x_i and ending between two adjacent points. It cuts the circle into two parts C_1 and C_2, which share only the point x_i, where C_1 lies clockwise of x_i. This contributes to the coproduct the term $C_1\wedge C_2$, and $\delta_{\mathrm{S}}C$ is the sum of these terms over all such cuts. That is,

$$\delta_SC=\sum_{\substack{\mathrm{cyc}\\x_0\in S}}\sum_{p=1}^n\left(\left(x_0\otimes x_p\otimes\cdots\otimes x_n\right)\otimes[h]\right)\wedge\left(\left(x_0\otimes x_1\otimes\cdots\otimes x_{p-1}\right)\otimes[h]\right),\tag{1.7}$$

where the outer sum is only taken over those cyclic reorderings where v_0 is a puncture. (See Figure 1.2, top.)

(2) Term δ_{Cas}: Consider a line inside the circle beginning between two points y_1 and z_1 and ending between

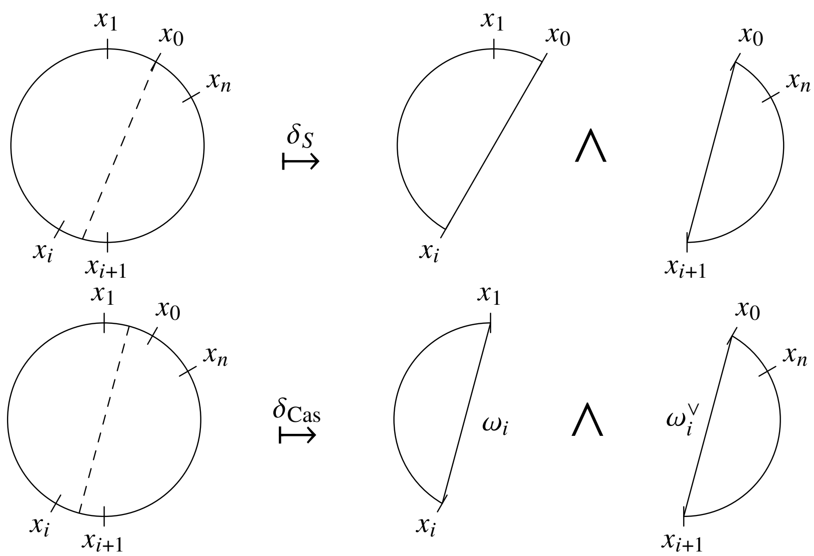


Figure 1.2. *Above:* The typical term in the δ_S component of the coproduct.
Below: The typical term in the δ_{Cas} component.

two points y_2 and z_2. It cuts the circle into two parts C_1 and C_2, in which y_1 and z_2 are adjacent and in which y_2 and z_1 are adjacent. We insert a point labeled ω between y_1 and z_2 on C_1 and a point labeled $\omega^\vee$ between z_2 and y_1 on C_2 to obtain C_1' and C_2', then take the sum over ω in a fixed symplectic basis $\{\omega_i\}_{i=0}^{2g(X)}$ of $H^1(X;\mathbb{C})$. This contributes the term $C_1' \wedge C_2'$, and δ_{Cas} is the sum of these terms over all such cuts. That is,

$$\delta_{\mathrm{Cas}} C = \sum_{p=0}^{n}\sum_{q=0}^{n}\sum_{i=1}^{2g(X)} \left(\left(x_p \otimes \cdots \otimes x_{q-1} \otimes \omega_i\right) \otimes [h]\right) \wedge \left(\left(x_q \otimes \cdots \otimes x_{p-1} \otimes \omega_i^\vee\right) \otimes [h]\right). \tag{1.8}$$

(See Figure 1.2, bottom.)

The term δ_{Cas} is absent if $X = \mathbb{P}^1$. If X has positive genus, $\mathcal{C}\mathcal{L}^\vee_{X,S,v_0}$ is graded by weight and filtered by depth, and the terms δ_{Cas} disappear in the associated graded $\mathrm{gr}^D \mathcal{C}\mathcal{L}^\vee_{X,S,v_0}$.

Period map and Hodge correlator functions

Recall that the Hodge correlator functions $\mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n)$ satisfy cyclic symmetry and shuffle relations, so we may also denote by $\mathrm{Cor}_{\mathcal{H}}$ the function

$$\mathrm{Cor}_{\mathcal{H}} : \mathcal{C}\mathcal{L}^\vee_{X,S,v_0} \to \mathbb{C},$$
$$(x_0 \otimes \cdots \otimes x_n)(1) \mapsto \mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n).$$

The dual to the Hodge correlator $\mathrm{Cor}_{\mathcal{H}} : \mathcal{CL}^{\vee}_{X,S,v_0} \to \mathbb{C}$ is called the *Green operator* $\mathbf{G}_{v_0}$. It can be viewed as a special derivation of $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_0) \otimes \mathbb{C}$, and defines a $\mathbb{R}$-mixed Hodge structure on $\pi_1^{\mathrm{nil}}(X \setminus S, v_0)$. An element $x \in \mathcal{CL}^{\vee}_{X,S,v_0}$ of type (p,q) provides a framing $\mathbb{R}(p,q) + \mathbb{R}(q,p) \to \mathrm{gr}^W_{p+q}\pi_1^{\mathrm{nil}}(X \setminus S, v_0)$, and $\mathrm{Cor}_{\mathrm{Hod}}(x)$ is the element of $\mathrm{Lie}^{\vee}_{\mathrm{Hod}}$ induced by this framing.

As made precise by a main result of [G9], $\mathrm{Cor}_{\mathcal{H}}$ factors through the Hodge correlator map to $\mathrm{Lie}^{\vee}_{\mathrm{Hod}}$ and the period map $\mathrm{Lie}^{\vee}_{\mathrm{Hod}} \to \mathbb{C}$, and the resulting mixed Hodge structure on π_1^{nil} coincides with the standard one.

Theorem 1.1 ([G9], Theorem 1.12)**.** *(a) For* $x \in \mathcal{CL}^{\vee}_{X,S,v_0}$, $\mathrm{Cor}_{\mathcal{H}}(x) = (2\pi i)^{-n} p(\mathrm{Cor}_{\mathrm{Hod}}(x))$*, where p is the canonical period map* $\mathrm{Lie}^{\vee}_{\mathrm{Hod}} \to \mathbb{R}$.

(b) The mixed Hodge structure on π_1^{nil} *determined by the dual Hodge correlator map coincides with the standard mixed Hodge structure on* π_1^{nil}.

Furthermore, let X/B be a smooth curve over a base B. For a collection of nonintersecting sections S and choice of relative tangent vector v_0, we can analogously define $\mathcal{CL}^{\vee}_{X/B,S,v_0}$. In this setting, for $x \in \mathcal{CL}^{\vee}_{X/B,S,v_0}$, [G9] constructs a connection on the fiberwise $\mathrm{Cor}_{\mathrm{Hod}}(x)$ that makes $\mathrm{Cor}_{\mathrm{Hod}}(x)$ a variation of mixed Hodge structures over B. We have the following essential fact, which follows from the Griffiths transversality condition:

Lemma 1.2. *If* $x \in \mathcal{CL}^{\vee}_{X/B,S,v_0}$ *of type* (p,q)*, and weight* $p+q > 2$*, has* $\delta(\mathrm{Cor}_{\mathrm{Hod}}(x)) = 0$*, and* $\mathrm{Cor}_{\mathcal{H}}(x|_b) = 0$ *at some* $b \in B$*, then* $\mathrm{Cor}_{\mathrm{Hod}}(x) = 0$.

Proof. If $\delta(\mathrm{Cor}_{\mathrm{Hod}}(x)) = 0$, then $\mathrm{Cor}_{\mathrm{Hod}}(x) \in \mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(p,q) + \mathbb{R}(q,p))$, which is one-dimensional and rigid by the Griffiths transversality condition. Hence the variation is constant and captured by the period map $\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(p,q) + \mathbb{R}(q,p)) \to \mathbb{C}$. □

Motivic correlators and motivic fundamental group

Let F be a number field. There is a semisimple abelian category $\mathcal{PM}_F$ of Grothendieck pure motives over F and a functor $H : \mathbf{SmProj}_F \to \mathcal{PM}_F$ assigning to every smooth projective variety over F the sum of its motivic cohomology objects:

$$H(X) = \bigoplus_{i=0}^{2\dim(X)} H^i(X).$$

This category is graded by the *weight*, where the weight of $H^i(X)$ is i. There is an invertible Tate object $\mathbb{Q}(1)$ of weight -2; we write $M(n)$ for the Tate twist $M \otimes \mathbb{Q}(1)^{\otimes n}$. The various realization functors respect the weight.

Beilinson's conjectures ([B3]) predict that there is a category $\mathcal{MM}_F$ of mixed motives over F. Every object in $\mathcal{MM}_F$ should have a weight filtration, and there should be a functor $\mathrm{gr}^W : \mathcal{MM}_F \to \mathcal{PM}_F$, where

$\mathcal{PM}_F$ is the category of pure motives over F. For every embedding $\sigma : F \to \mathbb{C}$, there should be a realization functor $r_\sigma : \mathcal{MM}_F \to \mathrm{MHS}_F$. For every simple object $M \in \mathcal{MM}_F$, there should be an injective regulator map

$$\mathrm{reg} : \mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{Q}(0), M) \to \bigoplus_{F\to\mathbb{C}/\text{complex conj.}} \mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{R}}(\mathbb{R}(0), r_\sigma(M))).$$

Following constructions from the theory of Tannakian categories (see [DM]), define the *fundamental (motivic) Lie algebra* $\mathrm{Lie}_{\mathrm{Mot}/F}$ to be the algebra of tensor derivations of the functor gr^W, a graded Lie algebra in the category $\mathcal{PM}_F$. The category $\mathcal{MM}_F$ is equivalent to the category of graded $\mathrm{Lie}_{\mathrm{Mot}/F}$-modules. The Hodge realization of a mixed motive should be a mixed Hodge structure; an embedding σ induces a map $r_\sigma : \mathrm{Lie}^\vee_{\mathrm{Mot}/F} \to \mathrm{Lie}^\vee_{\mathrm{Hod}}$.

Let X be a curve defined over F, $S \subset X(F)$ a finite set of punctures, and v_0 the distinguished tangent vector at $s \in S$. There is expected to be a *motivic fundamental group* $\pi_1^{\mathrm{Mot}}(X \setminus S, v_\infty)$, a prounipotent group scheme in the category $\mathcal{MM}_F$. The Hodge realization of its Lie algebra should be $\pi_1^{\mathrm{nil}}(X \setminus S, v_0)$. As it is an object in $\mathcal{MM}_F$, there is an action $\mathrm{Lie}_{\mathrm{Mot}/F} \to \mathrm{Der}\left(\mathrm{gr}^W \pi_1^{\mathrm{Mot}}\right)$.

The construction of the Hodge correlator coalgebra $\mathcal{CL}^\vee_{X,S,v_0}$ can be upgraded to the motivic setting, simply by replacing all the Hodge-theoretic objects by their motivic avatars. For example, the definition of the *motivic correlator coalgebra* mimics that of its Hodge realization:

$$\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0}\right)^\vee := \frac{T\left((\mathbb{Q}(1)^S)^\vee \oplus H^1(X)\right)}{\text{relations}} \otimes H_2(X),$$

a graded Lie coalgebra in the category of pure motives over F, where the relations imposed are the cyclic symmetry, first shuffles, and quotient by nonpositive weight. Then $\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0}$ is isomorphic to the algebra of special derivations of $\mathrm{gr}^W \pi_1^{\mathrm{Mot}}(X - S, v_0)$, and there is a *motivic correlator map*

$$\mathrm{Cor}_{\mathrm{Mot}} : \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0}\right)^\vee \to \mathrm{Lie}^\vee_{\mathrm{Mot}/F}.$$

We will write $\mathrm{Cor}_{\mathrm{Mot}}(x_0, \ldots, x_n)$ for $\mathrm{Cor}_{\mathrm{Mot}}((x_0 \otimes \cdots \otimes x_n)(1))$.

Fix an embedding $\sigma : F \to \mathbb{C}$. We have the composition of the realization functor with the period map:

$$\mathrm{Cor}_{\mathcal{H}} \circ r_\sigma : \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0}\right)^\vee \otimes \mathbb{C} \to \mathcal{CL}^\vee_{X,S,v_0} \otimes \mathbb{C} \to \mathbb{C}.$$

By Theorem 1.1, it coincides with the composition

$$\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee \to \mathrm{Lie}^\vee_{\mathrm{Mot}} \to \mathrm{Lie}^\vee_{\mathrm{Hod}} \to \mathbb{C}.$$

We can summarize all of the objects and maps as follows, where the top row is conjectural:

$$\begin{tikzcd}
\mathrm{Der}^S(\mathrm{gr}^W \pi_1^{\mathrm{Mot}}(X \setminus S, v_0))^\vee \arrow[r, dash] & (\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0})^\vee \arrow[r, "\mathrm{Cor}_{\mathrm{Mot}}"] \arrow[d, "r"] & \mathrm{Lie}^\vee_{\mathrm{Mot}/F} \arrow[d, "r"] \\
\mathrm{Der}^S(\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_0))^\vee \arrow[r, dash] & \mathcal{CL}^\vee_{X,S,v_0} \arrow[r, "\mathrm{Cor}_{\mathrm{Hod}}"] \arrow[dr, "\mathrm{Cor}_{\mathcal{H}}"'] & \mathrm{Lie}^\vee_{\mathrm{Hod}} \arrow[d, "p"] \\
& & \mathbb{C}.
\end{tikzcd} \tag{1.9}$$

Relations among the motivic correlators can be proven by showing that they hold in the Hodge realization under any complex embedding: there is the following fact, which is an immediate consequence of the (hypothetical) injectivity of the regulator and Lemma 1.2.

Lemma 1.3. *Suppose* $x \in \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee$ *is of type* (p,q) *with weight* $p+q > 2$, $\delta\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$, *and* $\mathrm{Cor}_{\mathcal{H}}(r(x)) = 0$ *for every embedding* $r : F \to \mathbb{C}$. *Then* $\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$.

This fact allows us to lift relations on Hodge correlators to relations on motivic correlators. In particular, all results in Chapter 3 – the second shuffle relations for Hodge correlators and the map from the Bianchi complexes to an algebra of Hodge correlators – should hold with "Hodge" replaced by "motivic".

1.2 Mixed Tate motives and correlators on rational curves

In this section, we trace in greater detail the constructions of §1.1 in the Tate case. This will mainly be used in Chapter 2.

1.2.1 Hodge correlators

Hodge correlator integrals

Let $X = \mathbb{P}^1(\mathbb{C})$. The Hodge correlator integrals $\mathrm{Cor}_{\mathcal{H}}$ defined in §1.1.1 can be thought of as functions of $n+1$ complex variables $\mathrm{Cor}_{\mathcal{H}}(z_0, \dots, z_n)$, defined by the expression (1.4) via the Green's function $G_\infty(x,y) = (2\pi i)^{-1} \log|x-y|$.

Let us consider the simplest examples. The Hodge correlator integral of weight 2 (semiweight[1] 1) is shown in Figure 1.3.

1. We will very soon reindex the weight filtrations for mixed Hodge-Tate structures, renaming the semiweight to the weight. In anticipation of this, on the next several pages, whenever weight is mentioned we also write the semiweight. (This convention will be used in this section and in Chapter 2.)

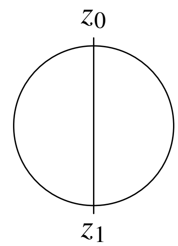


Figure 1.3. Hodge correlator with two arguments: $\mathrm{Cor}_{\mathcal{H}}(z_0, z_1) = (2\pi i)^{-1} \log |z_0 - z_1|$.

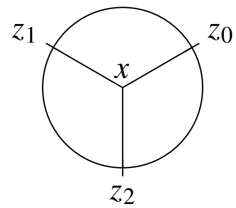


Figure 1.4. Hodge correlator with three arguments.

In weight 4 (semiweight 2), the Hodge correlator integrals are given by

$$\mathrm{Cor}_{\mathcal{H}}(z_0, z_1, z_2) = -\frac{1}{8} \int_x (2\pi i)^{-3} \log |x - z_0| \; d^{\mathbb{C}} \log |x - z_1| \wedge d^{\mathbb{C}} \log |x - z_2| \, .$$

This integral is described by the Feynman diagram in Figure 1.4.

Recall the single-valued version of the dilogarithm, called the Bloch-Wigner function:

$$\mathcal{L}_2(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \log |z| \arg(1 - z)$$

The weight-4 (semiweight-2) Hodge correlator integral can be calculated explicitly as

$$\mathrm{Cor}_{\mathcal{H}}(z_0, z_1, z_2) = (2\pi i)^{-2} \mathcal{L}_2 \left(\frac{z_1 - z_0}{z_2 - z_0} \right). \tag{1.10}$$

The Hodge correlator integrals satisfy *dihedral symmetry* relations:

$$\begin{aligned} \mathrm{Cor}_{\mathcal{H}}(z_0, z_1, \ldots, z_n) &= \mathrm{Cor}_{\mathcal{H}}(z_1, \ldots, z_n, z_0) \\ &= (-1)^{n+1} \mathrm{Cor}_{\mathcal{H}}(z_n, \ldots, z_1, z_0). \end{aligned}$$

One can show directly using (1.3) that the Hodge correlators are invariant under an additive shift of the arguments. In weight > 2 (semiweight > 1), they are also invariant under a multiplicative shift:

$$\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n) = \mathrm{Cor}_{\mathcal{H}}(z_0 + a, \ldots, z_n + a),$$
$$\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n) = \mathrm{Cor}_{\mathcal{H}}(az_0, \ldots, az_n) \quad (a \in \mathbb{C}^*, n > 1).$$

Mixed Hodge theory

A mixed Hodge structure is a *mixed Hodge-Tate structure* if the Hodge number $h^{p,q} = 0$ unless $p = q$. When considering mixed Hodge-Tate structures, we reindex the Hodge filtration by semiweight (so $\mathbb{R}(1)$ has weight -1, rather than -2). Mixed Hodge-Tate structures are iterated extensions of the one-dimensional pure mixed Hodge-Tate structures of weight $-n$, denoted $\mathbb{R}(n)$.

Let $\mathrm{MHT}_{\mathbb{R}}$ of be the tensor category of $\mathbb{R}$-mixed Hodge-Tate structures (a subcategory of $\mathrm{MHS}_{\mathbb{R}}$) and $\mathrm{HTS}_{\mathbb{R}}$ the category of $\mathbb{R}$-pure Hodge-Tate structures (a subcategory of $\mathrm{HS}_{\mathbb{R}}$). Every object of $\mathrm{MHT}_{\mathbb{R}}$ is filtered by weight, and $\mathrm{MHT}_{\mathbb{R}}$ is generated by the simple objects $\mathbb{R}(n)$, the pure Hodge-Tate structures of weight $-n$. The cohomology of a punctured projective line is a mixed Hodge-Tate structure, nontrivial in weights 0 and 1.

The *Galois Lie algebra* of the category of mixed Hodge-Tate structures, $\mathrm{Lie}_{\mathrm{HT}}$, is the algebra of tensor derivations of the functor $\mathrm{gr}^W : \mathrm{MHT}_{\mathbb{R}} \to \mathrm{HTS}_{\mathbb{R}}$. It is a graded Lie algebra in the category $\mathrm{HTS}_{\mathbb{R}}$, and $\mathrm{MHT}_{\mathbb{R}}$ is equivalent to the category of graded $\mathrm{Lie}_{\mathrm{HT}}$-modules in $\mathrm{HTS}_{\mathbb{R}}$. Let $\mathrm{Lie}_{\mathrm{HT}}^{\vee}$ be its graded dual.

Let $\mathrm{Lie}_{\mathrm{HT}} = \mathrm{Der}^{\otimes}\mathrm{gr}^W$ be the graded Lie algebra in $\mathrm{HTS}_{\mathbb{R}}$ of tensor derivations of the functor gr^W. That is, every mixed Hodge-Tate structure X determines an action

$$\mathrm{Lie}_{\mathrm{HT}} \to \mathrm{Der}\left(\mathrm{gr}^W X\right).$$

Let $\mathrm{Lie}_{\mathrm{HT}}^{\vee}$ be the graded dual of $\mathrm{Lie}_{\mathrm{HT}}$.

$\mathrm{Lie}_{\mathrm{HT}}$ is free on

$$\bigoplus_{n<0} \mathrm{Ext}^1_{\mathrm{MHT}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(n))^{\vee} \otimes_{\mathrm{End}\mathbb{R}(n)} \mathbb{R}(n).$$

A *framing* of a mixed Hodge-Tate structure V of weight n consists of a pair of morphisms $\mathbb{R}(0) \to \mathrm{gr}^W_0 V$, $\mathrm{gr}^W_{-2n} V \to \mathbb{R}(n)$. The isomorphism classes of framed $\mathbb{R}$-mixed Hodge-Tate structures generate a Hopf algebra $\mathcal{H}_{\bullet}$, with the structure defined by [BGSV], which is canonically isomorphic to the dual of the universal enveloping algebra of $\mathrm{Lie}_{\mathrm{HT}}$. An element of $\mathrm{Lie}_{\mathrm{HT}}^{\vee}$ of weight n is represented by a framed $\mathbb{R}$-mixed Hodge-Tate structure of weight n, modulo products in $\mathcal{H}$, that is,

$$\mathrm{Lie}_{\mathrm{HT}/B}^{\vee} \cong \frac{\mathcal{H}}{\mathcal{H}_{>0} \cdot \mathcal{H}_{>0}}. \tag{1.11}$$

The $\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(n))$ are trivial for $n \geq 0$ and 1-dimensional for $n < 0$, in which case

$$\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(n)) = (\mathbb{R}(n) \otimes \mathbb{C})/\mathbb{R}(n) = \mathbb{R}(n) \otimes_{\mathbb{R}} i\mathbb{R}.$$

According to [G9], a choice of generators n_w of $\mathrm{Lie}_{\mathrm{HT}} \otimes \mathbb{C}$ satisfying $n_w = -\overline{n}_w$ amounts to a map

$$\mathrm{Lie}_{\mathrm{HT}}^{\vee} \to \bigoplus_{n<0} \mathrm{Ext}^1_{\mathrm{MHT}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(n)) \otimes \mathbb{R}(n)^{\vee} = \bigoplus_{n<0} \mathbb{R}(n) \otimes_{\mathbb{R}} i\mathbb{R},$$

and thus defines a canonical period map

$$p : \mathrm{Lie}_{\mathrm{HT}}^{\vee} \to \mathbb{R}.$$

Such generators were originally defined by Deligne for the larger category of $\mathbb{R}$-mixed Hodge structures ([D4]). However, we use the different set of generators proposed by Goncharov ([G9]), the *Green's operators* G_w. They have the property that, for Hodge structures varying over a base, the Griffiths transversality condition needed to define variations of Hodge structures is expressed by a Maurer-Cartan differential equation on the G_w, which is essential to the construction of Hodge correlators. Contrary to this, the differential equations for Deligne's generators are difficult to write.

Recall that a variation of $\mathbb{R}$-mixed Hodge-Tate structures on a complex variety B is a variation of the linear data of $\mathbb{R}$-mixed Hodge-Tate structure that satisfies the Griffiths transversality condition. A consequence of the transversality condition is that for $n > 1$, $\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(n))$ is rigid in the category of variations of mixed Hodge-Tate structures over B: if the coproduct of a variation of Hodge-Tate structures of weight $w > 1$ is 0, then the variation is isomorphic to a constant one.

Pronilpotent fundamental group

Let $X = \mathbb{P}^1(\mathbb{C})$, $S \subset X$ a finite set of punctures containing ∞, and $v_\infty = \frac{-1}{z^2}\frac{d}{dz}$ a distinguished tangent vector at ∞. Let $\pi_1 = \pi_1(X \setminus S, \infty)$ be the classical fundamental group. The group algebra $A = \mathbb{Q}[\pi_1]$ is a free group generated by loops around the points of $S \setminus \{\infty\}$. Let $\mathcal{I} = \ker(A \to \mathbb{Q})$ be the augmentation ideal. Then form a Hopf algebra

$$A^{\mathrm{nil}}(X \setminus S, v_\infty) := \varprojlim \Big(\cdots \to A/\mathcal{I}^{n+1} \to A/\mathcal{I}^n \to \cdots \to A \Big),$$

with coproduct defined by $g \to g \otimes g$ for $g \in \pi_1$. The subset of primitive elements is denoted $\pi_1^{\mathrm{nil}}(X \setminus S, v_\infty)$. It is actually a pronilpotent Lie algebra, the Maltsev completion of π_1.

There is a canonical weight filtration on $H_1(X \setminus S, \mathbb{Q})$, where the loops around punctures lie in weight -1.

This induces a weight filtration W on A^{nil}, and we have

$$\mathrm{gr}^W A^{\mathrm{nil}}(X \setminus S, v_\infty) = T(\mathrm{gr}^W H_1(X \setminus S, \mathbb{Q})).$$

L_{X,S,v_∞} be the free Lie algebra generated by $\mathbb{C}\,[S \setminus \{\infty\}]$. Then there is a canonical isomorphism

$$L_{X,S,v_\infty} \cong \mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_\infty) \otimes \mathbb{C}.$$

Let $X = \mathbb{P}^1(\mathbb{C})$, $S \subset X$ a finite set of punctures containing ∞, and $v_\infty = \frac{-1}{z^2}\frac{\partial}{\partial z}$ a distinguished tangent vector at ∞. The pronilpotent completion $\pi_1^{\mathrm{nil}}(X \setminus (S \cup \{\infty\}), v_\infty)$ of the fundamental group $\pi_1(X \setminus S, \infty)$ carries a mixed Hodge-Tate structure, depending on v_∞, and thus there is a map

$$\mathrm{Lie}_{\mathrm{HT}} \to \mathrm{Der}\left(\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_\infty)\right).$$

Hodge correlator coalgebra

The *Hodge correlator coalgebra* for a rational curve $X \setminus S$ is defined as

$$\mathcal{CL}^\vee_{X,S,v_\infty} := \frac{T(\mathbb{C}\,[S \setminus \{\infty\}]^\vee)}{\text{relations}} \otimes H_2(X).$$

Note that $H_2(X) \cong \mathbb{R}(1)$. If $[h] \in H_2(X)$ is the fundamental class, we write $x(1)$ for $x \otimes [h]$.

The relations are the following:

(1) Cyclic symmetry: $x_0 \otimes \cdots \otimes x_n = x_1 \otimes \cdots \otimes x_n \otimes x_0$.

(2) (First) shuffle relations:

$$\sum_{\sigma \in \Sigma_{p,q}} x_0 \otimes x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(p+q)} = 0.$$

(3) Set to 0 the elements of nonpositive weight (elements $(x_0)(1)$ and $(\mathbf{1})(1)$).

The Lie cobracket in $\mathcal{CL}^\vee_{X,S,v_0}$, given by the sum of terms (1.7) and (1.8), simplifies to

$$\delta\left((x_0 \otimes \cdots \otimes x_n)(1)\right) = \sum_{\mathrm{cyc}} \sum_{i=1}^{n-1} \left((x_0 \otimes x_1 \otimes \cdots \otimes x_i)(1)\right) \wedge \left((x_0 \otimes x_{i+1} \otimes \cdots \otimes x_n)(1)\right). \tag{1.12}$$

The algebra $\mathcal{CL}_{X,S,v_\infty}$ acts by derivations on L_{X,S,s_0}, and the action

$$\mathcal{CL}_{X,S,v_\infty} \to \mathrm{Der}\left(L_{X,S,v_\infty}\right)$$

is injective. Its image consists of the *special derivations* $\mathrm{Der}^S\left(L_{X,S,v_\infty}\right)$, those which act by 0 on the loop around ∞ and preserve the conjugacy classes of all the loops around punctures $s \in S \setminus \{\infty\}$.

Dualizing this map composed with the action of $\mathrm{Lie}_{\mathrm{HT}}$, we get the *Hodge correlator map* of Lie coalgebras:

$$\mathrm{Cor}_{\mathrm{Hod}} : \mathcal{CL}^\vee_{X,S,v_\infty} \to \mathrm{Lie}^\vee_{\mathrm{HT}}.$$

We will also write $\mathrm{Cor}_{\mathrm{Hod}}(x_0, \ldots, x_n)$ for $\mathrm{Cor}_{\mathrm{Hod}}\left((x_0 \otimes \cdots \otimes x_n)(1)\right)$.

Period map and Hodge correlator functions

Recall that the Hodge correlator functions $\mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n)$ satisfy cyclic symmetry and shuffle relations, so we may also denote by $\mathrm{Cor}_{\mathcal{H}}$ the function

$$\begin{aligned} \mathrm{Cor}_{\mathcal{H}} : \mathcal{CL}^\vee_{X,S,v_\infty} &\to \mathbb{C}, \\ (x_0 \otimes \cdots \otimes x_n)(1) &\mapsto \mathrm{Cor}_{\mathcal{H}}(x_0, \ldots, x_n). \end{aligned}$$

Recall that the Green operator $\mathbf{G}_{v_\infty}$ is the dual to the Hodge correlator map $\mathrm{Cor}_{\mathcal{H}} : \mathcal{CL}^\vee_{X,S,v_\infty} \to \mathbb{C}$. It acts as a special derivation of $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_\infty) \otimes \mathbb{C}$, and defines a $\mathbb{R}$-mixed Hodge structure on $\pi_1^{\mathrm{nil}}(X \setminus S, v_\infty)$. An element $x \in \mathcal{CL}^\vee_{X,S,v_\infty}$ provides a framing $\mathbb{R}(n) \to \mathrm{gr}^W_{2n} \pi_1^{\mathrm{nil}}(X \setminus S, v_\infty)$, and $\mathrm{Cor}_{\mathrm{Hod}}(x)$ is the element of $\mathrm{Lie}^\vee_{\mathrm{HT}}$ induced by this framing.

We then have the following variant of Theorem 1.1:

Theorem 1.4 ([G9], Theorem 1.12)**.** *(a) Let $x \in \mathcal{CL}^\vee_{X,S,v_\infty}$ be homogeneous of weight n. Then* $\mathrm{Cor}_{\mathcal{H}}(x) = (2\pi i)^{-n} p(\mathrm{Cor}_{\mathrm{Hod}}(x))$*, where p is the canonical period map* $\mathrm{Lie}^\vee_{\mathrm{HT}} \to \mathbb{R}$.

(b) The mixed Hodge structure on π_1^{nil} determined by the dual Hodge correlator map coincides with the standard mixed Hodge structure on π_1^{nil}.

Correlators in families

The construction of the Hodge correlator coalgebra can be performed over a base. Let $X \to B$ be a smooth family of genus-0 curves. Generalizing from the case of B a point, one simply replaces the punctures s by *nonintersecting* sections $s : B \to X$ and the tangential base point by a nonvanishing section $v_\infty : B \to T^1_{X/B}$ factoring through a distinguished section $s_\infty : B \to X$. This construction yields a family of coalgebras

$$\left(\mathcal{CL}^\vee_{X_t,\{(s_i)_t\},(v_\infty)_t}\right)_{t \in B}. \tag{1.13}$$

We will denote this coalgebra by $\mathcal{CL}^{\vee}_{X/B,S,v_\infty}$ when the objects X, S, v_∞ vary over B.

The Green's function $(2\pi i)^{-1}\log|x-y|$, used in the definition of the Hodge correlator, becomes a distribution on $X \times_B X$ with logarithmic singularities along the relative divisors $x = s_\infty$, $y = s_\infty$, and $x = y$. As we explain below, the higher-weight correlators also determine smooth variations over the base. In particular, the period map $\mathrm{Cor}_{\mathcal{H}} : \mathcal{CL}^{\vee}_{X,S,v_\infty} \to \mathbb{C}$ is upgraded to a map

$$\mathrm{Cor}_{\mathcal{H}} : \mathcal{CL}^{\vee}_{X/B,S,v_\infty} \to \mathcal{A}^0_B,$$

and the map $\mathrm{Cor}_{\mathrm{Hod}}$ to a map

$$\mathrm{Cor}_{\mathrm{Hod}} : \mathcal{CL}^{\vee}_{X/B,S,v_\infty} \to \mathrm{Lie}^{\vee}_{\mathrm{HT}/B}$$

to the fundamental Lie coalgebra of the category of variations of $\mathbb{R}$-mixed Hodge-Tate structures.

The case of specialization at intersecting sections, as well as degeneration to nodal curves, is related to the behavior of the Hodge structure on π_1^{nil} at the boundary of the moduli space of Riemann surfaces with n punctures. We will examine this question in §2.3.

As X, S, v_0 vary over the moduli space $\mathcal{M}'_{0,n}$ of Riemann surfaces of genus 0 with n distinct marked points and a tangential base point v_0, we get a family $\mathbf{V}$ of framed $\mathbb{R}$-mixed Hodge structures on $\pi_1^{\mathrm{nil}}(X \setminus S, s_0)$. Theorem 1.4 is generalized to the following.

Theorem 1.5 ([G9], Theorem 1.12)**.** *(a) There is a flat connection on* $\mathbf{V}$ *making it a variation of mixed Hodge structures over* $\mathcal{M}'_{0,n}$.

(b) This variation coincides with the standard variation of mixed Hodge structures on π_1^{nil}.

A consequence of Theorem 1.5 is that the coalgebra structure on $\mathcal{CL}^{\vee}_{X,S,s_0}$ should translate into differential equations on the periods over $\mathcal{M}'_{0,n}$. We now describe these equations.

Extend the period map $\mathrm{Cor}_{\mathcal{H}}$ to a map defined on homogeneous elements by

$$\begin{aligned}
\mathrm{Cor}_{\mathcal{H}} : \wedge^2 \mathcal{CL}_{X/B,S,v_\infty} &\to \mathcal{A}^1_B, \\
C_1 \wedge C_2 &\mapsto \frac{2w_2-1}{2(w-1)}\mathrm{Cor}_{\mathcal{H}}(C_2)\, d^{\mathbb{C}}_B \mathrm{Cor}_{\mathcal{H}}(C_1) \\
&\quad - \frac{2w_1-1}{2(w-1)}\mathrm{Cor}_{\mathcal{H}}(C_1)\, d^{\mathbb{C}}_B \mathrm{Cor}_{\mathcal{H}}(C_2),
\end{aligned}$$

where $w_i = \mathrm{wt}\, C_i$ and $w = w_1 + w_2$. Then we have a diagram that commutes in weight > 1:

$$\begin{tikzcd} \mathcal{CL}_{X/B,S,v_\infty} \arrow[r, "\delta"] \arrow[d, "\mathrm{Cor}_{\mathcal{H}}"'] & \bigwedge^2_B \mathcal{CL}_{X/B,S,v_\infty} \arrow[d, "\mathrm{Cor}_{\mathcal{H}}"] \\ \mathcal{A}^0_B \arrow[r, "d_B"] & \mathcal{A}^1_B. \end{tikzcd} \tag{1.14}$$

We emphasize that we have so far required the sections to be nonintersecting. In §2.3 we will prove a specialization theorem, which allows to pass to the boundary of $\mathcal{M}'_{0,n}$. It will imply a statement about periods:

Theorem 1.6. *The Hodge correlators* $\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n)$ *are continuous on* $\mathbb{C}^{n+1} \setminus \{z_0 = \cdots = z_n\}$.

Distribution relations

The formula expressing how the Hodge correlators transform under endomorphisms of X appears in [G9], Lemma 12.3. We translate this result to our setting.

Consider the map $[l] : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^l$ ($l \in \mathbb{Z}_{>0}$). Let $S' = [l]^{-1}(S)$. It induces a map

$$\begin{gathered} [l]^* : \mathcal{CL}_{X,S,v_\infty} \to \mathcal{CL}_{X,S',v_\infty}, \\ (z_0 \otimes \cdots \otimes z_n)(1) \mapsto \frac{1}{l}(z'_0 \otimes \cdots \otimes z'_n)(1), \end{gathered}$$

where

$$z'_i = \begin{cases} \sum_{y_i^l = z_i} (y_i) & z_i \neq 0 \\ l \cdot (0) & z_i = 0 \end{cases}.$$

The factor $\frac{1}{l}$ comes from the degree of the induced map on $H_2(X)$.

Then the diagram commutes:

$$\begin{tikzcd} \mathcal{CL}_{X,S,v_\infty} \arrow[r, "{[l]^*}"] \arrow[dr, "\mathrm{Cor}_{\mathrm{Hod}}"'] & \mathcal{CL}_{X,S',v_\infty} \arrow[d, "\mathrm{Cor}_{\mathrm{Hod}}"] \\ & \mathrm{Lie}^\vee_{\mathrm{HT}}. \end{tikzcd}$$

1.2.2 Motivic correlators

Mixed motives

Let F be a number field and $\mathcal{MTM}_F$ the category of mixed Tate motives over F. It should be a full tensor subcategory of the conjectural mixed motive category $\mathcal{MM}_F$. It is generated by objects $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ for $n \in \mathbb{Z}$, where $\mathbb{Q}(1)$ is the Tate motive, pure of weight -1. This induces a canonical weight filtration on objects of $\mathcal{MTM}_F$. There is a functor $\mathrm{gr}^W : \mathcal{MTM}_F \to \mathcal{PM}_F$, where $\mathcal{PM}_F$ is the category of pure motives over F.

Such a category with desirable properties has been constructed by [DG]. If X is a rational curve, then $H(X)$ is a mixed Tate motive. The simple objects of $\mathcal{MTM}_F$ are $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, $n \in \mathbb{Z}$, and every object of $\mathcal{MTM}_F$ is an iterated extension of these objects. They satisfy

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Q}(m), \mathbb{Q}(n)) &= 0, \quad m < n; \\
\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) &= \begin{cases} 0 & n \leq 0 \\ K_{2n-1}(F) \otimes \mathbb{Q} & n > 0 \end{cases}, \\
\mathrm{Ext}^i(\mathbb{Q}(0), \mathbb{Q}(n)) &= 0, \quad i > 1.
\end{aligned}$$

When we speak about mixed Tate motives, the associated graded objects of the weight filtration are trivial in odd weight, so we reindex the filtration by semiweight (so $\mathbb{Q}(1)$ has weight -1, rather than -2).

Fundamental Lie algebra and period map

The Tannakian reconstruction theorem implies that there is a negatively graded Lie algebra $\mathrm{Lie}_{\mathrm{MT}/F}$ in the category $\mathcal{PM}_F$, the *fundamental (motivic Tate) Lie algebra*, such that $\mathcal{MTM}_F$ is canonically equivalent to the category of finite-dimensional graded $\mathrm{Lie}_{\mathrm{MT}/F}$-modules in $\mathcal{PM}_F$. That is, for any $X \in \mathcal{MTM}_F$, there is an action by derivations $\mathrm{Lie}_{\mathrm{MT}/F} \to \mathrm{Der}(\mathrm{gr}^W X)$. We prefer to study its graded dual $\mathrm{Lie}^\vee_{\mathrm{MT}/F}$.

Let $X = \mathbb{P}^1$, $S \subset X(F)$ a finite set of punctures containing ∞, and v_∞ the distinguished tangent vector at ∞. Deligne and Goncharov's *motivic fundamental group* ([DG]) $\pi_1^{\mathrm{Mot}}(X \setminus S, v_\infty)_{\mathrm{un}}$ is a prounipotent group scheme in the category $\mathcal{MTM}_F$. The Hodge realization of its Lie algebra is $\pi_1^{\mathrm{nil}}(X \setminus S, v_\infty)$. As it is an object in $\mathcal{MTM}_F$, there is an action $\mathrm{Lie}_{\mathrm{MT}/F} \to \mathrm{Der}\left(\mathrm{gr}^W \pi_1^{\mathrm{Mot}}\right)$.

An embedding $\sigma : F \to \mathbb{C}$ induces a *realization functor* $r : \mathcal{MTM}_F \to \mathrm{MHT}_{\mathbb{R}}$ and a map $r : \mathrm{Lie}^\vee_{\mathrm{MT}/F} \to \mathrm{Lie}^\vee_{\mathrm{HT}}$. This means that there is a period map $p \circ r_{\mathrm{Hod}} : \mathrm{Lie}^\vee_{\mathrm{MT}/F} \to \mathbb{R}$.

For every integer $n > 0$, there is the Beilinson regulator map

$$\mathrm{reg} : \mathrm{Ext}^1_{\mathcal{MTM}_F}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \bigoplus_{F \to \mathbb{C}/\mathrm{conj.}} \mathrm{Ext}^1_{\mathrm{MHT}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(n)).$$

By Beilinson's theorem ([B1]) it coincides for $n > 1$ with the Borel regulator on $K_{2n-1}(F)$. Borel's theorem states that this regulator map is injective [B6]. So there is an injective map on the first cohomology of the fundamental Lie coalgebras

$$\ker(\mathrm{Lie}^\vee_{\mathrm{MT}/F} \xrightarrow{\delta} \wedge^2 \mathrm{Lie}^\vee_{\mathrm{MT}/F}) \to \bigoplus_{F \to \mathbb{C}/\mathrm{conj.}} \ker(\mathrm{Lie}^\vee_{\mathrm{HT}} \xrightarrow{\delta} \wedge^2 \mathrm{Lie}^\vee_{\mathrm{HT}}).$$

In particular, the conjectural Lemma 1.3 simplifies to the following basic theorem, which plays a crucial role in Chapter 2:

Theorem 1.7. *If $x \in \mathrm{Lie}^\vee_{\mathrm{MT}/F}$ is of weight at least 2 with $\delta(x) = 0$ and $p(r_{\mathrm{Hod}}(x)) = 0$ for every embedding $r : F \longrightarrow \mathbb{C}$, then $x = 0$.*

Motivic correlator coalgebra

The construction of the Hodge correlator coalgebra $\mathcal{CL}^\vee_{X,S,v_\infty}$ can be upgraded to the motivic setting. The definition of the *motivic correlator coalgebra* mimics that of its Hodge realization:

$$\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee := \frac{T\left((\mathbb{Q}(1)^{S\setminus\{\infty\}})^\vee\right)}{\text{relations}} \otimes H_2(X),$$

a graded Lie coalgebra in the category of pure motives over F, where the relations imposed are the cyclic symmetry, first shuffles, and quotient by nonpositive weight. Then $\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0}$ is isomorphic to the algebra of special derivations of $\mathrm{gr}^W \pi_1^{\mathrm{Mot}}(X - S, v_\infty)$, and there is a map

$$\mathrm{Cor}_{\mathrm{Mot}} : \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee \to \mathrm{Lie}^\vee_{\mathrm{MT}/F}.$$

We will write $\mathrm{Cor}_{\mathrm{Mot}}(x_0, \ldots, x_n)$ for $\mathrm{Cor}_{\mathrm{Mot}}((x_0 \otimes \cdots \otimes x_n)(1))$.

Let us describe how motivic correlators are related to Hodge correlators. Fix an embedding $r : F \longrightarrow \mathbb{C}$. The Hodge realization provides coalgebra maps $\mathrm{Lie}^\vee_{\mathrm{MT}/F} \longrightarrow \mathrm{Lie}^\vee_{\mathrm{HT}}$ and

$$r : \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee \otimes \mathbb{C} \to \mathcal{CL}^\vee_{X,S,v_\infty} \otimes \mathbb{C},$$

and thus a period map

$$\mathrm{Cor}_{\mathcal{H}} \circ r : \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee \otimes \mathbb{C} \to \mathcal{CL}^\vee_{X,S,v_\infty} \otimes \mathbb{C} \to \mathbb{C}.$$

By Theorem 1.4, it coincides with the composition

$$\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee \to \mathrm{Lie}^\vee_{\mathrm{Mot}} \to \mathrm{Lie}^\vee_{\mathrm{HT}} \to \mathbb{C}.$$

We can summarize the main objects and maps defined in this section in a variant of diagram (1.9):

$$\begin{tikzcd}
\mathrm{Der}^S(\mathrm{gr}^W \pi_1^{\mathrm{Mot}}(X \setminus S, v_\infty))^\vee \arrow[r, dash] & (\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty})^\vee \arrow[r, "\mathrm{Cor}_{\mathrm{Mot}}"] \arrow[d, "r"] & \mathrm{Lie}^\vee_{\mathrm{MT}/F} \arrow[d, "r"] \\
\mathrm{Der}^S(\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X \setminus S, v_0))^\vee \arrow[r, dash] & (\mathcal{CL}^\vee_{X,S,v_\infty}) \arrow[r, "\mathrm{Cor}_{\mathrm{Hod}}"] \arrow[dr, "(2\pi i)^w \mathrm{Cor}_{\mathcal{H}}"'] & \mathrm{Lie}^\vee_{\mathrm{HT}} \arrow[d, "p"] \\
 & & \mathbb{R}.
\end{tikzcd}$$

In this setting, a variant of Lemma 1.3 holds. Under certain conditions, relations on motivic correlators can be proven by showing that they hold in the Hodge realization under any complex embedding. This is a key fact in the proof of the motivic upgrade of our relations on Hodge correlators in Chapter 2:

Lemma 1.8. *Let $X \setminus S$ be a rational curve over F. Suppose $x \in \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee$ has weight > 1, $\delta\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$, and $\mathrm{Cor}_{\mathcal{H}}(r(x)) = 0$ for every embedding $r : F \to \mathbb{C}$. Then $\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$.*

Proof. $\mathrm{Cor}_{\mathrm{Mot}}(x)$ is an element of $\mathrm{Lie}^\vee_{\mathrm{MT}/F}$ with coproduct 0. The canonical period of its Hodge realization in $\mathrm{Lie}^\vee_{\mathrm{HT}}$ coincides with the correlator period $\mathrm{Cor}_{\mathcal{H}}(r(x)) = 0$. By Theorem 1.7, it is 0. □

This does not hold in weight 1. For example, choose z to be an element of F that is not a root of unity, but has norm 1 under every complex embedding (e.g., $F = \mathbb{Q}(i)$ and $z = \frac{1}{5}(3 + 4i)$). Then $((0) \otimes (z))(1)$ has coproduct 0 and period $\log|\sigma(z)| = 0$ under both of the embeddings $\mathbb{Q}(i) \xrightarrow{\sigma} \mathbb{C}$. However, the object $\mathrm{Cor}_{\mathrm{Mot}}(0, z)$ is not 0 as an element of $\mathrm{Ext}^1_{\mathcal{MTM}/F}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$.

Dependence on S

If $S \subseteq S'$, there is an induced inclusion $\iota : (\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0})^\vee \to (\mathcal{CL}^{\mathrm{Mot}}_{X,S',v_0})^\vee$.

The following diagram commutes:

$$\begin{tikzcd}
 & (\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty})^\vee \arrow[d, "\iota"] \arrow[ddl, "\mathrm{Cor}_{\mathrm{Mot}}"'] \arrow[ddr, "\mathrm{Cor}_{\mathcal{H}} \circ r"] & \\
 & (\mathcal{CL}^{\mathrm{Mot}}_{X,S',v_\infty})^\vee \arrow[dl, "\mathrm{Cor}_{\mathrm{Mot}}"] \arrow[dr, "\mathrm{Cor}_{\mathcal{H}} \circ r"'] & \\
\mathrm{Lie}^\vee_{\mathrm{MT}/F} \arrow[rr, "p \circ r"'] & & \mathbb{C}.
\end{tikzcd}$$

This allows us to write down elements of $(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_0})^\vee$ without explicitly specifying S.

Distribution relations

Suppose $x_i \in F$ are such that $x^l - x_i$ splits in F for all i. Then the distribution relations defined above for Hodge correlators hold for motivic correlators as well, that is:

$$\mathrm{Cor}_{\mathrm{Mot}}(x_0, \dots, x_n) = \frac{1}{l} \sum_{y_i^l = x_i} \mathrm{Cor}_{\mathrm{Mot}}(y_0, \dots, y_n),$$

where $y_i = 0$ is taken with multiplicity l if $x_i = 0$.

1.2.3 Multiple polylogarithms

We review the properties of multiple polylogarithms ([G3]).

It is well known that these functions obey a family of *double shuffle relations*, similar to our relations for the Hodge correlators that we will show in Chapter 2. However, they do not enjoy some of their other properties. They are multi-valued and do not satisfy dihedral symmetry relations. The shuffle relations between multiple polylogarithms involve products, while for Hodge correlators they are linear.

Definition and properties

The multiple polylogarithms are defined by

$$\mathrm{Li}_{n_1,\dots,n_r}(z_1, \dots, z_r) = \sum_{0<k_1<\cdots<k_r} \frac{z_1^{k_1} \dots z_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}, \quad n_1, \dots, n_r > 0. \tag{1.15}$$

(The *depth* of this formal expression is r and the *weight* is $w := n_1 + \cdots + n_r$.) These series converge for $|z_i| < 1$ and have analytic continuations to multivalued functions with singularities on $\mathbb{C}^r$. The multivalued structure is encoded by a smooth variation of mixed Hodge-Tate structures of weight w over a dense open subset of $\mathbb{C}^r$.

When $r = 1$, the multiple polylogarithms are the classical polylogarithms $\mathrm{Li}_n(z)$. Their monodromy and associated mixed Hodge-Tate structures are well understood ([H]).

We can form an algebra L generated over $\mathbb{Q}$ by the multiple polylogarithms, filtered by the weight and the depth. The expression (1.15) yields expansions for products of polylogarithms, which shows that L has a well-defined multiplication. For example,

$$\begin{aligned}\mathrm{Li}_{n_1}(z_1)\mathrm{Li}_{n_2}(z_2) &= \left(\sum_{0<k_1} \frac{z_1^{k_1}}{k_1^{n_1}}\right)\left(\sum_{0<k_2} \frac{z_2^{k_2}}{k_2^{n_2}}\right) = \left[\sum_{0<k_1<k_2} + \sum_{0<k_2<k_1} + \sum_{0<k_1=k_2}\right] \frac{z_1^{k_1} z_1^{k_2}}{k_1^{n_1} k_2^{n_2}} \\ &= \mathrm{Li}_{n_1,n_2}(z_1, z_2) + \mathrm{Li}_{n_2,n_1}(z_2, z_1) + \mathrm{Li}_{n_1+n_2}(z_1 z_2).\end{aligned}$$

Notice that the left side and all terms on the right side have weight $n_1 + n_2$; however, the left side and the first two terms on the right side have depth 2, while $\mathrm{Li}_{n_1+n_2}(z_1z_2)$ has depth 1.

The general relation is:

$$\begin{aligned} &\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r)\mathrm{Li}_{n_{r+1},\dots,n_{r+s}}(z_{r+1},\dots,z_{r+s}) \\ &= \sum_{\sigma\in\Sigma_{r,s}} \mathrm{Li}_{n_{\sigma^{-1}(1)},\dots,n_{\sigma^{-1}(r+s)}}(z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(r+s)}) + \text{lower-depth terms,} \end{aligned} \tag{1.16}$$

where $\Sigma_{r,s} \subset S_{r+s}$ is the set of (r,s)-shuffles.

Expressions (1.16) are called *first shuffle relations* for multiple polylogarithms. It is convenient to express them with generating functions. Let

$$L\big(z_1,\dots,z_r \mid t_1 : \cdots : t_r\big) \sum_{n_i>0} \mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r) \prod_{t=1}^{r} t_i^{n_i-1};$$

then

$$\begin{aligned} &L\big(z_1,\dots,z_r \mid t_1 : \cdots : t_r\big)L\big(z_{r+1},\dots,z_{r+s} \mid t_{r+1},\dots,t_{r+s}\big) = \\ &= \sum_{\sigma\in\Sigma_{r,s}} L\big(z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(r+s)} \mid t_{\sigma^{-1}(1)} : \cdots : t_{\sigma^{-1}(r+s)}\big) + \text{lower-depth terms.} \end{aligned} \tag{1.17}$$

To describe the lower-depth terms in the right side of (1.16), we need the notion of *quasishuffle*. Let $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$ be two ordered sets. A quasishuffle of A and B is a sequence of slots $\{1,\dots,M\}$ and a placement of each element of $A \cup B$ in a slot, such that each slot is filled with one of: (1) some $a_i \in A$, (2) some $b_j \in B$, or (3) a pair $\{a_i, b_j\}$, and such that the sequence of slots containing the $a_1,\dots,a_r$ and the sequence of slots containing the $b_1,\dots,b_s$ are ordered left to right. If a_i and b_j share a slot, they are said to *collide*. If no elements collide, the quasishuffle is said to be a shuffle.

Let $A = \{1,\dots,r\}$ and $B = \{r+1,\dots,r+s\}$ with the natural orders. Then, equivalently, the quasishuffles are the surjective maps $\{1,\dots,r+s\} \xrightarrow{\sigma} \{1,\dots,M_\sigma\}$ that are strictly increasing on $1,\dots,r$ and $r+1,\dots,r+s$. An index $i \in \{1,\dots,r\}$ collides with an index $j \in \{r+1,\dots,r+s\}$ whenever $\sigma(i) = \sigma(j)$. Let $\overline{\Sigma}_{r,s}$ be the set of quasishuffles.

A quasishuffle σ is a shuffle if $M_\sigma = r+s$. We naturally identify $\Sigma_{r,s}$ with the the subset of shuffles in $\overline{\Sigma}_{r,s}$.

We then have:

$$\begin{aligned}
&\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r)\mathrm{Li}_{n_{r+1},\ldots,n_{r+s}}(z_{r+1},\ldots,z_{r+s}) = \\
&= \sum_{\sigma\in\widetilde{\Sigma}_{r,s}} \mathrm{Li}_{\tilde{n}_{\sigma^{-1}(1)},\ldots,\tilde{n}_{\sigma^{-1}(M_\sigma)}}(\tilde{z}_{\sigma^{-1}(1)},\ldots,\tilde{z}_{\sigma^{-1}(M_\sigma)}),
\end{aligned} \tag{1.18}$$

where

$$\tilde{n}_{\sigma^{-1}(i)} = \sum_{\sigma(j)=i} n_j, \quad \tilde{z}_{\sigma^{-1}(i)} = \prod_{\sigma(j)=i} z_j.$$

Such relations are easily proved by interpreting the terms as cells of the simplicial decomposition of the product of an r-simplex and an s-simplex.

Iterated integrals

The analytic continuation of the multiple polylogarithms has a presentation in terms of iterated integrals. Let

$$I_{n_1,\ldots,n_r}(z_1 : z_2 : \cdots : z_{r+1}) = \int_\gamma \underbrace{\frac{dt}{z_1-t}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_1}\circ\cdots\circ\underbrace{\frac{dt}{z_r-t}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_r},$$

where $\gamma : [0,1]\to\mathbb{C}$ is a path from 0 to z_{r+1}. Here, for 1-forms $\omega_1,\ldots,\omega_r$,

$$\int_\gamma \omega_1\circ\cdots\circ\omega_r := \int_{0\le t_1\le\cdots\le t_r\le 1} \bigwedge_{i=1}^{m} \gamma^*\omega_i(t_i)$$

is Chen's iterated path integral ([C]). Then ([G3], Theorem 2.1)

$$\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r) = I_{n_1,\ldots,n_r}(1 : z_1 : z_1z_2 : \cdots : z_1\ldots z_r). \tag{1.19}$$

Iterated path integrals also satisfy a shuffle product formula, whose terms correspond to the top-dimensional cells of a decomposition of the product of two simplices:

$$\int_\gamma \omega_1\circ\cdots\circ\omega_r \int_\gamma \omega_{m+1}\circ\cdots\circ\omega_{m+n} = \sum_{\sigma\in\Sigma_{m,n}} \omega_{\sigma^{-1}(1)}\circ\cdots\circ\omega_{\sigma^{-1}(m+n)}.$$

This gives a different kind of shuffle relations (*second shuffle relations*) on the iterated integrals $I_{n_1,\ldots,n_r}$, which can also be expressed in terms of generating functions. Let

$$\begin{aligned} &L'\big(z_1 : \cdots : z_{r+1} \mid t_1, \ldots, t_r\big) = \\ &= \sum_{n_i>0} I_{n_1,\ldots,n_r}\,(z_1 : \cdots : z_{r+1})\, t_1^{n_1-1}(t_1+t_2)^{n_2-1} \ldots (t_1 + \cdots + t_r)^{n_r-1}, \end{aligned} \tag{1.20}$$

so

$$L\big(z_1, \ldots, z_r \mid t_1 : \cdots : t_r\big) = L'\big(1 : z_1 : \cdots : z_1 \ldots z_r \mid t_1, t_2 - t_1, \ldots, t_r - t_{r-1}\big). \tag{1.21}$$

Then

$$\begin{aligned} &L'\big(z_1 : \cdots : z_r : 1 \mid t_1, \ldots, t_r\big) L'\big(z_{r+1} : \cdots : z_{r+s} : 1 \mid t_{r+1}, \ldots, t_{r+s}\big) = \\ &\qquad = \sum_{\sigma \in \Sigma_{r,s}} L'\big(z_{\sigma^{-1}(1)} : \cdots : z_{\sigma^{-1}(r+s)} : 1 \mid t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(r+s)}\big). \end{aligned} \tag{1.22}$$

Double shuffle relations

Note the similarity between (1.17) and (1.22). There is a duality between the relations with homogeneous and inhomogeneous arguments z_i and t_i. Together, they form systems of *double shuffle relations*.

The combinatorics of such relations are studied by [G3, G5], allowing them to describe a connection between an algebra of values of the multiple polylogarithms at roots of unity and the geometry of some locally symmetric spaces for $\mathrm{GL}_n(\mathbb{Z})$ ($n = 2, 3$; and recently for $n = 4$ in [G10]).

Relation to Hodge correlators

In depth 1, the Hodge correlators are related to the multiple polylogarithms. We have seen this in weights 1 and 2. In higher weight, define a single-valued version of the polylogarithm by

$$\mathcal{L}_n(z) = \begin{cases} \mathrm{Re} & n \text{ odd} \\ \mathrm{Im} & n \text{ even} \end{cases} \left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot \mathrm{Li}_{n-k}(z) \right) \quad (n \geq 2),$$

where β_k, close relatives of the Bernoulli numbers, are the coefficients of the Taylor expansion $\frac{2x}{e^{2x}-1} = \sum \beta_k x^k$. Then

$$\mathrm{Cor}_{\mathcal{H}}(1, \underbrace{0, \ldots, 0}_{n-1}, z) = -(2\pi i)^{-n} \binom{2n-2}{n-1}^{-1} \sum_{\substack{0 \leq k \leq n-2 \\ k \text{ even}}} \binom{2n-k-3}{n-1} \frac{2^{k+1}}{(k+1)!} \mathcal{L}_{n-k}(z) \log^k |z|\,. \tag{1.23}$$

The precise relationship between the multiple polylogarithms and Hodge correlators in depth > 1 is unknown.

Remark

A formula relating iterated integrals and Hodge correlators, and a different interpretation of the quasishuffle relations (Chapter 2), was very recently found by Rudenko ([R2], §4).

Chapter 2

Shuffle relations for Hodge and motivic correlators on $\mathbb{P}^1$

The results of this chapter have appeared in [M2].

2.1 Introduction and main results

2.1.1 Summary

The Hodge correlators $\mathrm{Cor}_{\mathcal{H}}(z_0, z_1, \ldots, z_n)$ are functions of several complex variables, defined by an explicit integral formula in [G9]. They satisfy some linear relations: the dihedral symmetry relations, the distribution relations, and the shuffle relations.

We found new relations, called *second shuffle relations*. When $z_i \in \{0\} \cup \mu_N$, where μ_N are the N-th roots of unity, they should give almost all relations: the results of [G10] suggest that the other relations are sporadic, i.e., cannot be described by universal formulae.

When the z_i run through a finite subset S of $\mathbb{C}$, the Hodge correlators are the canonical real periods of the mixed Hodge-Tate structures on the pronilpotent completion of the fundamental group $\pi_1^{\mathrm{nil}}(\mathbb{CP}^1 \setminus (S \cup \{\infty\}), v_\infty)$, with the tangential base point at ∞. The latter is a Lie algebra in the category of mixed $\mathbb{Q}$-Hodge-Tate structures. The Hodge correlators describe the $\mathbb{R}$-mixed Hodge structure on this Lie algebra tensored over $\mathbb{Q}$ by $\mathbb{R}$.

The category of mixed $\mathbb{Q}$-Hodge-Tate structures is canonically equivalent to the category of representations of a graded Lie algebra over $\mathbb{Q}$. Let us take its image in the representation defining $\pi_1^{\mathrm{nil}}(\mathbb{CP}^1 \setminus (S \cup \{\infty\}), v_\infty)$, and consider the graded dual Lie coalgebra $\mathrm{Lie}_{\mathrm{HT}}^{\vee}(S)$. The Hodge correlators were lifted in [G9] to canonical elements

$$\mathrm{Cor}_{\mathrm{Hod}}(z_0, \ldots, z_n) \in \mathrm{Lie}^{\vee}_{\mathrm{HT}}(S). \tag{2.1}$$

The real numbers $\mathrm{Cor}_{\mathcal{H}}$ are the canonical real periods of these elements. We prove that our new relations can be lifted to relations on the elements (2.1).

Let $S \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. The Lie algebra $\pi_1^{\mathrm{nil}}(\mathbb{CP}^1 \setminus (S \cup \{\infty\}), v_\infty)$ is the Betti realization of the motivic fundamental group $\pi_1^{\mathrm{Mot}}(\mathbb{P}^1 \setminus (S \cup \{\infty\}), v_\infty)$. The latter is a Lie algebra in the category of mixed Tate motives over $\overline{\mathbb{Q}}$, defined in [DG]. This category is identified with the category of representations of the motivic Galois Lie algebra. Just like in the Hodge case, we take the image of this Lie algebra in the representation provided by the motivic fundamental group, and consider the graded dual Lie coalgebra $\mathrm{Lie}^{\vee}_{\mathrm{MT}}(S)$. In [G9], the elements (2.1) were lifted to elements

$$\mathrm{Cor}_{\mathrm{Mot}}(z_0, \ldots, z_n) \in \mathrm{Lie}^{\vee}_{\mathrm{MT}}(S). \tag{2.2}$$

We prove that our relations can be upgraded to linear relations on these elements.

The universal enveloping algebra for the Lie coalgebra $\mathrm{Lie}^{\vee}_{\mathrm{MT}}(S)$ was described in [G5] via motivic multiple polylogarithms. The motivic double shuffle relations for them were proved in [G6]. The explicit relation between motivic correlators and multiple polylogarithms is an interesting open problem.

The multiple polylogarithms obey a similar system of double shuffle relations, but the dihedral symmetry relation holds only at roots of unity. The combinatorics of those relations, originally described by [G3]-[G5], were studied further by [R1].

The motivic correlator description of $\pi_1^{\mathrm{Mot}}(\mathbb{P}^1 \setminus (S \cup \{\infty\}), v_\infty)$ has several advantages. Most importantly, motivic correlators are defined for any algebraic curve, not only $\mathbb{A}^1 \setminus S$, and the double shuffle relations admit a generalization to elliptic curves, at least in the depth 2 (see Chapter 3). The motivic correlators obey double shuffle and cyclic symmetry relations at all points. Motivic correlators describe elements of the Lie coalgebra rather than its universal enveloping algebra. Finally, they give the best way to describe the mysterious connection between the Lie coalgebra $\mathrm{Lie}^{\vee}_{\mathrm{MT}}(\{0\} \cup \mu_N)$ and modular manifolds ([G10]).

Structure

In the remainder of the introduction, we will state the main results: Theorems 2.1, 2.6, and 2.7. Their proof has three parts.

The first and longest part (§2.2) is a combinatorial calculation, the construction of the quasidihedral coalgebra.

The second part (§2.3) is a specialization theorem that allows us to make continuity arguments about the

Hodge correlators.

The third part (§2.4) is a series of formal arguments to deduce the three realizations of the main result.

2.1.2 Hodge correlator integrals and shuffle relations

The main result of this chapter is a set of functional equations on the Hodge correlator integrals and the Hodge-theoretic and motivic upgrades of these relations. Let us first state the results for the Hodge correlator integrals.

Second shuffle relation

Recall the first shuffle relation for the Hodge correlator integrals (1.5):

$$\sum_{\sigma\in\Sigma_{i,j}} \mathrm{Cor}_{\mathcal{H}}(x_0, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(i+j)}) = 0.$$

The shuffle relations may be considered "easy" because they hold on the level of the sum over trees of the *integrands* in (1.3).

We found another relation on the Hodge correlators. Together, the two relations form the *double shuffle relations*. To state the new relations, we must introduce some notation.

Because of the multiplicative invariance (in weight > 1) of Hodge correlators, it is possible and convenient to introduce an inhomogeneous notation for them, where the arguments are represented by the quotients between successive nonzero values and the number of 0s between them. That is, given $w_0, \ldots, w_k \in \mathbb{C}^*$ such that $w_0 w_1 \ldots w_k = 1$, define

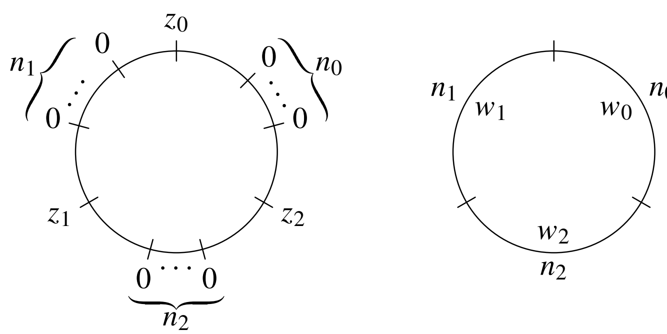
$$\begin{aligned}&\mathrm{Cor}^*_{\mathcal{H}}(w_0|n_0, w_1|n_1, \ldots, w_k|n_k) :=\\ &= \mathrm{Cor}_{\mathcal{H}}(\underbrace{0,\ldots,0}_{n_0}, 1, \underbrace{0,\ldots,0}_{n_1}, w_1, \underbrace{0,\ldots,0}_{n_2}, w_1w_2, \ldots, \underbrace{0,\ldots,0}_{n_k}, w_1\ldots w_k).\end{aligned} \tag{2.3}$$

This definition is illustrated in Figure 2.1.

The depth of an expression $\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n)$ is one less than the number of arguments in the multiplicative notation, that is, k in the formula above.

Our new shuffle relation states:

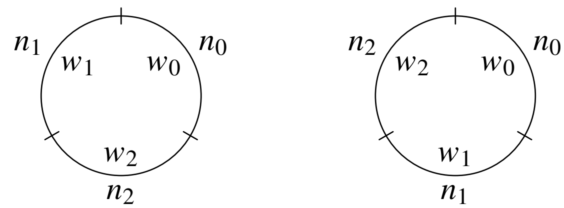
$$\sum_{\sigma\in\Sigma_{r,s}} \mathrm{Cor}^*_{\mathcal{H}}(w_{\sigma^{-1}(1)}|n_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(r+s)}|n_{\sigma^{-1}(r+s)}, w_0|n_0) + \text{lower-depth terms} = 0. \tag{2.4}$$



$$w_i = \frac{z_i}{z_{i-1}}$$

Figure 2.1. Definition of $\mathrm{Cor}^*_{\mathcal{H}}$: $\mathrm{Cor}_{\mathcal{H}}(z_0, \underbrace{0, \ldots, 0}_{n_1}, z_1, \underbrace{0, \ldots, 0}_{n_2}, z_2, \underbrace{0, \ldots, 0}_{n_0}) \equiv \mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, w_2|n_2, w_0|n_0)$.

That is, we shuffle two ordered sets of expressions $(w_i \mid n_i)$, while leaving the segment $(w_0 \mid n_0)$ fixed. For example the $(1, 1)$-shuffle relation begins:

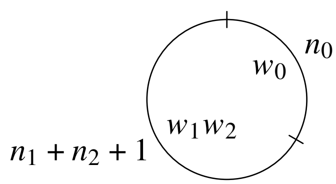


$$\mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, w_2|n_2, w_0|n_0) \quad + \mathrm{Cor}^*_{\mathcal{H}}(w_2|n_2, w_1|n_1, w_0|n_0)$$

We explicitly describe the lower-depth terms in (2.4). They come in two kinds:

(1) Terms coming from the (r, s)-quasishuffles that are not proper shuffles (see §1.2.3). Whenever the segments $(w_i \mid n_j)$ and $(w_j \mid n_j)$ collide, we get a new segment $(w_i w_j \mid n_i + n_j + 1)$ in their place – a 0 is inserted – and the term picks up a negative sign.

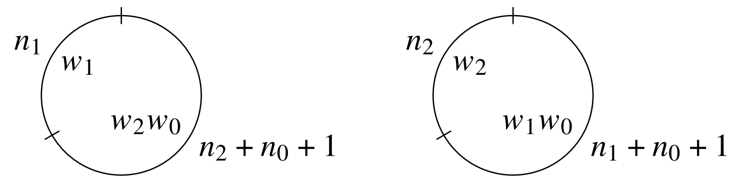
For the $(1, 1)$-shuffle relation, there is only one quasishuffle that is not a shuffle. In this quasishuffle, the two segments $(w_1 \mid n_1)$ and $(w_2 \mid n_2)$ collide:



$$-\mathrm{Cor}^*_{\mathcal{H}}(w_1 w_2|n_1 + n_2 + 1, w_0|n_0)$$

(2) Two additional terms: one where the segments $w_1, \ldots, w_r$ appear in order and the remaining segments $w_{r+1}, \ldots, w_{r+s}, w_0$ collapse; another where the segments $w_{r+1}, \ldots, w_{r+s}$ appear in order and $w_1, \ldots, w_r, w_0$ collapse. These terms come with a negative sign.

For the $(1,1)$-shuffle relation:



$$-\mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, w_2w_0|n_2+n_0+1) \quad - \mathrm{Cor}^*_{\mathcal{H}}(w_2|n_2, w_1w_0|n_1+n_0+1)$$

In summary, the $(1,1)$-shuffle relation states, for $w_0, w_1, w_2 \in \mathbb{C}^*$ and $w_0w_1w_2 = 1$,

$$\begin{aligned}\mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, w_2|n_2, w_0|n_0) + \mathrm{Cor}^*_{\mathcal{H}}(w_2|n_2, w_1|n_1, w_0|n_0)& \\ - \mathrm{Cor}^*_{\mathcal{H}}(w_1w_2|n_1+n_2+1, w_0|n_0)& \\ - \mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, w_2w_0|n_2+n_0+1)& \\ - \mathrm{Cor}^*_{\mathcal{H}}(w_2|n_2, w_1w_0|n_1+n_0+1)& = 0.\end{aligned}$$

It is already a nontrivial relation which is not easy to prove from the definition (1.3) even for $n_0 = n_1 = n_2 = 0$.

By formula (1.10), Hodge correlators in weight 2 are expressed in a simple way in terms of the Bloch-Wigner function $\mathcal{L}_2$. The $(1,1)$-shuffle relation with $n_0 = n_1 = n_2 = 0$ is equivalent to the five-term relation,

$$\mathcal{L}_2\left(\frac{1-w_1}{1-w_1w_2}\right) + \mathcal{L}_2\left(\frac{1-w_2}{1-w_1w_2}\right) + \mathcal{L}_2(1-w_1w_2) + \mathcal{L}_2(w_1) + \mathcal{L}_2(w_2) = 0.$$

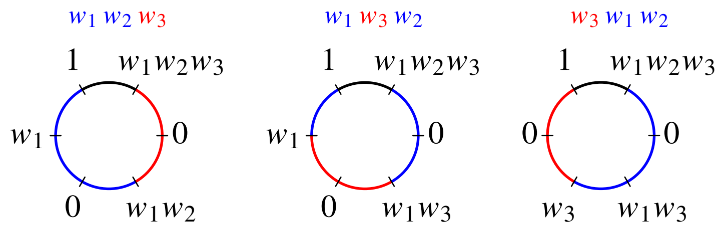
According to [B5], this is essentially the only functional equation for $\mathcal{L}_2$. It follows that the dihedral symmetry and shuffle relations are the *only* relations between the Hodge correlators in weight 2.

For further illustration, let us write out the $(2,1)$-shuffle relation for the Hodge correlator

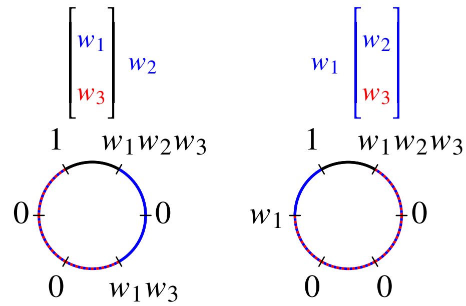
$$\mathrm{Cor}^*_{\mathcal{H}}(w_1|0, w_2|1|w_3|1, w_4|0),$$

where w_1 and w_2 will be shuffled with w_3:

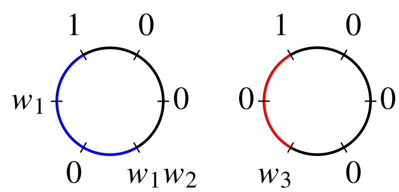
(0) There are three terms from the shuffles:



(1) There are two terms from the quasishuffles that are not shuffles:



(2) There are two additional terms:



The full relation is then

$$
\begin{aligned}
&\mathrm{Cor}^*_{\mathcal{H}}(w_1|0, w_2|1, w_3|1, w_4|0) + \mathrm{Cor}^*_{\mathcal{H}}(w_1|0, w_3|1, w_2|1, w_4|0) + \mathrm{Cor}^*_{\mathcal{H}}(w_3|1, w_1|0, w_2|1, w_4|0) \\
&\qquad - \mathrm{Cor}^*_{\mathcal{H}}(w_1w_3|2, w_2|1, w_4|0) - \mathrm{Cor}^*_{\mathcal{H}}(w_1|0, w_2w_3|3, w_4|0) \\
&\qquad - \mathrm{Cor}^*_{\mathcal{H}}(w_1|0, w_2|1, (w_1w_2)^{-1}|2) - \mathrm{Cor}^*_{\mathcal{H}}(w_3|1, w_3^{-1}|3) \qquad = 0,
\end{aligned}
$$

where the $3+2+2$ terms in the three rows match the $3+2+2$ pictures above.

We now write out the general relation:

Theorem 2.1. *(a) Suppose that $r, s > 1$ and that not all $n_i = 0$ or not all $w_i = 1$. Then the Hodge correlators satisfy the relation:*

$$
\begin{aligned}
&\sum_{\sigma\in\overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma} \mathrm{Cor}^*_{\mathcal{H}}(w_{\sigma^{-1}(1)}|n_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(M_\sigma)}|n_{\sigma^{-1}(M_\sigma)}, w_0|n_0) \\
&- \mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1, \dots, w_r|n_r, w_{\{r+1,\dots,r+s,0\}}|n_{\{r+1,\dots,r+s,0\}}) \\
&- \mathrm{Cor}^*_{\mathcal{H}}(w_{r+1}|n_{r+1}, \dots, w_{r+s}|n_{r+s}, w_{\{1,\dots,r,0\}}|n_{\{1,\dots,r,0\}}) \qquad = 0,
\end{aligned}
$$

where

$$n_I = \sum_{i \in I} (n_i + 1) - 1, \quad w_I = \prod_{i \in I} w_i.$$

(b) The Hodge correlators satisfy all specializations of this relation as any subset of the w_i ($1 \le i \le n$) approaches 0.

Applications

Theorem 2.1 gives simple proofs of certain results of [GR].

Corollary 2.2 ([GR], Proposition 2.8)**.** *For $n > 2$, every Hodge correlator of weight n is a linear combination of Hodge correlators of weight n and depth at most $n-2$. Explicitly, for $z_1, \dots, z_n \in \mathbb{C}^*$, we have*

$$\begin{aligned}\mathrm{Cor}_{\mathcal{H}}(z_1, \dots, z_n, 0) = &\sum_{i=1}^{n} \mathrm{Cor}_{\mathcal{H}} \left(z_1, \dots, z_{i-1}, z_i, z_i \frac{z_1}{z_n}, \dots, z_{n-1} \frac{z_1}{z_n}, z_n \frac{z_1}{z_n} \right) \\ &- \sum_{i=2}^{n} \mathrm{Cor}_{\mathcal{H}} \left(z_1, \dots, z_{i-1}, 0, z_i \frac{z_1}{z_n}, \dots, z_{n-1} \frac{z_1}{z_n}, z_n \frac{z_1}{z_n} \right) \\ &- \mathrm{Cor}_{\mathcal{H}} \left(z_1, z_1 \frac{z_1}{z_n}, 0, \dots, 0 \right). \end{aligned} \tag{2.5}$$

In weight 3, we deduce the Hodge correlator version of relations (27) and (29) from [GR].

Corollary 2.3. *The Hodge correlators in weight 3 satisfy the relations:*

$$\mathrm{Cor}_{\mathcal{H}}(1, 0, 0, x) + \mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1-x) + \mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1-x^{-1}) = \mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1), \tag{2.6}$$

$$\begin{aligned}\mathrm{Cor}_{\mathcal{H}}(0, x, 1, y) = &-\mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1-x^{-1}) - \mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1-y^{-1}) - \mathrm{Cor}_{\mathcal{H}} \left(1, 0, 0, \frac{y}{x} \right) \\ &- \mathrm{Cor}_{\mathcal{H}} \left(1, 0, 0, \frac{1-y}{1-x} \right) + \mathrm{Cor}_{\mathcal{H}} \left(1, 0, 0, \frac{1-y^{-1}}{1-x^{-1}} \right) + \mathrm{Cor}_{\mathcal{H}}(1, 0, 0, 1). \end{aligned} \tag{2.7}$$

We have noted that the double shuffle and dihedral symmetry relations give all relations between Hodge correlators in weight 2. In weight 3, the Hodge correlators of depth 1 are expressed in terms of the single-valued trilogarithm $\mathcal{L}_3$ (see the end of §1.2.3). By the results of [GR], the relations (2.7) imply the general functional equation for $\mathcal{L}_3$ ([G1]). We conclude that the double shuffle relations for Hodge correlators imply all functional equations for $\mathcal{L}_2$ and $\mathcal{L}_3$.

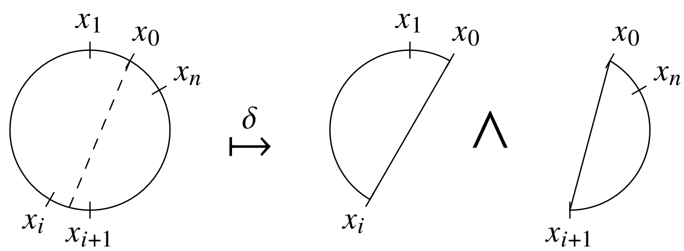


Figure 2.2. The typical term of the coproduct in the cyclic Lie coalgebra.

2.1.3 Quasidihedral Lie coalgebras

Let G be an abelian group. We use the multiplicative notation for G; the identity element is $1 \in G$. Typically, G will be the multiplicative group of a field $F^\times$ or the group of N-th roots of unity μ_N. We adjoin to G a formal element 0, where $0 \cdot g = 0$ for $g \in G \cup \{0\}$.

We define the *quasidihedral Lie coalgebra* $\mathcal{D}(G)$. It generalizes the dihedral Lie coalgebra of [G4]. The aim of the construction of $\mathcal{D}(G)$ is twofold:

(1) It is the main combinatorial ingredient in the proof of the double shuffle relations for correlators.

(2) The Lie coalgebra $\mathcal{D}(G)$ describes the coproduct of motivic correlators.

Cyclic Lie coalgebra

Let V be the $\mathbb{Q}$-vector space with basis indexed by $G \cup \{0\}$.

Let $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ be the tensor algebra of V over $\mathbb{Q}$. We impose a grading by weight, where $V^{\otimes n}$ has weight $n - 1$. Then define the cyclic Lie coalgebra, as a vector space, by

$$C(G) = \frac{T(V)}{\text{cyclic symmetry}}.$$

It is positively graded and generated in weight n by elements $x_0 \otimes \cdots \otimes x_n$ modulo the relation $x_0 \otimes \cdots \otimes x_n = x_1 \otimes \cdots \otimes x_n \otimes x_0$. We can represent these elements by elements of $G \cup \{0\}$ written counterclockwise at marked points on a circle.

The coproduct on $C(G)$ is defined on such a generator by splitting the circle into two arcs that share exactly one point. That is, consider a line inside the circle, starting at a marked point and ending between two marked points. It splits the circle into two parts, representing generators x' and x'', and the coproduct of $x_0 \otimes \cdots \otimes x_n$ is the sum of $x' \wedge x''$ over all such cuts (Figure 2.2).

To be precise, the coproduct is defined by

$$\delta\left(x_0\otimes\cdots\otimes x_n\right)=\sum_{\mathrm{cyc}}\sum_{i=1}^{n-1}\left(x_0\otimes x_1\otimes\cdots\otimes x_i\right)\wedge\left(x_0\otimes x_{i+1}\otimes\cdots\otimes x_n\right). \tag{2.8}$$

It respects the weight grading and satisfies the co-Jacobi identity.

We will write elements of $C(G)$ as

$$C(x_0,\ldots,x_n)=x_0\otimes\cdots\otimes x_n.$$

Also introduce a notation, analogous to that for Hodge correlators, for $w_0,\ldots,w_k\in G$ with $w_0\ldots w_k=1$:

$$\begin{aligned}&C^*(w_0|n_0,w_1|n_1,\ldots,w_k|n_k):=\\&=C(\underbrace{0,\ldots,0}_{n_0},1,\underbrace{0,\ldots,0}_{n_1},w_1,\underbrace{0,\ldots,0}_{n_2},\ldots,w_1\ldots w_{k-1},\underbrace{0,\ldots,0}_{n_k},w_1\ldots w_k).\end{aligned}$$

Relations

A *first shuffle* in $C(G)$ is an element of the form

$$\sum_{\sigma\in\Sigma_{r,s}}C(x_0,x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)},\ldots,x_{\sigma^{-1}(r+s)}).$$

Define

$$\widetilde{\mathcal{D}}(G)=\frac{C(G)}{\text{first shuffles, scaling relations, distribution relations}}.$$

The scaling relations we impose are:

(1) In weight 1, we have $C(0,0)=0$ and $C(ab,ac)=C(0,a)+C(b,c)$ for $a\in G$.

(2) In weight >1, multiplicative invariance:

$$C(x_0,\ldots,x_n)=C(ax_0,\ldots,ax_n),\quad a\in G.$$

The distribution relations are the following. For $l\in\mathbb{Z}_{>0}$, let G_l denote the l-torsion of G. Suppose that G_l is finite and l divides $|G_l|$, and suppose $x_0,\ldots,x_n\in G\cup\{0\}$ are divisible by l (note 0 is always divisible by l). Let m be the number of 0s among the x_i. Then the relation is

$$C(x_0,\ldots,x_n)=\frac{l^m}{|G_l|}\sum_{y_i^l=x_i}C(y_0,\ldots,y_n), \tag{2.9}$$

except in the case that $n = 1$ and $x_0 = x_1$.

The following is immediate from the constructions of [G4] (Theorem 4.3).

Theorem 2.4. *The first shuffles, scaling relations, and distribution relations generate a coideal in $C(G)$. The coproduct on $C(G)$ descends to a well-defined coproduct on $\widetilde{\mathcal{D}}^\vee(G)$.*

Abusing notation, denote also by C and C^* the images in $\widetilde{\mathcal{D}}(G)$ of the elements C, C^* in $C(G)$.

A *second shuffle* in $C(G)$ is an element of the form suggested by Theorem 2.1:

$$\sum_{\sigma\in\overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma} C^*(w_{\sigma^{-1}(1)}|n_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(M_\sigma)}|n_{\sigma^{-1}(M_\sigma)}, w_0|n_0)$$
$$- C^*(w_1|n_1, \dots, w_r|n_r, w_{\{r+1,\dots,r+s,0\}}|n_{\{r+1,\dots,r+s,0\}})$$
$$- C^*(w_{r+1}|n_{r+1}, \dots, w_{r+s}|n_{r+s}, w_{\{1,\dots,r,0\}}|n_{\{1,\dots,r,0\}}),$$

where

$$n_I = \sum_{i\in I}(n_i + 1) - 1, \quad w_I = \prod_{i\in I} w_i.$$

Define the *quasidihedral Lie coalgebra*

$$\mathcal{D}(G) = \frac{\widetilde{\mathcal{D}}(G)}{\text{second shuffles}}.$$

Then we prove:

Theorem 2.5. *The second shuffles form a coideal in $\widetilde{\mathcal{D}}(G)$. The coproduct on $\widetilde{\mathcal{D}}(G)$ descends to a well-defined coproduct on $\mathcal{D}(G)$.*

Theorem 2.5 provides us with a Lie coalgebra generated by sequences of elements of $G \cup \{0\}$ that satisfies dihedral symmetry, scaling, and the two shuffle relations.

Let $C^\circ(G)$ the subspace of $C(G)$ generated by elements $C(x_0, \dots, x_n)$ where not all x_i are equal. It is a subcoalgebra, which we call the *restricted cyclic Lie coalgebra*. The image of $C^\circ(G)$ in $\mathcal{D}(G)$ is the *restricted quasidihedral Lie coalgebra*, denoted $\mathcal{D}^\circ(G)$.

The Hodge correlators satisfy cyclic symmetry, first shuffle, distribution, and scaling relations. Equivalently, the function $\mathrm{Cor}^*_{\mathcal{H}}$ factors through $\widetilde{\mathcal{D}}(\mathbb{C}^*)$ and a map

$$C^*(w_0|n_0, \dots, w_k|n_k) \mapsto \mathrm{Cor}^*_{\mathcal{H}}(w_0|n_0, \dots, w_k|n_k).$$

An equivalent form of Theorem 2.1 is that, restricted to the set of arguments where not all $w_i = 1$ or not all

$n_i = 0$, this function factors through the quotient $\mathcal{D}^\circ(\mathbb{C}^*)$.

Depth filtration

The Lie coalgebra $\mathcal{D}(G)$ is filtered by the *depth*, where a generator has depth d if it includes $d+1$ elements of G (not counting 0s). In the associated graded coalgebra $\mathrm{gr}^D\mathcal{D}(G)$, the second shuffle relations lose their lower-depth terms.

2.1.4 Relations for Hodge and motivic correlators

Hodge correlators

Recall from §1.2 that the Hodge correlators are objects in the fundamental Lie coalgebra of the category of $\mathbb{R}$-mixed Hodge structures, and are Hodge-theoretic upgrades of the Hodge correlator functions. Specifically, given any collection of complex numbers $z_0, \ldots, z_n$, the Hodge correlators $\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n)$ were upgraded to elements of the Tannakian Lie coalgebra $\mathrm{Lie}^\vee_{\mathrm{HT}}$ of the category of $\mathbb{R}$-mixed Hodge structures:

$$\mathrm{Cor}_{\mathrm{Hod}}(z_0, \ldots, z_n) \in \mathrm{Lie}^\vee_{\mathrm{HT}}. \tag{2.10}$$

The upgraded Hodge correlators (2.10) satisfy the dihedral and first shuffle relations, and their coproduct in the coalgebra $\mathrm{Lie}^\vee_{\mathrm{HT}}$ has a simple combinatorial description (1.12).

One of the main results of this chapter is that the elements (2.10) satisfy the second shuffle relations. In other words, they provide a map of Lie coalgebras $\mathcal{D}^\circ(\mathbb{C}^*) \to \mathrm{Lie}^\vee_{\mathrm{HT}}$. Last us state the main result for the Hodge correlators, on the level of the map $\mathrm{Cor}_{\mathrm{Hod}}$. Here we use the multiplicative notation $\mathrm{Cor}^*_{\mathrm{Hod}}$, whose definition is analogous to that of $\mathrm{Cor}^*_{\mathcal{H}}$ (see (2.3)).

Theorem 2.6. *(a) Restricted to the subspace of $C\mathcal{L}^\vee_{X,S,v_\infty}$ generated by elements $(x_0 \otimes \cdots \otimes x_n)(1)$ with not all x_i equal, the map $\mathrm{Cor}_{\mathrm{Hod}}$ factors through $\mathcal{D}^\circ(\mathbb{C}^*)$. (Here $S \subset \mathbb{P}^1(\mathbb{C})$ is any finite set of punctures containing all points appearing in the relation in (b).)*

(b) Suppose that $r, s > 1$ and that not all $n_i = 0$ or not all $w_i = 1$. Then the Hodge correlators satisfy the relation:

$$\begin{aligned}
&\sum_{\sigma \in \overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma} \mathrm{Cor}^*_{\mathrm{Hod}}(w_{\sigma^{-1}(1)}|n_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(M_\sigma)}|n_{\sigma^{-1}(M_\sigma)}, w_0|n_0) \\
&- \mathrm{Cor}^*_{\mathrm{Hod}}(w_1|n_1, \ldots, w_r|n_r, w_{\{r+1,\ldots,r+s,0\}}|n_{\{r+1,\ldots,r+s,0\}}) \\
&- \mathrm{Cor}^*_{\mathrm{Hod}}(w_{r+1}|n_{r+1}, \ldots, w_{r+s}|n_{r+s}, w_{\{1,\ldots,r,0\}}|n_{\{1,\ldots,r,0\}}) \qquad = 0,
\end{aligned}$$

where

$$n_I = \sum_{i \in I}(n_i + 1) - 1, \quad w_I = \prod_{i \in I} w_i.$$

(c) *The Hodge correlators satisfy all specializations of this relation as any subset of the* w_i $(1 \le i \le n)$ *approaches 0.*

While Theorem 2.1 was an equality between functions, Theorem 2.6 is a relation in the fundamental Lie coalgebra of mixed Hodge-Tate structures. Theorem 2.1 follows immediately from Theorem 2.6 by applying the period map.

Motivic correlators over a number field

We are ready to state the most general version of the result by upgrading the constructions of the previous section from mixed Hodge structures to mixed motives over a number field.

Theorem 2.7. *Let* F *be a number field and* $X = \mathbb{P}^1$.

(a) *Restricted to the subspace of* $\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee$ *generated by elements* $(x_0 \otimes \cdots \otimes x_n)(1)$ *with not all* x_i *equal, the map* $\mathrm{Cor}_{\mathrm{Mot}}$ *factors through* $\mathcal{D}^\circ(F^\times)$. *(Here* $S \subset \mathbb{P}^1(F)$ *is any finite set of punctures containing all points appearing in the relation in (b).)*

(b) *Suppose that* $r, s > 1$ *and that not all* $n_i = 0$ *or not all* $w_i = 1$. *Then the motivic correlators satisfy the same relation as in Theorem 2.6, with* $\mathrm{Cor}^*_{\mathrm{Hod}}$ *replaced by* $\mathrm{Cor}^*_{\mathrm{Mot}}$.

(c) *The motivic correlators satisfy all specializations of this relation as any subset of the* w_i $(1 \le i \le n)$ *approaches 0.*

2.2 Construction of the quasidihedral Lie coalgebra

2.2.1 Definitions

For an abelian group G, we defined the Lie coalgebra $\widetilde{\mathcal{D}}(G)$ as the quotient of the tensor algebra of $\mathbb{Q}[G \cup \{0\}]$ by cyclic symmetry, first shuffle, distribution, and scaling relations.

Recall Theorem 2.5:

Theorem. *The second shuffles form a coideal in* $\widetilde{\mathcal{D}}(G)$. *The coproduct on* $\widetilde{\mathcal{D}}(G)$ *descends to a well-defined coproduct on* $\mathcal{D}(G)$.

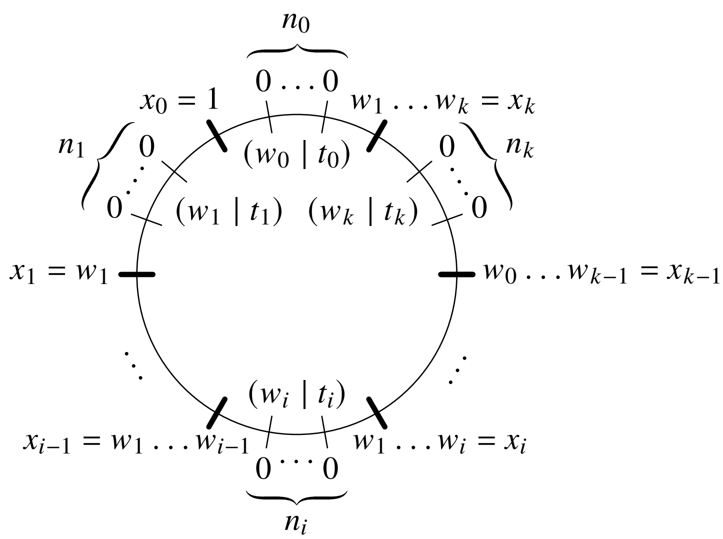


Figure 2.3. Definition of the generating function $\mathbf{\Lambda}^*$ (2.11).

The proof of this theorem is the goal of this section.

The extra term in the scaling relation in weight 1, and the presence of terms of lower depth in the coproduct formula (2.8), makes the proof more difficult than that in [G4]'s construction of the dihedral Lie coalgebra. We find Theorem 2.5 to be a small combinatorial miracle. Unfortunately, we do not know a simpler proof.

Generating functions

The second shuffle relations can be expressed in a compact form in terms of generating functions. This simplifies their proof.

We package the elements of $\widetilde{\mathcal{D}}(G)$ into a generating function as follows:

$$\mathbf{\Lambda}^*\big(w_0, \dots, w_k \mid t_0, \dots, t_k\big) := \sum_{n_i \geq 0} C^*(w_0|n_0, \dots, w_k|n_k) \prod_{i=0}^{k} t_i^{n_i}, \tag{2.11}$$

where $\prod_{i=0}^{k} w_i = 1$ and the t_i are formal variables (Figure 2.3).

We allow multisets of variables to appear in place of the t_i: if $S_i = \{t_{i,1}, \dots, t_{i,d_i}\}$, then

$$\begin{aligned}\mathbf{\Lambda}^*\big(w_0, \dots, w_k \mid S_1, \dots, S_k\big) &= \sum_{n_i \geq 0} \sum_{\substack{n_{i,j} \geq 0 \\ \sum_{j=1}^{d_i} n_{i,j} = n_i - d_i + 1}} C^*(w_0|n_0, \dots, w_k|n_k) \prod_{i=0}^{k} \prod_{j=1}^{d_i} t_{i,j}^{n_{i,j}} \\ &= \sum_{n_{i,j} \geq 0} C^*(w_0|N_0, \dots, w_k|N_k) \prod_{i=0}^{k} \prod_{j=1}^{d_i} t_{i,j}^{n_{i,j}}, \end{aligned} \tag{2.12}$$

where in the last expression $N_i = n_{i,1} + 1 + n_{i,2} + 1 + \cdots + 1 + n_{i,d_j}$. The corresponding operation on the

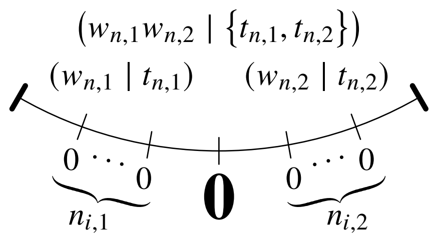


Figure 2.4. Definition of the generating function with multiset arguments.

correlator coefficients is combining adjacent segments of 0s, with additional 0s being inserted between them, such as

$$\begin{matrix}(n_{i,1}\text{ 0s indexed by } t_{i,1})\\(n_{i,2}\text{ 0s indexed by } t_{i,2})\end{matrix} \to (n_{i,1}+1+n_{i,2}\text{ 0s indexed by }\{t_{i,1}, t_{i,2}\}).$$

(See Figure 2.4.)

There is a useful identity

Lemma 2.8.

$$\begin{aligned}\mathbf{\Lambda}^*(\ldots, w, \cdots \mid \ldots, \{t\} \sqcup T, \ldots) - \mathbf{\Lambda}^*(\ldots, w, \cdots \mid \ldots, \{u\} \sqcup T, \ldots)&\\ = (t-u)\mathbf{\Lambda}^*(\ldots, w, \cdots \mid \ldots, \{t,u\} \sqcup T, \ldots).&\end{aligned} \tag{2.13}$$

Proof. Clear by comparing the coefficients of $t^r u^s$. □

Theorem 2.5 can then be expressed in terms of the generating functions:

Theorem. *The subspace of $\widetilde{\mathcal{D}}(G)$ [[$t_1, \ldots, t_k$]] generated by elements of the form*

$$\begin{aligned}&\sum_{\sigma \in \overline{\overline{\Sigma}}_{r,s}} (-1)^{r+s-M_\sigma} \mathbf{\Lambda}^*\left(w_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(M_\sigma)}, w_0 \mid S_{\sigma^{-1}(1)}, \ldots, S_{\sigma^{-1}(M_\sigma)}, S_0\right)\\ &- \mathbf{\Lambda}^*\left(w_1, \ldots, w_r, w_{\{r+1,\ldots,r+s,0\}} \mid S_1, \ldots, S_r, S_{\{r+1,\ldots,r+s,0\}}\right)\\ &- \mathbf{\Lambda}^*\left(w_{r+1}, \ldots, w_{r+s}, w_{\{1,\ldots,r,0\}} \mid S_{r+1}, \ldots, S_{r+s}, S_{\{1,\ldots,r,0\}}\right) \qquad = 0,\end{aligned}$$

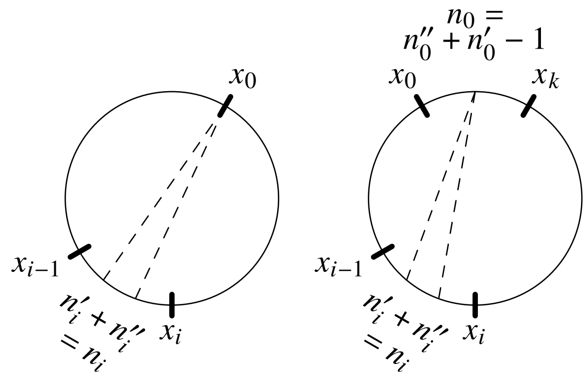
where

$$S_I = \bigsqcup_{i \in I} S_i, \quad w_I = \prod_{i \in I} w_i$$

forms a coideal.

Coproduct

Let us write down the formula defining the coproduct (2.8) in terms of the elements C^*.



(a) (b)

Figure 2.5. The two types of cuts contributing terms to δC.

Lemma 2.9. *Let* $C = C^*(w_0|n_0, \dots, w_k|n_k)$ *and suppose* $\mathrm{wt}(C) > 2$. *Then*

$$\delta C = \sum_{\mathrm{cyc}} \Bigg(\sum_{i=0}^{k} \sum_{n_i'+n_i''=n_i} C^*(\underbrace{w_i \dots w_k}|n_i', w_0|n_0, \dots, w_{i-1}|n_{i-1}) \wedge$$
$$\wedge\, C^*(w_{i+1}|n_{i+1}, \dots, w_k|n_k, \underbrace{w_0 w_1 \dots w_i}|n_i'') \Bigg) \tag{2.14}$$
$$+ \sum_{\mathrm{cyc}} \Bigg(\sum_{i=1}^{k} \sum_{\substack{n_i'+n_i''=n_i \\ n_0'+n_0''=n_0+1}} C^*(w_1|n_1, \dots, w_{i-1}|n_{i-1}, \underbrace{w_i \dots w_k w_0}|n_i' + n_0'') \wedge$$
$$\wedge\, C^*(\underbrace{w_0 \dots w_i}|n_0' + n_i'', w_{i+1}|n_{i+1}, \dots, w_k|n_k) \Bigg) \tag{2.15}$$
$$+ \sum_{i=0}^{k} L_i \wedge C(0, w_i), \tag{2.16}$$

where

$$L_i = \begin{cases} C^*(w_0|n_0, \dots, w_i|n_i - 1, \dots, w_k|n_k), & n_i > 0, \\ \begin{array}{l} \quad C^*(w_0|n_0, \dots, \underbrace{w_{i-1} w_i}|n_{i-1}, w_{i+1}|n_{i+1}, \dots, w_k|n_k) \\ +C^*(w_0|n_0, \dots, w_{i-1}|n_{i-1}, \underbrace{w_i w_{i+1}}|n_{i+1}, \dots, w_k|n_k) \end{array}, & n_i = 0 \end{cases}, \tag{2.17}$$

and the sums are taken over cyclic permutations of the indices $0, \dots, k$.

If $\mathrm{wt}(C) = 2$*, this formula holds modulo terms of the form* $C(0, a) \wedge C(0, b)$.

Proof. Classify the terms $C' \wedge C''$ of δC by the common point of the two resulting parts C' and C''. Let $x_i = w_1 \dots w_i$ be the point counterclockwise from the segment w_i. Up to cyclic symmetry, any cut is either:

(a) a cut from x_0 to the segment w_i (between x_{i-1} and x_i) (Figure 2.5(a));

(b) cut from a 0 on the segment w_0 (between x_k and x_0) to the segment w_i (Figure 2.5(b)).

We first write the terms arising from these cuts modulo elements of form $C(0, x)$.

Case (a) contributes the terms (2.14) and case (b) contributes the terms (2.15), noting that modulo elements of the form $C(0,a)$ the C^* have cyclic symmetry.

Now we handle the terms (2.16). Let $w = n_0+n_1+\cdots+n_k+k$ be the weight. Consider the (weight $w-1$) $\wedge$ (weight 1) terms of the coproduct.

Such elements, of form $C_{w-1} \wedge C$, fall into two cases, depending on which point is present in C but not in C_{w-1}.

(1) 0 on the segment w_i (from x_{i-1} to x_i).

(2) x_i.

If $w > 2$, the C_{w-1} are invariant under scaling. If $w = 2$, then the cyclic permutation of the arguments $w_0, \ldots, w_{i-1}w_i, w_{i+1}, \ldots, w_k$ and $w_0, \ldots, w_{i-1}, w_i w_{i+1}, \ldots, w_k$ in (2.17) modifies those terms by an element of weight 1, so the expressions in (2.16) are determined up to (weight 1) $\wedge$ (weight 1).

In case (1), we have

$$C_{w-1} = (w_0|n_0, \ldots, w_i|n_i - 1, \ldots, w_k|n_k).$$

The only nonzero terms that appear are $(C_{w-1} \wedge (-C(0,x_i))$ (cut clockwise of x_i) and $C_{w-1} \wedge C(0,x_{i-1})$ (cut counterclockwise of x_i).

On the other hand, (2.14) produces no terms for these two cuts (they correspond to to $i = 1$ and $i = k$). Thus this case contributes the terms

$$C_{w-1} \wedge (C(0,x_i) - C(0,x_{i-1})) = C_{w-1} \wedge C(0,w_i),$$

which are the $n_i > 0$ terms in (2.16).

In case (2),

$$C_{w-1} = C^*(x_0|n_0, \ldots, x_i x_{i+1}|n_i + n_{i+1}, \ldots, x_k|n_k).$$

Let C_1' and C_1'' be the elements formed by x_i and the point clockwise and counterclockwise from x_i, respectively. Then the resulting terms are $-C_{w-1} \wedge C_1'$ and $C_{w-1} \wedge C_1''$.

If $n_i = 0$, then $C_1' = C(x_i, x_{i-1}) = C^*(w_i|0, w_i^{-1}|0) + C(0,x_i)$, while (2.14) contributes $C^*(w_i^{-1}|0, w_i|0) \wedge C_{w-1}$. Thus we get an added term

$$-C_{w-1} \wedge (C(0,x_i) - C(0,w_i)).$$

If $n_i \neq 0$, then $C_1' = C(0,x_i)$, while (2.15) contributes 0. Thus we get a term $-C_{w-1} \wedge C(0,x_i)$.

Similarly, we get terms $C_{w-1} \wedge (C(0, x_i) + C(0, w_{i+1}))$ if $n_{i+1} = 0$ and $C_{w-1} \wedge C(0, x_i)$ if $n_{i+1} > 0$.

Collecting terms, the total contribution from this case is $C_{w-1} \wedge (M_i' + M_i'')$, where

$$M_i' = \begin{cases} C(0, x_i) & n_i = 0, \\ 0 & n_i \neq 0 \end{cases}, \quad M_i'' = \begin{cases} C(0, x_{i+1}) & n_{i+1} = 0, \\ 0 & n_{i+1} \neq 0 \end{cases}. \tag{2.18}$$

Reindexing, we get exactly the $n_i = 0$ terms of (2.16). $\square$

We remark that if a cyclic permutation is applied to the arguments in (2.14), so that it is written

$$C^*(w_0|n_0, \ldots, w_{i-1}|n_{i-1}, \underbrace{w_i \ldots w_k}|n_i')\wedge$$
$$\wedge\, C^*(\underbrace{w_0 w_1 \ldots w_i}|n_i'', w_{i+1}|n_{i+1}, \ldots, w_k|n_k)\Big),$$

then the $n_i = 0$ terms in (2.16) disappear.

Then there is the following formula for the coproduct of generating functions:

Lemma 2.10. *Suppose $k > 2$ and let $X = \mathbf{\Lambda}^*(w_0, \ldots, w_k \mid t_0, \ldots, t_k)$. Then*

$$\begin{aligned} \delta X = \sum_{\text{cyc}} \Bigg(\sum_{i=0}^{k} &\mathbf{\Lambda}^*(\underbrace{w_i \ldots w_k}, w_0, \ldots, w_{i-1}, \mid t_i, t_1, \ldots, t_{i-1}) \\ &\wedge \mathbf{\Lambda}^*(w_{i+1}, \ldots, w_k, \underbrace{w_0 \ldots w_i} \mid t_{i+1}, \ldots, t_k, t_i)\Bigg) \end{aligned} \tag{2.19}$$

$$\begin{aligned} + \sum_{\text{cyc}} \Bigg(\sum_{i=1}^{k} &t_0 \mathbf{\Lambda}^*(w_1, \ldots, w_{i-1}, \underbrace{w_i \ldots w_k w_0} \mid t_1, \ldots, t_{i-1}, \{t_i, t_0\}) \\ &\wedge \mathbf{\Lambda}^*(\underbrace{w_0 \ldots w_i}, w_{i+1}, \ldots, w_k \mid \{t_i, t_0\}, t_{i+1}, \ldots, t_k) \end{aligned} \tag{2.20}$$

$$+ \sum_{i=1}^{k} L_i \wedge \log w_i, \tag{2.21}$$

where

$$L_i = t_i \mathbf{\Lambda}^*(w_0, \ldots, w_k \mid t_0, \ldots, t_k) \tag{2.22}$$

$$+ \mathbf{\Lambda}^*(w_0, \ldots, \underbrace{w_{i-1} w_i}, w_{i+1}, \ldots, w_k \mid t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k) \tag{2.23}$$

$$+ \mathbf{\Lambda}^*(w_0, \ldots, w_{i-1}, \underbrace{w_i w_{i+1}}, \ldots, w_k \mid t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k). \tag{2.24}$$

If $k = 2$, this formula holds modulo terms of the form $C(0, a) \wedge C(0, b)$.

Proof. Directly reinterpret Lemma 2.9 via the definition (2.11) by summing the expressions (2.14), (2.15), (2.16) over choices of $\{n_i\}_{i=0}^{k}$ taken with a monomial $\prod_i t_i^{n_i}$.

The expressions (2.14) and (2.15) yield (2.19) and (2.20) in an obvious manner.

The $n_i > 0$ cases in (2.17) give the terms with (2.22), and the $n_i = 0$ cases give (2.23)-(2.24). $\square$

We also remark that if a cyclic permutation is applied to the arguments in (2.19), so that it is written

$$\begin{gathered}\mathbf{\Lambda}^*\big(w_0, \dots, w_{i-1}, \underline{w_i \dots w_k} \mid t_1, \dots, t_{i-1}, t_i\big) \\ \wedge\, \mathbf{\Lambda}^*\big(\underline{w_0 \dots w_i}, w_{i+1}, \dots, w_k \mid t_i, t_{i+1}, \dots, t_k\big)\end{gathered}$$

then the terms (2.23) and (2.24) disappear.

Dual generating function and homogeneity

For a more complete analogy with the generating functions L, L^* for multiple polylogarithms (§1.2.3), we define a dual generating function $\mathbf{\Lambda}$:

$$\mathbf{\Lambda}\big(x_0, \dots, x_k \mid t_0, \dots, t_k\big) := \sum_{n_i \geq 0} C(x_0, \underbrace{0, \dots, 0}_{n_0}, x_1 \dots, x_k, \underbrace{0, \dots, 0}_{n_k}) \prod_{i=0}^{k} (t_0 + \dots + t_i)^{n_i}, \tag{2.25}$$

where the formal variables t_i satisfy the relation $\sum_{i=0}^{k} t_i = 0$. The pair of generating functions $\mathbf{\Lambda}^*, \mathbf{\Lambda}$ resemble those used by [G4] in the definition of the dihedral Lie coalgebra.

The duality is made clear by the following statement:

Lemma 2.11. *(a) The generating functions are related by*

$$\mathbf{\Lambda}^*\big(w_0, \dots, w_k \mid t_0, \dots, t_k\big) = \mathbf{\Lambda}\big(1, w_0, \dots, w_0 \dots w_{k-1} \mid t_0, t_1 - t_0, \dots, t_k - t_{k-1}\big). \tag{2.26}$$

(b) For $k > 1$, the generating functions $\mathbf{\Lambda}^*$ are homogeneous in the t_i (invariant under a shift $t_i \mapsto t_i + t$), and the $\mathbf{\Lambda}$ are homogeneous in the x_i (invariant under a shift $x_i \mapsto x_i \cdot x$).

(c) Both generating functions are invariant under cyclic permutation of the indices.

Proof. Part (a) is clear from the definitions. For $\mathbf{\Lambda}^*$, (c) is clear from the scaling relations imposed in $\widetilde{\mathcal{D}}(G)$. For $\mathbf{\Lambda}$, (b) is also immediate. Part (c) for $\mathbf{\Lambda}$ would follow easily from (a) and (b,c) for $\mathbf{\Lambda}^*$, recalling that $t_1 + \dots + t_k = 0$.

The nontrivial part is (b) for $\mathbf{\Lambda}^*$. We must show

$$\mathbf{\Lambda}^*\big(w_0, \dots, w_k \mid t_0 + t, \dots, t_k + t\big) = \mathbf{\Lambda}^*\big(w_0, \dots, w_k \mid t_0, \dots, t_k\big).$$

Consider the coefficient of $t^n \cdot \prod_i t_i^{n_i}$ on each side. If $k = 0$, the coefficients on both sides are equal. If $k > 0$, the coefficient on the left side is precisely a first shuffle relation (where the n 0s indexed by the variable t are shuffled with all other points, with the point 1 remaining fixed), while the right side is 0. $\square$

The first shuffle relation imposed in $\widetilde{\mathcal{D}}(G)$ can be expressed in terms of the $\mathbf{\Lambda}$:

Lemma 2.12. *The generating functions $\mathbf{\Lambda}$ obey a shuffle relation for $r, s > 1$:*

$$\sum_{\sigma \in \Sigma_{r,s}} \mathbf{\Lambda}\big(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(r+s)}, x_0 \mid t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(r+s)}, t_0\big) = 0. \tag{2.27}$$

Proof. Similar to the previous lemma. It follows from the shuffle relation on the coefficients, where we fix x_0 and shuffle the $x_1, \dots, x_r$ and the zeros indexed by $t_1, \dots, t_r$ with the other points. $\square$

2.2.2 Proof of Theorem 2.5

Summary of the proof

The proof of the Theorem 2.5 will be by induction on the depth of the second shuffles.

Define

$$\mathrm{QSh}^{r,s}(w_1|S_1, \dots, w_n|S_n, w_0|S_0) =$$
$$= \sum_{\sigma \in \overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma} \mathbf{\Lambda}^*\big(w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(M_\sigma)}, w_0 \mid S_{\sigma^{-1}(1)}, \dots, S_{\sigma^{-1}(M_\sigma)}, S_0\big),$$

where $w_i \in G$ with $\prod_i w_i = 1$, and

$$\overline{\mathrm{QSh}}^{r,s}(w_1|S_1, \dots, w_n|S_n, w_0|S_0) = \mathrm{QSh}^{r,s}(w_1|S_1, \dots, w_n|S_n, w_0|S_0) \tag{2.28}$$
$$- \mathbf{\Lambda}^*\big(w_1, \dots, w_r, w_{\{r+1,\dots,r+s,0\}} \mid S_1, \dots, S_r, S_{\{r+1,\dots,r+s,0\}}\big) \tag{2.29}$$
$$- \mathbf{\Lambda}^*\big(w_{r+1}, \dots, w_{r+s}, w_{\{1,\dots,r,0\}} \mid S_{r+1}, \dots, S_{r+s}, S_{\{1,\dots,r,0\}}\big). \tag{2.30}$$

We must show that the elements $\overline{\mathrm{QSh}}$ form a coideal, i.e., their coproducts vanish modulo other elements of this form.

To make the notation more transparent, when r and s are fixed, we will relabel

$$\begin{aligned} T_1, \ldots, T_r &= S_1, \ldots, S_r, \\ U_1, \ldots, U_s &= S_{r+1}, \ldots, S_{r+s}, \\ V &= S_0, \\ a_1, \ldots, a_r &= w_1, \ldots, w_r \\ b_1, \ldots, b_s &= w_1, \ldots, w_{r+s}, \\ c &= w_0, \end{aligned}$$

so that we consider elements

$$\overline{\mathrm{QSh}}^{r,s}(a_1|T_1, \ldots, a_r|T_r, b_1|U_1, \ldots, b_s|U_s, c|V).$$

The main steps will be the following:

Step 0. Fix the a_i and b_j. Show that it suffices to assume $|T_i| = |U_i| = |V| = 1$. Denote the three terms (2.28), (2.29), (2.30) by Q, R_A, and R_B, respectively.

Step 1. Show that $\delta(Q - R_A - R_B)$ is zero modulo shuffle relations of lower depth and elements of the form $C(0, x)$ (Lemma 2.14).

(a) Group the terms of δQ according to a combinatorial classification and reduce them using shuffle relations of lower depth (Lemma 2.16).

(b) Group the terms of $\delta(R_A)$ and $\delta(R_B)$ in the same way and show that they coincide with the terms found in (a) (Lemma 2.22).

Step 2. Show that the (weight 1)$\wedge$(weight ≥ 1) component of $\delta(Q - R_A - R_B)$ is 0, modulo shuffle relations of lower depth (Lemma 2.15).

Throughout the proof, in a term $\mathbf{\Lambda}^*\big(w_1, \ldots, w_k, w_0 \mid s_1, \ldots, s_k, s_0\big)$ appearing in the definition of $\overline{\mathrm{QSh}}$, call the segment $\big(w_0 \mid s_0\big)$ the *distinguished segment* (i.e., $\big(c \mid v\big)$ in (2.28) and the collapsed segments in (2.29) and (2.30)). In the following lemmas, we will always use the following classification of terms of the coproduct of a generating function (see Figure 2.6).

(1) Terms $g \wedge h$ where one of the parts g or h contains the distinguished segment (i.e., the distinguished segment is not cut). In this case, we always write the term in the form $\pm g \wedge h$, where g contains the distinguished segment.

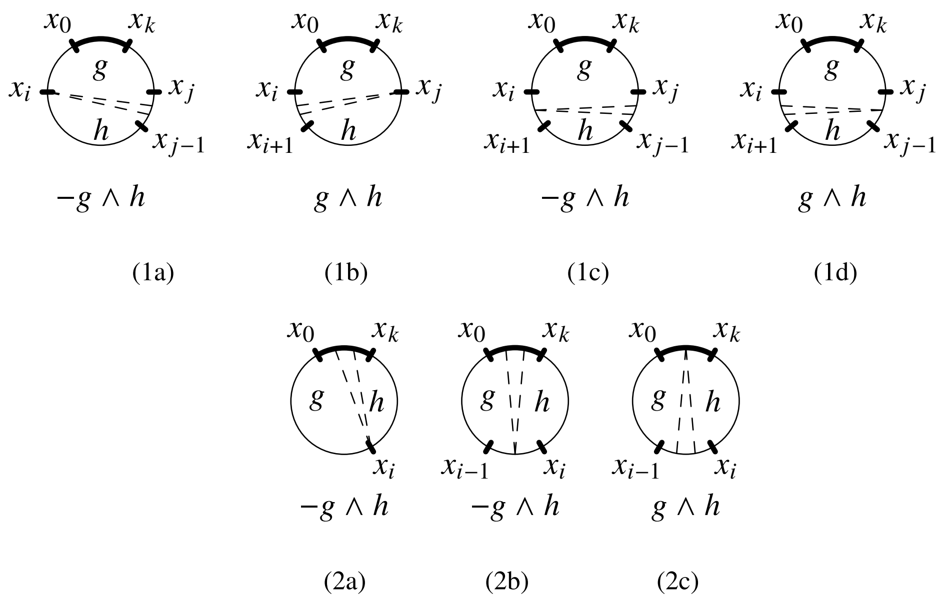


Figure 2.6. Classification of terms in the coproduct used in the proof of Theorem 2.5.

(a) Cut from a point x_i to the segment $\left(w_j \mid s_j\right)$ $(0 \le i < j \le k)$.

(b) Cut from a point x_j to the segment $\left(w_{i+1} \mid s_{i+1}\right)$ $(0 \le i < j \le k)$.

(c) Cut from a 0 on the segment $\left(x_{i+1} \mid t_{s+1}\right)$ to the segment $\left(w_j \mid t_j\right)$ $(0 \le i < j \le k)$.

(d) Cut from a 0 on the segment $\left(x_j \mid s_j\right)$ to the segment $\left(w_{i+1} \mid t_{i+1}\right)$ $(0 \le i < j \le k)$.

(2) Terms $g \wedge h$ where the distinguished segment is cut. In this case, we always write $\pm g \wedge h$, where g contains the point x_0 and h the point x_k.

(a) Cut from a point x_i to the distinguished segment.

(b) Cut from a 0 on the segment $\left(w_i \mid s_i\right)$ to the distinguished segment $(0 < i < k)$.

(c) Cut from a 0 on the distinguished segment to the segment $\left(s_i \mid t_i\right)$ $(0 < i < k)$.

Step 0

As stated in Step 0 above, we fix $m > 0$ and $n > 0$, the a_i, b_j, c, and the T_i, U_j, V, and let Q, R_A, R_B be the three terms of the expression defining $\overline{\mathrm{QSh}}$: (2.28), (2.29), and (2.30), respectively.

We may assume $T_i = \{t_i\}$, $U_j = \left\{u_j\right\}$, and $V = \{v\}$, by the following:

Lemma 2.13 (Step 0)**.** *The shuffle relations for* $|T_i| = \left|U_j\right| = |V| = 1$ *imply the shuffle relations for general index sets.*

Proof. Obvious by induction using (2.13). ◻

Lemma 2.14 (Step 1)**.** *Modulo shuffle relations of lower depth and elements $C(0,x)$, $\delta(Q - R_A - R_B) = 0$.*

Lemma 2.15 (Step 2)**.** *Modulo lower-depth shuffle relations and terms $C(0,x) \wedge C(0,y)$,*

$$\delta(Q - R_A - R_B) = \left[\sum_{i=1}^{m} C(0,a_i)(t_i - v) + \sum_{j=1}^{n} C(0,b_j)(u_j - v)\right] \wedge (Q - R_A - R_B). \tag{2.31}$$

Proof of Step 1

Lemma-Computation 2.16 (Step 1(a))**.** *Modulo shuffle relations of lower depth and elements $C(0,x)$, δQ is given by the sum of expressions (2.74)-(2.78) below.*

Group all terms of δQ by the type of cut as defined in the outline above. Some computational lemmas will simplify the contributions to δQ coming from the cuts of each type. The contribution of cuts (1a/b/c/d) is computed in Lemma 2.17, and cuts (2a/b/c) are dealt with in Lemma 2.21.

Lemma-Computation 2.17. *The contribution of cuts of type (1a/b/c/d) to δQ, modulo shuffle relations of lower depth and elements $C(0,x)$, is given by (2.57) below.*

The cuts of types (1a) and (1b) contribute terms of the form (2.19), while cuts of types (1c) and (1d) contribute terms of the form (2.20) below.

Consider the upper parts g of terms $\pm g \wedge h$ as shown in Figure 2.6; by cyclic invariance modulo $C(0,x)$ we may write

$$g = \mathbf{\Lambda}^*\big(w_1, \ldots, w_l, c \mid S_1, \ldots, S_l, V\big).$$

Let $\big(w_p \mid S_p\big)$ be the new segment arising from the cut (that is, the bracketed segment in (2.19) or (2.20)).

We say that a_i appears in g if either the segment $\big(a_i \mid t_i\big)$ or some $\big(a_i b_j \mid \{t_i, u_j\}\big)$ is present in g as one of the $\big(w_l \mid S_l\big)$ $(l \neq i)$, and similarly for b_j. Then the set of segments that *do not* appear in g ("appear below g") is determined by the $w_1, \ldots, \widehat{w_p}, \ldots, w_l$ and consists of consecutively indexed elements a_i and b_j, i.e., $a_{i_0}, \ldots, a_{i_1}$ and $b_{j_0}, \ldots, b_{j_1}$, where by convention $i_0 = i_1 + 1$ if no a_i appear, and likewise for j_0, j_1.

Group the terms $g \wedge h$ by the sequence of segments $w_1, \ldots, \widehat{w_p}, \ldots, w_l$. To shorten notation, write

$$\widetilde{g}\,(S) = \mathbf{\Lambda}^*\big(w_1, \ldots, w_p, \ldots, w_l, c \mid S_1, \ldots, S_p = S, \ldots, S_l, V\big).$$

There are three cases:

(1) $i_1 - i_0 > 0$ and $j_1 - j_0 > 0$: at least two a_i and two b_j appear below g (Lemma-Computation 2.18).

(2) $i_1 - i_0 = -1$ or $j_1 - j_0 = -1$: only a_i's or only b_j's appear below g (Lemma-Computation 2.19).

(3) $i_1 - i_0 = 0$ or $j_1 - j_0 = 0$: only one a_i or only one b_j appear below g (Lemma-Computation 2.20).

We compute the contribution of each case in the next three lemmas.

Lemma-Computation 2.18. *Case 1 ($i_1 - i_0 > 0$ and $j_1 - j_0 > 0$) contributes 0 to δQ.*

Proof. Consider a term $g \wedge h$ coming from a cut in Case 1. Let $i_0' \geq i_0$ be minimal such that $a_{i_0'}$ appears in h, and $i_1' \leq i_1$ be maximal such that $a_{i_1'}$ appears in h. Define j_0', j_1' in the analogous way. For example, for cuts of type (1a), $i_0' = i_0$; for cuts of type (1c),

$$i_0' = \begin{cases} i_0 & \text{if } \left(w \mid S\right) \text{ is } \left(b_{i_0} \mid u_{i_0}\right) \\ i_0 + 1 & \text{if } \left(w \mid S\right) \text{ is } \left(a_{i_0} \mid t_{i_0}\right) \text{ or } \left(a_{i_0} b_{i_0} \mid \left\{t_{i_0}, u_{j_0}\right\}\right) \end{cases},$$

where $\left(w \mid S\right)$ is the segment that contains the vertex of the cut. Notice that $i_0' - i_0 \leq 1$ and $j_0 - j_0' \leq 1$, and $i_1 - i_0 > 0$ implies $i_1' - i_0' \geq -1$.

Group all terms of δQ coming from Case 1 by the type of cut and by i_0', j_0', i_1', j_1'. These groups can be expressed in terms of

$$\widetilde{g}\,(S_1) \wedge \mathrm{QSh}\Big(a_{i_0'}, \ldots, a_{i_1'}, b_{j_0'}, \ldots, b_{j_1'}, \left(a_{i_0'} \ldots a_{i_1'} \cdot b_{j_0'} \ldots b_{j_1'}\right)^{-1} \mid t_{i_0'}, \ldots, t_{i_1'}, u_{j_0'}, \ldots, u_{j_1'}, S_2\Big)$$

for some S_1, S_2. Indeed, the arrangements of segments that may occur in the lower part of the cut, given i_0, j_0 and i_1', j_1', are precisely the quasishuffles. Applying the lower-weight shuffle relations, this expression becomes

$$\begin{aligned} &\widetilde{g}\,(S_1) \wedge \left(\mathbf{\Lambda}^*\left(a_{i_0'}, \ldots, a_{i_1'}, \left(a_{i_0'} \ldots a_{i_1'}\right)^{-1} \mid t_{i_0'}, \ldots, t_{i_1'}, \left\{u_{j_0'}, \ldots, u_{j_1'}\right\} \sqcup S_2\right)\right) \\ +&\widetilde{g}\,(S_1) \wedge \left(\mathbf{\Lambda}^*\left(b_{j_0'}, \ldots, b_{j_1'}, \left(b_{j_0'} \ldots b_{j_1'}\right)^{-1} \mid u_{j_0'}, \ldots, u_{j_1'}, \left\{t_{i_0'}, \ldots, t_{i_1'}\right\} \sqcup S_2\right)\right). \end{aligned} \tag{2.32}$$

Fix i_0', i_1', j_0', j_1', and introduce the notation

$$\begin{aligned} \widetilde{f_A}(i_0', i_1', S) &= \mathbf{\Lambda}^*\left(a_{i_0'}, \ldots, a_{i_1'}, \left(a_{i_0'} \ldots a_{i_1'}\right)^{-1} \mid t_{i_0'}, \ldots, t_{i_1'}, \left\{u_{j_0+1}, \ldots, u_{j_1-1}\right\} \sqcup S\right), \\ \widetilde{f_B}(j_0', j_1', S) &= \mathbf{\Lambda}^*\left(b_{j_0'}, \ldots, b_{j_1'}, \left(b_{j_0'} \ldots b_{j_1'}\right)^{-1} \mid u_{j_0'}, \ldots, u_{j_1'}, \left\{t_{i_0+1}, \ldots, t_{i_1-1}\right\} \sqcup S\right). \end{aligned}$$

The expressions in (2.32) can be rewritten with $\widetilde{f_A}$ and $\widetilde{f_B}$.

Now let us collect these terms coming from different cuts and show that they yield 0. By symmetry, it suffices to show this for three kinds of terms $\widetilde{f}_A(i'_0, i'_1, j'_0, j'_1, S_2)$: where $i'_0 = i_0$ and $i'_1 = i_1$; where $i'_0 = i_0$ and $i'_1 = i_1 - 1$; and where $i'_0 = i_0 + 1$ and $i'_1 = i_1 - 1$.

Look at the terms with $i'_0 = i_0$ and $i'_1 = i_1$ (all a_i that are not in g are in f_A). They arise from cuts (1a) and (1b) where the cut segment is b_{i_0+1} or b_{i_1-1} and from cuts (1c) and (1d) where the cut segment and the segment containing the vertex are b_{i_0+1} and b_{i_1-1}, or vice versa. These cases give:

$$\begin{aligned} -\widetilde{g}\left(u_{j_1}\right) \wedge \widetilde{f}_A(i_0, i_1, \{u_{j_0}\} \sqcup \{u_{j_1}\}), \\ \widetilde{g}\left(u_{j_0}\right) \wedge \widetilde{f}_A(i_0, i_1, \{u_{j_1}\} \sqcup \{u_{j_0}\}), \\ (u_{j_1} - u_{j_0})\widetilde{g}\left(\{u_{j_0}, u_{j_1}\}\right) \wedge \widetilde{f}_A(i_0, i_1, \{u_{j_0}, u_{j_1}\}), \end{aligned}$$

the sum of which is 0 by (2.13).

The terms with $i'_0 = i_0$ and $i'_1 = i_1 - 1$ (all a_i that are not in g, except the last, are in f_A) come from three sources:

- cuts of type (1a) where the cut segment x_2 is either a_{i_1} or $a_{i_1} b_{j_1}$;
- cuts of type (1c) and (1d) where the segment x_1 containing the vertex and the segment x_2 that is cut are b_{j_0} and a_{i_1}, or vice versa;
- cuts of type (1c) and (1d) where the segment x_1 containing the vertex and the segment x_2 that is cut are b_{j_0} and $a_{i_1} b_{j_1}$, or vice versa.

A similar computation shows their total contribution is 0.

Finally, consider terms with $i'_0 = i_0 + 1$ and $i'_1 = i_1 - 1$ (all a_i not in g except the first and last are in f_A). They arise from cuts of type (1c) and (1d), where the segment x_1 is either a_{i_0} or $a_{i_0} b_{j_0}$ and the segment x_2 is either a_{i_1} or $a_{i_1} b_{j_1}$, yielding four cases:

$$(x_1, x_2) = (a_{i_0}, a_{j_1}), (a_{i_0} b_{j_0}, a_{j_1}), (a_{i_0}, a_{j_1} b_{j_1}), (a_{i_0} b_{j_0}, a_{i_1} b_{j_1}).$$

The sum of their contributions is also 0. □

Lemma-Computation 2.19. *The contribution of Case 2 ($i_1 - i_0 = -1$) to δQ is given by the sum of expressions (2.33)-(2.36) below.*

Proof. Suppose that $i_1 - i_0 = -1$. The cuts of types (1a), (1b), (1c), and (1d) contribute

$$-\widetilde{g}\left(u_{j_1}\right) \wedge \widetilde{f}_B(j_0, j_1 - 1, u_{j_1}), \tag{2.33}$$

$$\widetilde{g}\left(u_{j_0}\right) \wedge \widetilde{f}_B(j_0 + 1, j_1, u_{j_0}), \tag{2.34}$$

$$-u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0 + 1, j_1 - 1, \left\{u_{j_0}, u_{j_1}\right\}), \tag{2.35}$$

$$u_{j_1}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0 + 1, j_1 - 1, \left\{u_{j_0}, u_{j_1}\right\}), \tag{2.36}$$

respectively. □

By symmetry, analogous expressions will result if $j_1 - j_0 = -1$.

Lemma-Computation 2.20. *The contribution of Case 3 ($i_1 - i_0 = 0$) to the δQ is given by the sum of expressions (2.45) and (2.46) below.*

Proof. Suppose $i_1 - i_0 = 0$, so only one segment a_i occurs below g.

If $j_1 - j_0 = 0$, then it is easy to see that only cuts of type (1a) and (1b) contribute nonzero terms, and that the (1a) terms cancel with the (1b) terms. So assume $j_1 - j_0 > 0$.

The cuts of type (1a) fall into three classes depending on which segment is cut: (i) a_{i_0}, (ii) b_{j_1}, or (iii) $a_{i_0}b_{j_1}$. The first two contribute

$$-\widetilde{g}\left(t_{i_0}\right) \wedge \widetilde{f}_B(j_0, j_1, t_{i_0}), \tag{2.37}$$

$$\begin{aligned} &-\widetilde{g}\left(u_{j_1}\right) \wedge \mathrm{QSh}^{1, j_1 - j_0}\big(a_{i_0}, b_{j_0}, \ldots, b_{j_1 - 1}, \left(a_{i_0}b_{j_0} \ldots b_{j_1 - 1}\right)^{-1} \mid \\ &\qquad\qquad t_{i_0}, u_{j_0}, \ldots, u_{j_1 - 1}, u_{j_1}\big) \\ \equiv &-\widetilde{g}\left(u_{j_1}\right) \wedge \Big(\widetilde{f}_B(j_0, j_1 - 1, \left\{t_{i_0}, u_{j_1}\right\}) \\ &\qquad + \mathbf{\Lambda}^*\big(a_{i_0}, a_{i_0}^{-1} \mid t_{i_0}, \left\{u_{j_0}, \ldots, u_{j_1}\right\}\big)\Big), \end{aligned} \tag{2.38}$$

respectively, where we have used that the sequences that may occur in the lower part of the cut are precisely the shuffles of a_i and b_j appearing below g, except the cut segment b_{j_1}. Finally, the third class gives

$$\begin{aligned} \frac{1}{t_{i_0} - u_{j_1}}\Big(&\widetilde{g}\left(t_{i_0}\right) \wedge \widetilde{f}_B(j_0, j_1 - 1, t_{i_0}) \\ &-\widetilde{g}\left(u_{j_1}\right) \wedge \widetilde{f}_B(j_0, j_1 - 1, u_{j_1})\Big), \end{aligned} \tag{2.39}$$

where we have applied (2.13) to break the generating functions with $\left\{t_{i_0}, u_{j_1}\right\}$ into ones with only t_{i_0} or u_{j_1}.

The cuts of type (1c) fall into five classes, depending on the segment where the vertex of the cut lies and the segment that is cut: (i) vertex on a_{i_0} and b_{j_1} is cut, (ii) vertex on b_{j_0} and b_{j_1} is cut, (iii) vertex on b_{j_0} and a_{i_0} is cut, (iv) vertex on b_{j_0} and $a_{i_0}b_{j_1}$ is cut, (v) vertex on $a_{i_0}b_{j_0}$ and b_{j_1} is cut. They contribute the following terms:

$$-t_{i_0}\widetilde{g}\left(\left\{t_{i_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0, j_1-1, \left\{t_{i_0}, u_{j_1}\right\}), \tag{2.40}$$

$$\begin{aligned} &-u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \mathrm{QSh}^{1,j_1-j_0-1}(a_{i_0}, b_{j_0+1}, \dots, b_{j_1-1}, (a_{i_0}b_{j_0+1}\dots b_{j_1-1})^{-1} \mid \\ &\qquad t_{i_0}, u_{j_0+1}, \dots, u_{j_1-1}, \left\{u_{j_0}, u_{j_1}\right\}) \\ &\equiv -u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \Big(\widetilde{f}_B(j_0+1, j_1-1, \left\{t_{i_0}, u_{j_0}, u_{j_1}\right\}) \\ &\qquad + \mathbf{\Lambda}^*(a_{i_0}, a_{i_0}^{-1} \mid t_{i_0}, \left\{u_{j_0}, \dots, u_{j_1}\right\})\Big), \end{aligned} \tag{2.41}$$

$$-u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, t_{i_0}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1, \left\{u_{j_0}, t_{j_0}\right\}), \tag{2.42}$$

$$\begin{aligned} &\frac{1}{t_{i_0}-u_{j_1}}\Big(u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, t_{i_0}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1-1, \left\{u_{j_0}, t_{i_0}\right\}) \\ &\qquad -u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1-1, \left\{u_{j_0}; u_{j_1}\right\})\Big), \end{aligned} \tag{2.43}$$

$$\begin{aligned} &\frac{1}{t_{i_0}-u_{j_0}}\Big(t_{i_0}\widetilde{g}\left(\left\{t_{i_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1-1, \left\{t_{i_0}, u_{j_1}\right\}) \\ &\qquad -u_{j_0}\widetilde{g}\left(\left\{u_{j_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1-1, \left\{u_{j_0}, u_{j_1}\right\})\Big). \end{aligned} \tag{2.44}$$

The cuts (1b) and (1d) contribute antisymmetric terms, i.e., u_{j_0} and u_{j_1} are exchanged and $\widetilde{f}_B(j_0+d_0, j_1-d_1, S)$ becomes $-\widetilde{f}_B(j_0+d_1, j_1-d_0, S)$.The entire contribution of case 3 is then the symmetrization of the sum of expressions (2.37)-(2.44).

The expression (2.37) with its symmetrization cancels to 0.

The remaining terms form the contribution of Case 3, and are simplified to

$$\widetilde{g}\left(\left\{t_{i_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0, j_1-1, u_{j_1}) - \widetilde{g}\left(\left\{t_{i_0}, u_{j_0}\right\}\right) \qquad \wedge \widetilde{f}_B(j_0+1, j_1, u_{j_0}) \tag{2.45}$$

$$-(u_{j_1}-u_{j_0})\widetilde{g}\left(\left\{t, u_{j_0}, u_{j_1}\right\}\right) \wedge \widetilde{f}_B(j_0+1, j_1-1, \left\{u_{j_0}, u_{j_1}\right\}). \tag{2.46}$$

Analogous expressions result if $j_1 - j_0 = 0$. □

Proof of Lemma 2.17. Let us collect the terms obtained from cases 2 and 3: (2.33)-(2.36), (2.45), and (2.46).

Consider first the expressions of the form $\widetilde{f}_B(j_0, j_1-1, u_{j_1})$, arising from (2.33) and (2.45). (The notation $\widetilde{f}_B()$, which by definition depends on i_0 and i_1, is unambiguous here since no a_i appear in the expression for

$\widetilde{f}_B()$ when $i_1 - i_0 \leq 0$.) We claim that for fixed j_0 and j_1, the sum of these terms over all g is precisely

$$-\mathrm{QSh}^{m,n-(j_1-j_0)}\big(a_1,\ldots,a_r,b_1,\ldots,\underline{b_{j_0}\ldots b_{j_1}},\ldots,b_s,c \mid t_1,\ldots,t_r,u_1,\ldots,u_{j_1},\ldots,u_s,v\big) \quad \wedge\ \widetilde{f}_B(j_0,j_1-1,u_{j_1}). \tag{2.47}$$

Indeed, the term that appears on the left side for a fixed g is $-\widetilde{g}\left(u_{j_1}\right)$ if $i_1 - i_0 = -1$ and $\widetilde{g}\left(\left\{t_{i_0},u_{j_1}\right\}\right)$ if $i_1 - i_0 = 0$. The quasishuffles for which the underlined segment collides with no a_i provides the terms with $i_1 - i_0 = -1$, while the quasishuffles for which the underlined segment collides with some a_i provide the terms with $i_0 = i_1 = i$.

In a similar way, the expressions with $u_{j_0}\widetilde{f}_B(j_0+1,j_1-1,\left\{u_{j_0},u_{j_1}\right\})$, coming from (2.46) and (2.35), yield

$$-u_0\mathrm{QSh}^{m,n-(j_1-j_0)}\big(a_1,\ldots,a_r,b_1,\ldots,\underline{b_{j_0}\ldots b_{j_1}},\ldots,b_s,c \mid t_1,\ldots,t_r,u_1,\ldots,\left\{u_{j_0},u_{j_1}\right\},\ldots,u_s,v\big) \quad \wedge\ \widetilde{f}_B(j_0+1,j_1-1,\left\{u_{j_0},u_{j_1}\right\}). \tag{2.48}$$

The expressions with $\widetilde{f}_B(j_0+1,j_1,u_{j_1})$ and $u_{j_1}\widetilde{f}_B(j_0+1,j_1-1,\left\{u_{j_0},u_{j_1}\right\})$ give the antisymmetric terms.

Applying the shuffle relations of lower depth to (2.47) and (2.48), we get the total contribution of cases 2 and 3 for fixed j_0 and j_1:

$$\begin{aligned}
&-\Big(\boldsymbol{\Lambda}^*\big(a_1,\ldots,a_r,b_1\ldots b_s\cdot c \mid t_1,\ldots,t_r,\left\{u_1,\ldots,u_{j_0-1},u_{j_1},u_{j_1+1},\ldots,u_s,v\right\}\big)\\
&\qquad+\boldsymbol{\Lambda}^*\big(b_1,\ldots,b_{j_0}\ldots b_{j_1},\ldots,b_s,a_1\ldots a_r\cdot c \mid u_1,\ldots,u_{j_1},\ldots,u_s,\{t_1,\ldots,t_r,v\}\big)\Big)\\
&\quad\wedge\ \boldsymbol{\Lambda}^*\big(b_{j_0},b_{j_0+1},\ldots,b_{j_1-1},\left(b_{j_0}\ldots b_{j_1-1}\right)^{-1} \mid u_{j_0},u_{j_0+1},\ldots,u_{j_1-1},u_{j_1}\big) &(2.49)\\
&+\Big(\boldsymbol{\Lambda}^*\big(a_1,\ldots,a_r,b_1\ldots b_s\cdot c \mid t_1,\ldots,t_r,\left\{u_1,\ldots,u_{j_0-1},u_{j_0},u_{j_1+1},\ldots,u_s,v\right\}\big)\\
&\qquad+\boldsymbol{\Lambda}^*\big(b_1,\ldots,b_{j_0}\ldots b_{j_1},\ldots,b_s,a_1\ldots a_r\cdot c \mid u_1,\ldots,u_{j_1},\ldots,u_s,\{t_1,\ldots,t_r,v\}\big)\Big)\\
&\quad\wedge\ \boldsymbol{\Lambda}^*\big(b_{j_0+1},\ldots,b_{j_1-1},b_{j_1},\left(b_{j_0+1}\ldots b_{j_1}\right)^{-1} \mid u_{j_0+1},\ldots,u_{j_1-1},u_{j_1},u_{j_0}\big) &(2.50)\\
&+(u_{j_1}-u_{j_0})\Big(\boldsymbol{\Lambda}^*\big(a_1,\ldots,a_r,b_1\ldots b_s\cdot c \mid t_1,\ldots,t_r,\left\{u_1,\ldots,u_{j_0-1},u_{j_0},u_{j_1},u_{j_1+1},\ldots,u_s,v\right\}\big)\\
&\qquad+\boldsymbol{\Lambda}^*\big(b_1,\ldots,b_{j_0}\ldots b_{j_1},\ldots,b_s,a_1\ldots a_r\cdot c \mid u_1,\ldots,\left\{u_{j_0},u_{j_1}\right\},\ldots,u_s,\{t_1,\ldots,t_r,v\}\big)\Big)\\
&\quad\wedge\ \boldsymbol{\Lambda}^*\big(b_{j_0+1},\ldots,b_{j_1-1},\left(b_{j_1}\ldots b_{j_1-1}\right)^{-1} \mid u_{j_0+1},\ldots,u_{j_1-1},\left\{u_{j_0},u_{j_1}\right\}\big). &(2.51)
\end{aligned}$$

Notice that this expression does not depend on i_0, i_1, and all but one of the segments in each generating function f depends only on the a_i or only on the b_j.

Reindexing leads to cancelation of all terms $f(a_1, \ldots, a_r, \ldots)$ except the term in (2.49) where $j_0 = 1$ and the term in (2.50) where $j_1 = n$. That is, if $j_0 \neq 1$ and $j_1 \neq n$, then this expression becomes

$$\begin{aligned}
F(j_0, j_1) := -\Big(&\mathbf{\Lambda}^*\big(b_1, \ldots, b_{j_0} \ldots b_{j_1}, \ldots, b_s, a_1 \ldots a_r \cdot c \mid u_1, \ldots, u_{j_1}, \ldots, u_s, \{t_1, \ldots, t_r, v\}\big) \\
&\wedge \mathbf{\Lambda}^*\big(b_{j_0}, b_{j_0+1}, \ldots, b_{j_1-1}, \big(b_{j_0} \ldots b_{j_1-1}\big)^{-1} \mid u_{j_0}, u_{j_0+1}, \ldots, u_{j_1-1}, u_{j_1}\big)\Big) \qquad (2.52) \\
+\Big(&\mathbf{\Lambda}^*\big(b_1, \ldots, b_{j_0} \ldots b_{j_1}, \ldots, b_s, a_1 \ldots a_r \cdot c \mid u_1, \ldots, u_{j_0}, \ldots, u_s, \{t_1, \ldots, t_r, v\}\big) \\
&\wedge \mathbf{\Lambda}^*\big(b_{j_0+1}, \ldots, b_{j_1-1}, b_{j_1}, \big(b_{j_0+1} \ldots b_{j_1}\big)^{-1} \mid u_{j_0+1}, \ldots, u_{j_1-1}, u_{j_1}, u_{j_0}\big)\Big) \qquad (2.53) \\
+\Big(&\mathbf{\Lambda}^*\big(b_1, \ldots, b_{j_0} \ldots b_{j_1}, \ldots, b_s, a_1 \ldots a_r \cdot c \mid u_1, \ldots, \{u_{j_0}, u_{j_1}\}, \ldots, u_s, \{t_1, \ldots, t_r, v\}\big) \\
&\wedge \mathbf{\Lambda}^*\big(b_{j_0+1}, \ldots, b_{j_1-1}, \big(b_{j_1} \ldots b_{j_1-1}\big)^{-1} \mid u_{j_0+1}, \ldots, u_{j_1-1}, \{u_{j_0}, u_{j_1}\}\big)\Big)(u_{j_1} - u_{j_0}). \qquad (2.54)
\end{aligned}$$

If $j_0 = 1$ or $j_1 = s$, the following terms remain, respectively:

$$\begin{aligned}
F_L(j_1) :=& -\mathbf{\Lambda}^*\big(a_1, \ldots, a_r, b_1 \ldots b_s \cdot c \mid t_1, \ldots, t_r, \{u_{j_1}, u_{j_1+1}, \ldots, u_s, v\}\big) \\
&\wedge \mathbf{\Lambda}^*\big(b_1, \ldots, b_{j_1-1}, \big(b_1 \ldots b_{j_1-1}\big)^{-1} \mid u_1, \ldots, u_{j_1-1}, u_{j_1}\big), \qquad (2.55) \\
F_R(j_0) :=& \mathbf{\Lambda}^*\big(a_1, \ldots, a_r, b_1 \ldots b_s \cdot c \mid t_1, \ldots, t_r, \{u_1, \ldots, u_{j_0}, v\}\big) \\
&\wedge \mathbf{\Lambda}^*\big(b_{j_0+1}, \ldots, b_s, \big(b_{j_0+1} \ldots b_s\big)^{-1} \mid u_{j_0+1}, \ldots, u_s, u_{j_0}\big). \qquad (2.56)
\end{aligned}$$

Identical terms $G(i_0, i_1)$, $G_L(i_1)$, $G_R(i_0)$ with the $\big(a_i \mid t_i\big)$ and $\big(b_j \mid u_j\big)$ exchanged appear in the cases $j_1 - j_0 = 0$ or -1.

So the total contribution of cuts of type 1 is

$$\begin{aligned}
&\sum_{\substack{1 \le j_0, j_1 \le s \\ j_1 - j_0 \ge -1}} F(j_0, j_1) + \sum_{\substack{1 \le i_0, i_1 \le r \\ i_1 - i_0 \ge -1}} G(i_0, i_1) \\
&+ \sum_{1 \le j_1 \le s} F_L(j_1) + \sum_{1 \le j_0 \le s} F_R(j_0) + \sum_{1 \le i_1 \le r} G_L(i_1) + \sum_{1 \le i_0 \le r} G_R(i_0) \qquad (2.57)
\end{aligned}$$

finishing the computation. □

Lemma-Computation 2.21. *The contribution of cuts of type (2) to δQ is given by the sum of expressions (2.73)-(2.77) below, plus symmetrical terms.*

Proof. A cut of type (2a/b/c) divides the circle into a left part g and a right part h (see Figure 2.6). Let i_0 be maximal such that a_{i_0} appears in g and i_1 minimal such that a_{i_1} appears in h, with $i_0 = -1$ or $i_1 = m+1$ if the corresponding segments do not appear. Define j_0, j_1 in the same manner, for the b_j.

Let

$$\begin{aligned}
f_L(i_0, j_0, S) &= \mathbf{\Lambda}^*\big(a_1, \ldots, a_{i_0}, (a_1 \ldots a_{i_0})^{-1} \mid t_1, \ldots, t_{i_0}, \{u_1, \ldots, u_{j_0}\} \sqcup S\big), \\
f_R(i_1, j_1, S) &= \mathbf{\Lambda}^*\big(a_{i_1}, \ldots, a_r, (a_{i_1} \ldots a_r)^{-1} \mid t_{i_1}, \ldots, t_r, \{u_{j_1}, \ldots, u_m\} \sqcup S\big),
\end{aligned}$$

and define $g_L(i_0, j_0, S)$ and $g_R(i_1, j_1, S)$ in a similar way for the $\big(b_j \mid u_j\big)$. (As usual, one interprets these expressions as 0 if the index set is empty.) Also let

$$\begin{aligned}
q_L(i_0, j_0, S) &= \mathrm{QSh}^{i_0, j_0}\big(a_1, \ldots, a_{i_0}, b_1, \ldots, b_{j_0}, (a_1 \ldots a_{i_0} \cdot b_1 \ldots b_{j_0})^{-1} \mid t_1, \ldots, t_{i_0}, u_1, \ldots, u_{j_0}, S\big), \\
&= f_L(i_0, j_0, S) + g_L(i_0, j_0, S) \\
q_R(i_1, j_1, S) &= \mathrm{QSh}^{r-i_1+1, s-j_1+1}\big(a_{i_1}, \ldots, a_r, b_{j_1}, \ldots, b_s, (a_{i_1} \ldots a_r \cdot b_{j_1} \ldots b_s)^{-1} \mid t_{i_1}, \ldots, t_r, u_{j_1}, \ldots, u_s, S\big) \\
&= f_R(i_1, j_1, S) + g_R(i_1, j_1, S).
\end{aligned}$$

Consider cuts (2a) for fixed i_0, i_1, j_0, j_1. For such cuts,

$$i_1 - i_0 = j_1 - j_0 = 1, \quad -1 \le i_0 \le r, \quad -1 \le j_0 \le s.$$

The g that occur in the resulting terms are exactly the quasishuffles of $\{a_i : i \le i_0\}$ and $\big\{b_j : j \le j_0\big\}$. The analogous statement holds for h. The contribution of cuts (2a) is

$$-q_L(i_0, j_0, v) \wedge q_R(i_0 + 1, j_0 + 1, v). \tag{2.58}$$

Now look at cuts (2b) and (2c). The non-distinguished segment containing the vertex or the cut is either a_{i_0+1} ($i_0 < r$), b_{j_0+1} ($j_0 < s$), or $a_{i_0+1}b_{j_0+1}$ ($i_0 < r, j_0 < s$). The terms coming from the sum of (2b) and (2c) are, for these three cases respectively,

$$(v - t_{i_0+1})q_L(i_0, j_0, \{t_{i_0+1}, v\}) \wedge q_R(i_0 + 2, j_0 + 1, \{t_{i_0+1}, v\}), \tag{2.59}$$

$$(v - u_{j_0+1})q_L(i_0, j_0, \{u_{j_0+1}, v\}) \wedge q_R(i_0 + 1, j_0 + 2, \{u_{j_0+1}, v\}), \tag{2.60}$$

$$\begin{aligned}
\frac{-1}{t_{i_0+1} - u_{j_0+1}}\Big(&(v - t_{i_0+1})q_L(i_0, j_0, \{t_{i_0+1}, v\}) \wedge q_R(i_0 + 2, j_0 + 2, \{t_{i_0+1}, v\}) \\
&-(v - u_{i_0+1})q_L(i_0, j_0, \{u_{j_0+1}, v\}) \wedge q_R(i_0 + 2, j_0 + 2, \{u_{j_0+1}, v\})\Big).
\end{aligned} \tag{2.61}$$

Let us assemble the terms of the form $f_L \wedge g_R$ and $g_L \wedge g_R$ coming from application of the shuffle relations to the q_L and q_R. (The terms $g_L \wedge f_R$ and $f_L \wedge f_R$ are symmetrical.)

The terms $f_L \wedge g_R$, for $-1 \leq i_0 < r$ and $-1 \leq j_0 < s$, are:

$$\begin{aligned}
&-f_L(i_0, j_0, v) \wedge g_R(i_0+1, j_0+1, v)\\
&+(v - t_{i_0+1}) f_L(i_0, j_0, \{t_{i_0+1}, v\}) \wedge g_R(i_0+2, j_0+1, \{t_{i_0+1}, v\}),\\
&+(v - u_{j_0+1}) f_L(i_0, j_0, \{u_{j_0+1}, v\}) \wedge g_R(i_0+1, j_0+2, \{u_{j_0+1}, v\}),\\
-\frac{1}{t_{i_0+1} - u_{j_0+1}}&\Big((v - t_{i_0+1}) f_L(i_0, j_0, \{t_{i_0+1}, v\}) \wedge g_R(i_0+2, j_0+2, \{t_{i_0+1}, v\})\\
&-(v - u_{i_0+1}) f_L(i_0, j_0, \{u_{j_0+1}, v\}) \wedge g_R(i_0+2, j_0+2, \{u_{j_0+1}, v\})\Big)\\
&= f_L(i_0, j_0, t_{i_0+1}) \wedge g_R(i_0+1, j_0+1, v)\\
&-f_L(i_0, j_0+1, t_{i_0+1}) \wedge g_R(i_0+1, j_0+2, v).
\end{aligned}$$

Summing this over j_0 leaves

$$f_L(i_0, 0, t_{i_0+1}) \wedge g_R(i_0+1, 1, v) - f_L(i_0, s, \{u_s, v\}) \wedge g_R(i_0+1, s+1, v) \tag{2.62}$$

$$= f_L(i_0, 0, t_{i_0+1}) \wedge g_R(i_0+1, 1, v) \qquad = -G_L(i_0+1). \tag{2.63}$$

If $i_0 = r$, $j_0 < s$, then from (2.58) and (2.60) we also have the terms

$$-f_L(r, j_0, v) \wedge g_R(r+1, j_0+1, v), \tag{2.64}$$

$$\begin{aligned}
&(v - u_{j_0+1}) f_L(r, j_0, \{u_{j_0+1}, v\}) \wedge g_R(r+1, j_0+2, \{u_{j_0+1}, v\})\\
&= f_L(r, j_0+1, v) \wedge \left(g_R(r+1, j_0+2, v) - g_R(r+1, j_0+2, u_{j_0+1})\right).
\end{aligned} \tag{2.65}$$

The last term $f_L(r, j_0+1, v) \wedge g_R(r+1, j_0+2, u_{j_0+1})$ is $F_R(j_0+1)$. The remaining term and (2.64) mostly cancel when summed over j_0, leaving only

$$\begin{aligned}
Z &:= -f_L(r, 0, v) \wedge g_R(r+1, 1, v) + f_L(r, s, v) \wedge g_R(r+1, s+1, v)\\
&= -\mathbf{\Lambda}^*\big(a_1, \ldots, a_r, \prod_j b_j \cdot c \mid t_1, \ldots, t_r, v\big) \wedge \mathbf{\Lambda}^*\big(b_1, \ldots, b_s, \prod_i a_i \cdot c \mid u_1, \ldots, u_s, v\big).
\end{aligned} \tag{2.66}$$

If $j_0 = s, i_0 < r$, there are terms

$$\begin{aligned} -f_L(i_0, s, v) \wedge g_R(i_0+1, s+1, v) &= 0, \\ (v - t_{i_0+1}) f_L(i_0, s, \{t_{i_0+1}, v\}) \wedge g_R(i_0+2, s+1, \{t_{i_0+1}, v\}) &= 0. \end{aligned} \tag{2.67}$$

Finally, $i_0 = r$, $j_0 = s$ also produces 0.

Thus the sum of terms $f_L \wedge g_R$ is

$$Z - \sum_{i_0=0}^{r-1} G_L(i_0+1) - \sum_{j_0=0}^{s-1} F_R(j_0+1). \tag{2.68}$$

Similarly, terms of the form $g_L \wedge f_R$ give

$$-Z - \sum_{j_0=0}^{s-1} F_L(j_0+1) - \sum_{i_0=0}^{r-1} G_R(i_0+1). \tag{2.69}$$

The terms $g_L \wedge g_R$ where $i_0 < r$, $j_0 < s$ are, similarly:

$$\begin{aligned} &-g_L(i_0, j_0, t_{i_0+1}) \wedge g_R(i_0+2, j_0+1, \{t_{i_0+1}, v\}) \\ &+g_L(i_0, j_0, \{t_{i_0+1}, u_{j_0+1}\}) \wedge g_R(i_0+2, j_0+2, \{t_{i_0+1}, v\}). \end{aligned} \tag{2.70}$$

If $i_0 = r$, $j_0 < s$, we get the terms

$$\begin{aligned} &-g_L(r, j_0, v) \wedge g_R(r+1, j_0+1, v), \\ &(v - u_{j_0+1}) g_L(r, j_0, \{u_{j_0+1}, v\}) \wedge g_R(r+1, j_0+2, \{u_{j_0+1}, v\}). \end{aligned} \tag{2.71}$$

If $j_0 = s, i_0 < r$, there are terms

$$\begin{aligned} -g_L(i_0, s, v) \wedge g_R(i_0+1, s+1, v) &= 0, \\ (v - t_{i_0+1}) g_L(i_0, s, \{t_{i_0+1}, v\}) \wedge g_R(i_0+2, s+1, \{t_{i_0+1}, v\}) &= 0. \end{aligned} \tag{2.72}$$

The case $i_0 = r$, $j_0 = s$ again contributes 0.

The terms $f_L \wedge f_R$ are symmetrical.

Assembling (2.69)-(2.71), the total contribution of cuts (2a/b/c) is

$$-\sum_{i=1}^{r} G_L(i) - \sum_{j=1}^{s} F_R(j) \tag{2.73}$$

$$+\sum_{i_0=0}^{r-1}\sum_{j_0=0}^{s-1}\Big(-g_L(i_0, j_0, t_{i_0+1}) \wedge g_R(i_0+2, j_0+1, \{t_{i_0+1}, v\}) \tag{2.74}$$

$$+g_L(i_0, j_0, \{t_{i_0+1}, u_{j_0+1}\}) \wedge g_R(i_0+2, j_0+2, \{t_{i_0+1}, v\})\Big) \tag{2.75}$$

$$+\sum_{j_0=0}^{s-1}\Big(-g_L(r, j_0, v) \wedge g_R(r+1, j_0+1, v) \tag{2.76}$$

$$+(v-u_{j_0+1})g_L(r, j_0, \{u_{j_0+1}, v\}) \wedge g_R(r+1, j_0+2, \{u_{j_0+1}, v\})\Big), \tag{2.77}$$

plus symmetrical terms. □

Proof of Lemma 2.16. Cancellation of (2.57) with (2.73) leaves

$$\sum_{\substack{1\le j_0, j_1 \le s\\ j_1 - j_0 \ge -1}} F(j_0, j_1) \tag{2.78}$$

plus the symmetrical term.

Thus δQ is the symmetrized sum of expressions (2.74)-(2.78). □

Lemma-Computation 2.22 (Step 1(b))**.** *Modulo elements $C(0,x)$, δR_B and δR_A are given by expression (2.91) below and its symmetric expression, respectively.*

Proof of Lemma 2.22. We compute δR_B.

Recall that the distinguished segment of R_B is $\prod_j a_i \cdot c$. We use the above classification of cuts of type (1a/b/c/d) and (2a/b/c).

Consider first the terms $f \wedge g$ coming from cuts of type (1). For each such term, let j_0, j_1 be the minimal and maximal indices of b_j that do not appear in g. For fixed j_0, j_1, the cuts of type (1a), (1b), and (1c/d) produce precisely the expressions (2.52), (2.53), and (2.54) above. Thus the contribution of cuts of type (1) is $F(j_0, j_1)$, and the total contribution is

$$\sum_{\substack{0\le j_0, j_1 \le s\\ j_1 - j_0 \ge -1}} F(j_0, j_1). \tag{2.79}$$

Next, we look at cuts of type (2). We will need a simplified formula for terms where either the vertex or the cut are on a segment indexed with $S = \{s_1, \ldots, s_k\}$. If $k = 1$ and the vertex is at a nonzero point, we get terms of the form

$$\mathbf{\Lambda}^*\big(\cdots \mid \dots, s_1\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid s_1, \dots\big).$$

Applying (2.13), it is easy to show by induction that, for general k, the resulting terms are

$$\sum_{i=1}^{k} \mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, \dots, s_i\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_i, \dots, s_k\}, \dots\big). \tag{2.80}$$

For example, if $k = 2$, this becomes

$$\begin{aligned}
&\mathbf{\Lambda}^*\big(\cdots \mid \dots, s_1\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_1, s_2\}, \dots\big) + \mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, s_2\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid s_2, \dots\big) \\
&= \frac{1}{s_1 - s_2}\Big(\mathbf{\Lambda}^*\big(\cdots \mid \dots, s_1\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid s_1, \dots\big) + \mathbf{\Lambda}^*\big(\cdots \mid \dots, s_2\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid s_2, \dots\big)\Big),
\end{aligned}$$

agreeing with the formula following directly from (2.13) that has been used in the previous computations.

Similarly, if the vertex is on some segment s', the term for $k = 1$,

$$s'\mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, s'\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_1, s'\}, \dots\big),$$

expands into

$$s' \sum_{i=1}^{k} \mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, \dots, s_i, s'\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_i, \dots, s_k, s'\}, \dots\big). \tag{2.81}$$

Finally, if the vertex is at a 0 on the segment s_i and the cut is on the segment s', we get terms

$$\begin{aligned}
&\sum_{i=1}^{k} s_i \mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, \dots, s_i, s'\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_i, \dots, s_k, s'\}, \dots\big) \\
&+ \sum_{i=1}^{k-1} \mathbf{\Lambda}^*\big(\cdots \mid \dots, \{s_1, \dots, s_i, s'\}\big) \wedge \mathbf{\Lambda}^*\big(\cdots \mid \{s_{i+1}, \dots, s_k, s'\}, \dots\big).
\end{aligned} \tag{2.82}$$

These identities can also be shown combinatorially, by interpreting the definition of the multiple generating functions in terms of collapsing segments.

For a term $f \wedge g$ coming from a cut of type (2), let j_0 be the maximal index of b_j appearing in f and j_1 the minimal index in g, so $j_1 - j_0 = 1$ for cuts (2a) and $j_1 - j_0 = 2$ for cuts (2b/c). By (2.80), for fixed j_0, the cuts of type (2a) contribute

$$-\sum_{i=1}^{m} g_L(i, j_0, \emptyset) \wedge g_R(i, j_0 + 1, v) \tag{2.83}$$

$$+ g_L(r, j_0, v) \wedge g_R(r + 1, j_0 + 1, v). \tag{2.84}$$

By (2.81), the cuts of type (2b) contribute

$$-\sum_{i=1}^{m} u_{j_0+1} g_L(i, j_0, u_{j_0+1}) \wedge g_R(i, j_0+2, \{u_{j_0+1}, v\}) \tag{2.85}$$

$$-u_{j_0+1} g_L(r, j_0, \{u_{j_0+1}, v\}) \wedge g_R(r+1, j_0+2, \{u_{j_0+1}, v\}). \tag{2.86}$$

By (2.82), the cuts of type (2c) contribute

$$\sum_{i=1}^{m} t_i g_L(i, j_0, u_{j_0+1}) \wedge g_R(i, j_0+2, \{u_{j_0+1}, v\}) \tag{2.87}$$

$$+v g_L(r, j_0, \{u_{j_0+1}, v\}) \wedge g_R(r+1, j_0+2, \{u_{j_0+1}, v\}) \tag{2.88}$$

$$+\sum_{i=1}^{m} g_L(i, j_0, u_{j_0+1}) \wedge g_R(i+1, j_0+2, \{u_{j_0+1}, v\}). \tag{2.89}$$

The sum of expressions (2.85), (2.87), and (2.89) simplifies to

$$g_L(i, j_0, u_{j_0+1}) \wedge g_R(i, j_0+2, v). \tag{2.90}$$

Then, letting $H_B(j_0)$ be the sum of expressions (2.83), (2.84), (2.86), (2.88), and (2.90), the coproduct of R_B is

$$\sum_{\substack{0 \le j_0, j_1 \le s \\ j_1 - j_0 \ge -1}} F(j_0, j_1) + \sum_{j=0}^{s-1} H_B(j). \tag{2.91}$$

The coproduct of R_A is the symmetric expression. □

Proof of Lemma 2.14. We now compare the results of the computations in Lemmas 2.16 and 2.22.

We have computed that δQ is the symmetrization of

$$(2.74) + (2.75) + (2.76) + (2.77) + (2.78)$$

and $\delta R_B + \delta R_A$ is the symmetrization of

$$(2.79) + \sum_{j=0}^{n-1} \left[(2.83) + (2.84) + (2.86) + (2.88) + (2.90)\right].$$

Obviously $(2.79) = (2.78)$. Now

$$(2.74) = \sum (2.83), \quad (2.75) = \sum (2.90), \quad (2.76) = \sum (2.84), \quad (2.77) = \sum \left[(2.86) + (2.88)\right],$$

which finishes the proof. □

Proof of Step 2

Here we show the terms of weight $(1) \wedge (w-1)$ coming from $\delta Q - \delta R_A - \delta R_B$ are 0.

Proof of Lemma 2.15. We first examine the relevant terms of δQ. Let us compute the coefficient L_i^A occurring with $C(0, a_i)$. These come from shuffles containing segment $\left(a_i \mid t_i\right)$, a segment $\left(a_i b_j \mid \left\{t_i, u_j\right\}\right)$, and the segment $\left(c \mid v\right)$ (where we write $c = \prod_i a_i^{-1} \prod_j b_j^{-1}$).

Inspect the generating functions of depth 1 $\mathbf{\Lambda}^*\left(w, w^{-1} \mid s_1, s_2\right)$ that appeared in the proof of Lemma 2.14. All generating functions in the lower half of cuts (1a/b/c/d) were written in a form where $\left(w \mid s_1\right)$ is the first segment counterclockwise of the distinguished segment, rather than with the segment counterclockwise of the vertex of the cut as in (2.19)).

So, by the remark following Lemma 2.10, the terms (2.24) vanish in the coproduct, so the terms arising from these cuts are canceled by the lower-depth shuffle relations in Lemma 2.14. Similarly, for cuts of type (2), we only have terms (2.22) contributing the coefficient of $C(0, a_i^{-1})$.

For quasishuffles in which $\left(a_i \mid t_i\right)$ appears, the terms (2.23) where some b_j appears immediately clockwise of a_i gives terms

$$\mathrm{QSh}_{(ij)}^{r,s}\left(a_1, \ldots, a_i, \ldots, a_r, b_1, \ldots, b_j, c \mid t_1, \ldots, \emptyset, \ldots, t_r, u_1, \ldots, u_j, \ldots, u_s, v\right), \tag{2.92}$$

where $\mathrm{QSh}_{(ij)}^{r,s}$ denotes the sum over only those quasishuffles where a_i collapses with b_j.

The terms (2.23) where either a_{i-1} or some $a_{i-1}b_j$ appears immediately clockwise of a_i sum to

$$\mathrm{QSh}^{r-1,s}\left(a_1, \ldots, a_{i-1}a_i, a_{i+1}, \ldots, a_r, b_1, \ldots, b_s, c \mid t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r, u_1, \ldots, u_s, v\right). \tag{2.93}$$

Finally, the terms (2.22) contribute to L_i^A the terms

$$t_i \mathrm{QSh}_{(i)}^{r,s}\left(a_1, \ldots, a_r, b_1, \ldots, b_s, c \mid t_1, \ldots, t_r, u_1, \ldots, u_s, v\right) \tag{2.94}$$

where $\mathrm{QSh}_{(i)}$ denotes the the quasishuffles in which a_i does not collapse with any b_j.

For quasishuffles in which some $\left(a_i b_j \mid \left\{t_i, u_j\right\}\right)$ appears, the terms (2.23) contribute 0, since they arise from cuts of segments containing no 0s.The terms (2.22) give

$$\frac{-1}{t_i-u_j}\Big(t_i\mathrm{QSh}^{r,s}_{(ij)}\big(a_1,\ldots,a_i,\ldots,a_r,b_1,\ldots,b_j,\ldots,b_s,c\ |t_1,\ldots,t_i,\ldots,t_r,u_1,\ldots,\emptyset,\ldots,u_s,v\big)$$

$$-u_j\mathrm{QSh}^{r,s}_{(ij)}\big(a_1,\ldots,a_i,\ldots,a_r,u_1,\ldots,u_j,\ldots,u_s,v\ |t_1,\ldots,\emptyset,\ldots,t_r,u_1,\ldots,u_j,\ldots,u_s,v\big)\Big)$$

$$=-t_i\mathrm{QSh}^{r,s}_{(ij)}\big(a_1,\ldots,a_i,\ldots,a_r,b_1,\ldots,b_s,c\ |t_1,\ldots,t_i,\ldots,t_r,u_1,\ldots,u_j,\ldots,u_s,v\big) \quad (2.95)$$

$$-\mathrm{QSh}^{r,s}_{(ij)}\big(a_1,\ldots,a_i,\ldots,a_r,b_1,\ldots,b_s,c\ |t_1,\ldots,\emptyset,\ldots,t_r,u_1,\ldots,u_j,\ldots,u_s,v\big). \quad (2.96)$$

For the segment $\big(c\mid v\big)$, which includes a factor of a_i^{-1}, we get a contribution of

$$-v\mathrm{QSh}^{r,s}\big(a_1,\ldots,a_r,b_1,\ldots,b_s,c\mid t_1,\ldots,t_r,u_1,\ldots,u_s,v\big)=-vQ. \quad (2.97)$$

from (2.22) and

$$-\mathrm{QSh}^{r-1,s}\big(a_1,\ldots,a_{m-1},b_1,\ldots,b_s,a_rc\ |t_1,\ldots,t_{r-1},u_1,\ldots,u_s,t_r\big)$$

$$-\mathrm{QSh}^{r,s-1}\big(a_1,\ldots,a_r,b_1,\ldots,b_{n-1},b_sc\ |t_1,\ldots,t_r,u_1,\ldots,u_{s-1},u_s\big)$$

$$+\mathrm{QSh}^{r-1,s-1}\big(a_1,\ldots,a_{m-1},b_1,\ldots,b_{n-1},a_rb_sc\ |t_1,\ldots,t_{r-1},u_1,\ldots,u_{s-1},\{t_r,u_s\}\big),$$

from (2.23), with three terms, depending on which segment (a_r, b_s, or a_rb_s) appears clockwise of c. By the lower-depth shuffle relations, this simplifies to

$$-\mathbf{\Lambda}^*\big(a_1,\ldots,a_r,\prod_j b_j\cdot c\ |t_1,\ldots,t_r,\{u_1,\ldots,u_s\}\big) \quad (2.98)$$

$$-\mathbf{\Lambda}^*\big(b_1,\ldots,b_s,\prod_i a_i\cdot c\ |u_1,\ldots,u_s,\{t_1,\ldots,t_r\}\big). \quad (2.99)$$

The terms (2.92) cancel with (2.96). Summing (2.95) over j and adding to (2.94) results in

$$t_i\mathrm{QSh}^{r,s}\big(a_1,\ldots,a_r,b_1,\ldots,b_s,c\mid t_1,\ldots,t_r,u_1,\ldots,u_s,v\big)=t_iQ. \quad (2.100)$$

Thus L_i^A is the sum of (2.93), (2.97), (2.98), (2.99) and (2.100). Applying lower-depth shuffle relations and

(2.13), this sum simplifies to

$$L_i^A = \mathbf{\Lambda}^*\big(a_1, \ldots, a_{i-1}a_i, a_{i+1}, \ldots, a_r, \prod_j b_j \cdot c \;|t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r, \{u_1, \ldots, u_s\} \sqcup \{v\}\big) \tag{2.101}$$

$$-\mathbf{\Lambda}^*\big(a_1, \ldots, a_r, \prod_j b_j \cdot c \;|t_1, \ldots, t_r, \{u_1, \ldots, u_s\}\big) \tag{2.102}$$

$$+(t_i - v)(Q - R_B). \tag{2.103}$$

Now let us compute the coefficient M_i^A occurring with $C(0, a_i)$ in $\delta(R_A)$. For the segment $\big(a_i \mid t_i\big)$ in R_A, (2.22) and (2.23) contribute the terms

$$t_i\mathbf{\Lambda}^*\big(a_1, \ldots, a_i, \ldots, a_r, \prod_j b_j \cdot c \;|t_1, \ldots, t_i, \ldots, t_r, \{u_1, \ldots, u_s\} \sqcup \{v\}\big) \qquad = t_i R_A, \tag{2.104}$$

$$\mathbf{\Lambda}^*\big(a_1, \ldots, a_{i-1}a_i, a_{i+1}, \ldots, a_r, \prod_j b_j \cdot c \;|t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r, \{u_1, \ldots, u_s\} \sqcup \{v\}\big), \tag{2.105}$$

where the second term appears only if $i > 1$.

The distinguished segment $\big(\prod_i a_i^{-1} \mid \{u_j\} \sqcup \{v\}\big)$ contributes only a term (2.22). By an argument similar to that in Lemma 2.22, this term can be written

$$-vR_A - \mathbf{\Lambda}^*\big(a_1, \ldots, a_r, \prod_j b_j \cdot c \mid t_1, \ldots, t_r, \{u_1, \ldots, u_s\}\big). \tag{2.106}$$

Combining (2.101)-(2.106), we find that

$$(L_i^A - M_i^A) = (t_i - v)(Q - R_A - R_B).$$

Therefore, adding the symmetric terms for the $C(0, b_j)$,

$$\delta(Q - R_A - R_B) = \left[\sum_{i=1}^{r} C(0, a_i)(t_i - v) + \sum_{j=1}^{s} C(0, b_j)(u_j - v)\right] \wedge (Q - R_A - R_B)$$

modulo lower-depth shuffle relations and elements $(\text{weight } 1) \wedge (\text{weight } 1)$. □

Conclusion

We are ready to use the coproduct we have computed to reduce the proof of the relations to a simple base case.

Proof of Theorem 2.5(a). We induct on the depth $r + s$. When $r = 0$ or $s = 0$, $\overline{\mathrm{QSh}}^{r,s}$ is identically 0.

If $r, s > 0$, taking coproduct on both sides of (2.31) and using that $\delta^2 = 0$, one deduces that $\delta(Q - R_A - R_B) = 0$ modulo shuffle relations of depth $< r + s$ and terms (weight 1) $\wedge$ (weight 1).

When no terms $C(0, x) \wedge C(0, y)$ are present in the coproduct, Lemma 2.13 and Lemma 2.15 imply that $\delta(Q - R_A - R_B)$ lies in the ideal generated by lower-depth relations.

These terms appear only in a base case: the constant term of the shuffle relation for $r = s = 1$. Showing the coproduct of this term is 0 amounts to proving the identity

$$\delta\left([C^*(a|0, b|0, c|0) + C^*(b|0, a|0, c|0) - C^*(ab|1, c|0)] - C^*(a|0, bc|1) - C^*(b|0, ac|1)\right) = 0. \quad (2.107)$$

We compute directly that the left side of (2.107) is

$$\begin{aligned}
&C(1,a) \wedge C(1,ab) + C(1,ab) \wedge (C(1,b) + C(0,a)) + (C(1,b) + C(0,a)) \wedge C(1,a) \\
&+ C(1,b) \wedge C(1,ab) + C(1,ab) \wedge (C(1,a) + C(0,b)) + (C(1,a) + C(0,b)) \wedge C(1,b) \\
&\qquad - C(1,ab) \wedge C(0,ab) + C(1,a) \wedge C(0,a) + C(1,b) \wedge C(0,b) \\
&= \quad C(1,ab) \wedge C(0,a) + C(0,a) \wedge C(1,a) \\
&\qquad + C(1,ab) \wedge C(0,b) + C(0,b) \wedge C(1,b) \\
&\qquad - C(1,ab) \wedge C(0,ab) + C(1,a) \wedge C(0,a) + C(1,b) \wedge C(0,b) \qquad = 0.
\end{aligned}$$

The theorem is proved. □

2.3 Specialization theorem for Hodge correlators

We now study how the Hodge correlators over a base B behave when the sections collide. This will require extending the theory of Hodge correlators to nodal curves.

The correlator Lie coalgebra for nodal curves

Recall the moduli space $\mathcal{M}'_{0,n}$ of n distinct points and a distinguished tangent vector on $\mathbb{P}^1$. Its Deligne-Mumford compactification $\overline{\mathcal{M}}'_{0,n}$ consists of the nodal curves of genus 0, i.e., those whose dual graph is a tree and in which every component is a punctured projective line. with n marked points and a distinguished tangent vector v_∞.

Let $X = \bigcup_i X_i$ be a genus 0 nodal curve with a set of punctures S. Let T be the dual tree of X, with vertices indexed by i corresponding to X_i, rooted at the component 0 with the base point $s_0 \in X_0$, oriented away from the root (write $i \to j$ if (i, j) is an edge). Choose a coordinate z_i on each X_i such that the point joining the component to its parent X_j is $(z_i = \infty, z_j = v_{ji})$, and the base point on X_0 is at $z_0 = \infty$ with tangent vector v_∞. Let S_i be the set of punctures on X_i. Let $N_i = \{v_{ij} : i \to j\}$.

We define the correlator Lie coalgebra for the nodal curve X by

$$\mathcal{CL}^\vee_{X,S,v_\infty} = \bigoplus_i \mathcal{CL}^\vee_{X_i, S_i \cup N_i, v_i}, \tag{2.108}$$

where v_i is the tangent vector $\frac{-1}{z_i^2}\frac{\partial}{\partial z_i}$ at $z_i = \infty$.

It coincides with the usual definition if X is smooth, justifying the notation. If X is not smooth, it is different from $\widetilde{\mathcal{CL}}^\vee_{X,S,s_0}$, the coalgebra naively defined as the tensor algebra of S modulo cyclic symmetry and shuffle relations with a $H_2(X)$ coefficient. They are related in the following way. For each i, there is a surjective coalgebra morphism to the component of the direct sum corresponding to X_i:

$$\widetilde{\mathcal{CL}}^\vee_{X,S,s_0} \xrightarrow{\pi_i} \mathcal{CL}^\vee_{X_i,S_i\cup N_i,v_i}.$$

To define it on a generator $(x_1 \otimes \cdots \otimes x_n) \otimes [X_i]$, let p be the common parent of the components containing the x_j. If $p \neq i$, the i-th component of the map is 0. Otherwise, set

$$\pi_i(x) = \begin{cases} x, & x \in X_i \\ z_i = v_{ij} \in N_i, & x \in X_k, \text{ where } \exists \text{ path } i \to j \to \cdots \to k \end{cases},$$

extended to preserve the tensor product. That is, points in X_i remain, while points in components below X_i collapse to the nearest node on X_i. Evidently this map preserves the coproduct and defining relations. Taking the direct sum of the maps π_i, we have produced a coalgebra morphism:

$$\pi : \widetilde{\mathcal{CL}}^\vee_{X,S,v_\infty} \to \mathcal{CL}^\vee_{X,S,v_\infty}.$$

It preserves the decomposition of the domain by $H_2(X) = \bigoplus_i H_2(X_i)$.

In particular, if (X, S, v_0) vary over a base $B \to \mathcal{M}'_{0,n}$, and the variation extends to $\overline{B} \to \overline{\mathcal{M}}'_{0,n}$, with $D = \overline{B} \setminus B$, then we have a *degeneration map*

$$\pi_D : \mathcal{CL}^\vee_{X/B,S,v_\infty} \to \widetilde{\mathcal{CL}}^\vee_{X/D,S,v_\infty} \to \mathcal{CL}^\vee_{X/D,S,v_\infty}, \tag{2.109}$$

where the first map simply applies the induced map on H_2 and the second map is the quotient defined above. The composition forgets the way in which the sections in S collided at boundary of B.

Specialization theorem

Recall that an element of $\mathcal{CL}^\vee_{X/B,S,v_\infty}$ over a base $B \to \mathcal{M}'_{0,n}$ determines, by the map $\mathrm{Cor}_{\mathrm{Hod}}$, a variation of Hodge structures over B, and, by the period map p, a smooth function on B. The maps $\mathrm{Cor}_{\mathrm{Hod}}$ and p also exist for X a nodal curve, extended by linearity from the definition (2.108).

Theorem 2.23. *Suppose $B \to \mathcal{M}'_{0,n}$ is a family of curves (X, S, v_∞) extending to $\overline{B} \to \overline{\mathcal{M}}'_{0,n}$, with $D = \overline{B} \setminus B$ a normal crossings divisor, and suppose $x \in \mathcal{CL}^\vee_{X/B,S,s_0}$ of weight $n > 1$.*

(a) The Deligne's canonical extension to D of the variation of framed mixed Hodge structures determined by $\mathrm{Cor}_{\mathrm{Hod}}(x)$ is independent of the normal vector to D. Thus there is a specialized map $\mathrm{Spec}_D \mathrm{Cor}_{\mathrm{Hod}}$: $\left(\mathcal{CL}_{X/B,S,v_\infty} \to \mathrm{Lie}^\vee_{\mathrm{HT}/D}\right)_{w>1}$.

(b) This specialized map coincides with the Hodge correlator of the degeneration map:

$$\begin{tikzcd} \left(\mathcal{CL}^\vee_{X/B,S,v_\infty}\right)_{w>1} \arrow[r, "\pi_D"] \arrow[d, "\mathrm{Cor}_{\mathrm{Hod}}"'] & \left(\mathcal{CL}^\vee_{X/D,S,v_\infty}\right)_{w>1} . \arrow[d, "\mathrm{Cor}_{\mathrm{Hod}}"] \\ (\mathrm{Lie}^\vee_{\mathrm{HT}/B}) \arrow[r, "\mathrm{Spec}_D"] & (\mathrm{Lie}^\vee_{\mathrm{HT}/D}). \end{tikzcd}$$

(c) Let $t = 0$ be a local equation for D. Then

$$\lim_{t\to 0} p(\mathrm{Cor}_{\mathrm{Hod}}(x_t)) = p(Cor_{\mathrm{Hod}}(x_{t=0})).$$

Proof. Let $x \in \mathcal{CL}^\vee_{X/B,S,v_\infty}$ be a generator of weight $w > 1$. For any v a normal vector to D, we get the specialized framed mixed Hodge-Tate structure $\mathrm{Spec}^v_D \mathrm{Cor}_{\mathrm{Hod}}(x)$.

We must show that:

(1) The periods of $\mathrm{Cor}_{\mathrm{Hod}}(x)$ extend continuously to D.

(2) The coproduct of $\mathrm{Spec}^v_D \mathrm{Cor}_{\mathrm{Hod}}(x)$ does not depend on the direction of specialization v at any smooth point of D.

(3) The periods of the specializations (i.e., the limits of the periods at D) coincide with the periods of the degeneration to D.

We will prove (1)-(3) by induction on the weight. First, let us see how they imply the result.

Assuming (2), the coproduct of $\mathrm{Spec}^v_D \mathrm{Cor}_{\mathrm{Hod}}(x)$ is independent of v. Because the coproduct commutes with Spec^v_D, this element is independent of v up to $\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(n))$, which is 1-dimensional and controlled by the period. By (1), the period is independent of the direction of specialization, which gives (a). By (3), it coincides with the period of the degeneration, which gives (b). Then (c) follows by the definitions from (b).

To show (1), we let $\{\varepsilon_i = 0\}$ be a set of smooth local equations for D and prove that $p(\mathrm{Cor}_{\mathrm{Hod}}(x))$ can be represented locally as a polynomial in the $\log \varepsilon_i$ such that the terms with $\log \varepsilon_i$ appearing in positive degree have coefficients vanishing along $\{\varepsilon_i = 0\}$ (*tame logarithmic singularities*). This will follow from the differential equations on the periods. Note that in weight 1, the period of $C(x, y)$ has a (not tame) logarithmic singularity along $x = y$. In weight > 1, we proceed by induction.

Consider a simple element $x = x_0 \otimes \cdots \otimes x_n \in C\mathcal{L}^\vee_{X/B,S,v_\infty}$ ($n > 1$). Suppose that not all x_i collide on D, so we must only consider the summand of the nodal $C\mathcal{L}^\vee_{X/D,S,s_0}$ corresponding to the component containing the base point. The terms of $\delta(x)$ can be grouped into those of two forms:

(i) $x' \wedge x''$, where not all sections in x' and in x'' collide to the same section on D;

(ii) $x' \wedge (x''_1 - x''_2)$, where not all sections in x' collapse on D, but x''_1 and x''_2 coincide on D.

By the inductive hypothesis, the specialization of $\delta(x)$ does not depend on the direction of specialization: for terms (i), x' and x'' satisfy (2), while in terms (ii) the $x''_1 - x''_2$ vanish under specialization to D. This gives (2).

For (1), from the differential equations on the periods (1.14), we see that $d_B p(\mathrm{Cor}_{\mathrm{Hod}}(x))$ is a sum of terms that are smooth over B with logarithmic singularities along D (from type (i)) and terms that vanish along D by the inductive hypothesis (from type (ii)). We conclude that $p^{v_\infty}(x)$ has tame logarithmic singularities along D.

If all x_i collide on D, we simply pass to their common parent component and apply the same argument.

We conclude with (3). We have shown that the specializations of $\mathrm{Cor}^{v_0}_{\mathrm{Hod}}$ and its coproduct to D exist at every point and their periods are independent of v, and thus the specialized period map $p \circ \mathrm{Cor}_{\mathrm{Hod}}$ is equal to the period of the degeneration up to adding a constant for each smooth component of the smooth locus of D. We must show the constant 0.

It is enough to show this for D a lowest-codimension boundary stratum in $\overline{\mathcal{M}}'_{g,n}$. We are done by the next lemma. □

Lemma 2.24. *Let I be a proper subset of $\{0, 1, \ldots, n\}$ ($n > 1$) and $x_0, \ldots, x_n \in \mathbb{C}^*$ with $x_i \neq x_j$ if $i \neq j$ and either $i, j \in I$ or $i, j \notin I$. Let*

$$x_i(t) = \begin{cases} tx_i & i \in I, \\ x_i & i \notin I \end{cases}.$$

Then

$$\mathrm{Cor}_{\mathcal{H}}(x_0(t), x_1(t), \ldots, x_n(t))$$

is continuous at $t = 0$*.*

Proof. For $n = 2$, this amounts to continuity of $\mathcal{L}_2$ at 1.

In the proof of Theorem 2.23 it was established that

$$\lim_{t\to 0} \mathrm{Cor}_{\mathcal{H}}(x_0(t), x_1(t), \ldots, x_n(t)) - \mathrm{Cor}_{\mathcal{H}}(x_0(0), x_1(0), \ldots, x_n(0))$$

is independent of the x_i, for generic x_i. Let us integrate this difference over $(x_0, \ldots, x_n) \in (S^1)^{n+1}$, with respect to the standard measures $\mu(x_i)$ of volume 1 on $S^1 = \{|z| = 1\} \subset \mathbb{C}$.

The limit is uniform in the directions x_i $(i \in I)$, and so

$$\int \lim_{t\to 0} \mathrm{Cor}_{\mathcal{H}}(x_0(t), x_1(t), \ldots, x_n(t)) \prod d\mu(x_i) = \lim_{t\to 0} \int \mathrm{Cor}_{\mathcal{H}}(x_0(t), x_1(t), \ldots, x_n(t)) \prod d\mu(x_i).$$

To conclude, it suffices to show that

$$\int \mathrm{Cor}_{\mathcal{H}}(x_0(t), x_1(t), \ldots, x_n(t)) \prod d\mu(x_i) = 0. \tag{2.110}$$

for all t.

For any tree T entering into the Feynman integral expression for (2.110), choose a pair of boundary vertices (without loss of generality, labeled x_0 and x_1) incident to a common internal vertex v with corresponding variable x_v, and let x_w be variable corresponding to the third vertex incident to v. Then the integral over the x_i contains the term

$$\int_{x_0, x_1} \left(\int \mathcal{L}_2 \left(\frac{x_w - x_0(t)}{x_w - x_1(t)} \right) \wedge (\text{terms independent of } x_0(t), x_1(t)) \right) d\mu(x_0)\, d\mu(x_1).$$

Exchanging the two integrals and noting that $\mathcal{L}_2(\frac{z-a}{z-b})$ changes sign under the involution

$$a \mapsto \overline{a} \frac{z^2}{|z|^2}, \quad b \mapsto \overline{b} \frac{z^2}{|z|^2},$$

we conclude that this expression is 0. □

The specialization theorem states that when the punctures labeling an element of $\mathcal{CL}^\vee$ collide, only the component nearest to the base point of the resulting nodal curve determines the limit Hodge correlator. We obtain as a corollary Theorem 1.6:

Theorem. *The Hodge correlators* $\mathrm{Cor}_{\mathcal{H}}(z_0, \ldots, z_n)$ *are continuous on* $\mathbb{C}^{n+1} \setminus \{z_0 = \cdots = z_n\}$.

For example, $\mathcal{L}_2$ is continuous with a tame logarithmic singularity at 1, but $\mathcal{L}_2\left(\frac{a-c}{b-c}\right)$ has no limit as $a, b, c \to 0$.

2.4 The second shuffle relations

2.4.1 Proofs of Theorems 2.1, 2.6, and 2.7

In this section we will prove the second shuffle relations for Hodge and motivic correlators.

Proof for Hodge correlators

Recall Theorem 2.6:

Theorem. *(a) Restricted to the subspace of* $\mathcal{CL}^{\vee}_{X,S,v_\infty}$ *generated by elements* $(x_0 \otimes \cdots \otimes x_n)(1)$ *with not all* x_i *equal, the map* $\mathrm{Cor}_{\mathrm{Hod}}$ *factors through* $\mathcal{D}^\circ(\mathbb{C}^*)$. *(Here* $S \subset \mathbb{P}^1(\mathbb{C})$ *is any finite set of punctures containing all points appearing in the relation in (b).)*

(b) Suppose that $r, s > 1$ *and that not all* $n_i = 0$ *or not all* $w_i = 1$. *Then the Hodge correlators satisfy the relation:*

$$\begin{aligned}
&\sum_{\sigma \in \overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma} \mathrm{Cor}^*_{\mathrm{Hod}}(w_{\sigma^{-1}(1)} | n_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(M_\sigma)} | n_{\sigma^{-1}(M_\sigma)}, w_0 | n_0) \\
&- \mathrm{Cor}^*_{\mathrm{Hod}}(w_1 | n_1, \ldots, w_r | n_r, w_{\{r+1,\ldots,r+s,0\}} | n_{\{r+1,\ldots,r+s,0\}}) \\
&- \mathrm{Cor}^*_{\mathrm{Hod}}(w_{r+1} | n_{r+1}, \ldots, w_{r+s} | n_{r+s}, w_{\{1,\ldots,r,0\}} | n_{\{1,\ldots,r,0\}}) \qquad = 0,
\end{aligned}$$

where

$$n_I = \sum_{i \in I} (n_i + 1) - 1, \quad w_I = \prod_{i \in I} w_i.$$

(c) The Hodge correlators satisfy all specializations of this relation as any subset of the w_i $(1 \le i \le n)$ *approaches 0.*

Proof. For fixed r, s, and n_i, consider the (r, s)-second shuffle relation in (b). It is a family of framed mixed Hodge-Tate structures over

$$S = \left\{(w_0, \ldots, w_n) \in (\mathbb{C}^*)^{n+1} : w_0 \ldots w_n = 1\right\}.$$

To show (b), it suffices to show the family is trivial as an element of $\mathrm{Lie}^{\vee}_{\mathrm{HT}}$ over every point of S, except at $(1,\ldots,1)$ if all $n_i = 0$. This is equivalent to (a) by the definitions, as the Hodge correlators are already known to satisfy the defining relations in $\widetilde{\mathcal{D}}^{\circ}(\mathbb{C}^*)$.

Each term of this relation is an element

$$\mathrm{Cor}_{\mathrm{Hod}}(1, z_1, \ldots, z_n),$$

where each z_k is either 0 or monomial in the w_i. By Theorem 1.5, it is a variation $\mathbf{V}$ of framed mixed Hodge-Tate structures over

$$T = \{(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n\} \setminus (\text{diagonals}) .$$

We first show by induction on the weight n that all such variations is trivial.

In the base case $n = 1$, there are no second shuffle relations.

For the induction hypothesis, suppose $n > 1$ and (b) holds in weights $1 < w < n$. Fix r, s, and n_i and let $\mathbf{V}$ be the variation defined above. By the induction hypothesis, $\delta\mathrm{Cor}_{\mathrm{Hod}}(\mathbf{V})$ vanishes, and thus, by rigidity, $\mathbf{V}$ is a constant variation, determined pointwise as an element of $\mathrm{Ext}^1(\mathbb{R}(0), \mathbb{R}(n))$ by the period. We show the period is 0.

The specialization theorem (§2.3) implies that the period of $\mathbf{V}$ is continuous away from the main diagonal in $\mathbb{C}^{n+1}$. Unless all $n_i = 0$ or all $w_i = 1$, in no term of the relation (b) do all points collide to the main diagonal. By Corollary 1.6, the specialization of the period at $w_1, \ldots, w_n = 0$ is equal to the substitution $w_i = 0$. Under this substitution, the period of each term of the relation becomes

$$\mathrm{Cor}_{\mathcal{H}}(1, 0, \ldots, 0) = 0.$$

Therefore, $\mathbf{V}$ is trivial over T.

Because T is dense in $\mathbb{C}^n$, the relation at all points – except $w_1 = \cdots = w_n = 1$ if all $n_i = 0$ – follows by the specialization theorem. This completes the proof of (b) and (c). □

Applying the period map, we immediately obtain Theorem 2.1:

Theorem. *(a) Suppose that $r, s > 1$ and that not all $n_i = 0$ or not all $w_i = 1$. Then the Hodge correlators* *satisfy the relation:*

$$\sum_{\sigma\in\overline{\Sigma}_{r,s}} (-1)^{r+s-M_\sigma}\mathrm{Cor}^*_{\mathcal{H}}(w_{\sigma^{-1}(1)}|n_{\sigma^{-1}(1)},\dots,w_{\sigma^{-1}(M_\sigma)}|n_{\sigma^{-1}(M_\sigma)},w_0|n_0)$$
$$-\mathrm{Cor}^*_{\mathcal{H}}(w_1|n_1,\dots,w_r|n_r,w_{\{r+1,\dots,r+s,0\}}|n_{\{r+1,\dots,r+s,0\}})$$
$$-\mathrm{Cor}^*_{\mathcal{H}}(w_{r+1}|n_{r+1},\dots,w_{r+s}|n_{r+s},w_{\{1,\dots,r,0\}}|n_{\{1,\dots,r,0\}}) = 0,$$

where

$$n_I = \sum_{i\in I}(n_i+1)-1, \quad w_I = \prod_{i\in I} w_i.$$

(b) *The Hodge correlators satisfy all specializations of this relation as any subset of the w_i $(1 \le i \le n)$ approaches 0.*

Proof for motivic correlators

Recall Theorem 2.7:

Theorem. *Let F be a number field and $X = \mathbb{P}^1$.*

(a) *Restricted to the subspace of $\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^\vee$ generated by elements $(x_0\otimes\cdots\otimes x_n)(1)$ with not all x_i equal, the map $\mathrm{Cor}_{\mathrm{Mot}}$ factors through $\mathcal{D}^\circ(F^\times)$. (Here $S\subset\mathbb{P}^1(F)$ is any finite set of punctures containing all points appearing in the relation in (b).)*

(b) *Suppose that $r,s>1$ and that not all $n_i=0$ or not all $w_i=1$. Then the motivic correlators satisfy the same relation as in Theorem 2.6, with $\mathrm{Cor}^*_{\mathrm{Hod}}$ replaced by $\mathrm{Cor}^*_{\mathrm{Mot}}$.*

(c) *The motivic correlators satisfy all specializations of this relation as any subset of the w_i $(1\le i\le n)$ approaches 0.*

Proof. Fix an embedding $F\xrightarrow{r}\mathbb{C}$. It induces a map $\mathcal{D}^\circ(F^\times)\to\mathcal{D}^\circ(F^\times)$, which we also denote by r.

Denoting by $\mathcal{CL}^{\vee\circ}$ the subalgebras generated by elements $(x_1\otimes\cdots\otimes x_n)(1)$ where not all x_i are equal, we have the diagram

$$\begin{tikzcd}
\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^{\vee\circ} \arrow[rr, "\mathrm{Cor}_{\mathrm{Mot}}"] \arrow[dr] \arrow[dd, "r"] & & \mathrm{Lie}^{\vee}_{\mathrm{MT}/F} \arrow[dd, "r"] & \\
& \mathcal{D}^{\circ}(F^{\times}) \arrow[ur, dashed] \arrow[dd, "r"] & & \\
\mathcal{CL}^{\vee\circ}_{X,S,v_\infty} \arrow[rr, "\mathrm{Cor}_{\mathrm{Hod}}"] \arrow[dr] & & \mathrm{Lie}^{\vee}_{\mathrm{HT}} \arrow[r, "p"] & \mathbb{R} \\
& \mathcal{D}^{\circ}(\mathbb{C}^{*}) \arrow[ur] & &
\end{tikzcd},$$

where the lower half commutes by Theorem 2.6 and the vertical maps are induced by r.

It is necessary to show the dashed arrow is well-defined, i.e., that $\mathrm{Cor}_{\mathrm{Mot}}$ vanishes on the kernel of the map $\left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^{\vee\circ} \to \mathcal{D}^{\circ}(F^{\times})$.

Commutativity of the diagram for every embedding r implies the result. We argue by induction.

In weight 1, then there are no first or second shuffles, and the shuffle relations are mapped to 0 by $\mathrm{Cor}_{\mathrm{Mot}}$. Indeed, we have $\mathrm{Cor}_{\mathrm{Mot}}(0,0) = 0$ and $\mathrm{Cor}_{\mathrm{Mot}}(ab, ac) = \mathrm{Cor}_{\mathrm{Mot}}(0, a) + \mathrm{Cor}_{\mathrm{Mot}}(b, c)$, since

$$\mathrm{Cor}_{\mathrm{Mot}}(a, b) = (a - b) \in (\mathrm{Lie}_{\mathrm{MT}/F})^{\vee}_{w=1} \cong F^{\times} \otimes \mathbb{Q}.$$

For the inductive step, if $x \in \left(\mathcal{CL}^{\mathrm{Mot}}_{X,S,v_\infty}\right)^{\vee\circ}$, homogeneous of weight > 1, vanishes in $\mathcal{D}^{\circ}(F^{\times})$, then $\mathrm{Cor}_{\mathrm{Hod}}(r(x)) = 0 \in \mathrm{Lie}^{\vee}_{\mathrm{HT}}$ under every embedding r, and $\partial\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$ by the inductive hypothesis. By Lemma 1.8, $\mathrm{Cor}_{\mathrm{Mot}}(x) = 0$. □

2.4.2 Applications

Additive shuffle relation

Specializing all w_i to 1 in the second shuffle relation, where all $n_i = 0$, we extract an *additive* second shuffle relation, which does not have lower-depth terms:

Corollary 2.25. *Let $m, n > 0$. The additive shuffle*

$$\sum_{\sigma \in \Sigma_{m,n}} \mathrm{Cor}_{\mathcal{H}}(\varepsilon_{\sigma^{-1}(1)}, \varepsilon_{\sigma^{-1}(1)} + \varepsilon_{\sigma^{-1}(2)}, \ldots, \varepsilon_{\sigma^{-1}(1)} + \cdots + \varepsilon_{\sigma^{-1}(m+n)}, 0).$$

is a constant independent of $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{C}^n \setminus 0$.

Proof. This is a specialization of the second shuffle relation with $w_i = 1 + t\varepsilon_i$ as $t \to 0$. The terms are the specializations of those in the second shuffle relation arising from proper shuffles. The specializations of the lower-depth terms are independent of the ε_i, as not all arguments of the correlators are equal at $t = 0$. $\square$

It is easy to see that this constant is 0 if $m + n$ is even. If $m + n$ is odd, it is equal, in particular, to a sum of Hodge correlators at roots of unity.

Proofs of Corollaries 2.2 and 2.3

Recall Corollary 2.2:

Corollary ([GR], Proposition 2.8)**.** *For $n > 2$, every Hodge correlator of weight n is a linear combination of Hodge correlators of weight n and depth at most $n-2$. Explicitly, for $z_1, \ldots, z_n \in \mathbb{C}^*$, we have*

$$\begin{aligned}\mathrm{Cor}_{\mathcal{H}}(z_1, \ldots, z_n, 0) = &\sum_{i=1}^{n} \mathrm{Cor}_{\mathcal{H}}\left(z_1, \ldots, z_{i-1}, z_i, z_i\frac{z_1}{z_n}, \ldots, z_{n-1}\frac{z_1}{z_n}, z_n\frac{z_1}{z_n}\right)\\ &- \sum_{i=2}^{n} \mathrm{Cor}_{\mathcal{H}}\left(z_1, \ldots, z_{i-1}, 0, z_i\frac{z_1}{z_n}, \ldots, z_{n-1}\frac{z_1}{z_n}, z_n\frac{z_1}{z_n}\right)\\ &- \mathrm{Cor}_{\mathcal{H}}\left(z_1, z_1 \cdot \frac{z_1}{z_n}, 0, \ldots, 0\right). \end{aligned} \tag{2.111}$$

Proof. By multiplicative invariance, we may assume $z_1 = 1$. Then this is precisely the $(n-1, 1)$-second shuffle relation applied to the segments

$$(z_2/z_1 \mid 0), (z_3/z_2 \mid 0), \ldots, (z_n/z_{n-1} \mid 0)$$

and

$$(z_1/z_n \mid 0),$$

where the segment $(1 \mid 0)$ is left fixed. Indeed, the two summations come from the n shuffles and the $n-1$ additional quasishuffles, with the remaining terms giving the left side and the last summand.

All terms on the right side have at least two coinciding arguments. After an additive shift, they have at least two arguments equal to 0, so they are equal to those of depth at most $n-2$. $\square$

Recall Corollary 2.3:

Corollary. *The Hodge correlators in weight 3 satisfy the relations:*

$$\mathrm{Cor}_{\mathcal{H}}(1,0,0,x) + \mathrm{Cor}_{\mathcal{H}}(1,0,0,1-x) + \mathrm{Cor}_{\mathcal{H}}(1,0,0,1-x^{-1}) = \mathrm{Cor}_{\mathcal{H}}(1,0,0,1), \tag{2.112}$$

$$\begin{aligned}\mathrm{Cor}_{\mathcal{H}}(0,x,1,y) = &-\mathrm{Cor}_{\mathcal{H}}(1,0,0,1-x^{-1}) - \mathrm{Cor}_{\mathcal{H}}(1,0,0,1-y^{-1}) - \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{y}{x}\right)\\ &- \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-y}{1-x}\right) + \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-y^{-1}}{1-x^{-1}}\right) + \mathrm{Cor}_{\mathcal{H}}(1,0,0,1).\end{aligned} \tag{2.113}$$

Proof. Apply the $(1,1)$-second shuffle relation to the segments $(x \mid 0)$ and $\left(x^{-1} \mid 1\right)$, keeping the segment $(1 \mid 1)$ fixed:

$$\begin{aligned}&\mathrm{Cor}_{\mathcal{H}}(1,x,0,0) + \mathrm{Cor}_{\mathcal{H}}(1,0,x^{-1},1) - \mathrm{Cor}_{\mathcal{H}}(1,0,0,1)\\ &\qquad - \mathrm{Cor}_{\mathcal{H}}(1,x,0,0) - \mathrm{Cor}_{\mathcal{H}}(1,0,x^{-1},0) \qquad = 0.\end{aligned}$$

Multiplicative invariance and the first shuffle relation imply

$$-\mathrm{Cor}_{\mathcal{H}}(1,x,0,0) - \mathrm{Cor}_{\mathcal{H}}(1,0,x^{-1},0) = \mathrm{Cor}(1,0,0,x).$$

Rearranging terms and applying additive invariance gives (2.112).

Now apply (2.111) to $\mathrm{Cor}_{\mathcal{H}}(x,1,y,0)$ and apply the dihedral symmetry and additive invariance to change all terms to the form $\mathrm{Cor}_{\mathcal{H}}(1,0,0,z)$:

$$\begin{aligned}\mathrm{Cor}_{\mathcal{H}}(0,x,1,y) = &\mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-y}{x-y}\right) + \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-x^{-1}}{1-y^{-1}}\right) + \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{x-1}{x-y}\right)\\ &- \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,1-y^{-1}\right) - \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,x^{-1}\right) - \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{x}{y}\right).\end{aligned}$$

Finally, by (2.112),

$$\mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-y}{x-y}\right) + \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{x-1}{x-y}\right) = \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,1\right) - \mathrm{Cor}_{\mathcal{H}}\left(1,0,0,\frac{1-x}{1-y}\right),$$

which gives the result. □

Chapter 3

Motivic π_1 of CM elliptic curves and geometry of Bianchi hyperbolic threefolds

The results of this chapter have appeared in [M1].

3.1 Introduction and main results

In this chapter we describe a connection between the realizations of motivic fundamental groups of CM elliptic curves and the geometry of Bianchi hyperbolic threefolds. The first instance of this connection was described by Goncharov in [G8].

3.1.1 Summary

Motivation

We aim to study the action of the motivic Galois group on the motivic fundamental group of an elliptic curve punctured at the $\mathfrak{p}$-torsion points with tangential base point v_0:

$$\mathrm{Gal}_{\mathrm{Mot}} \circlearrowright \pi_1^{\mathrm{Mot}}(E - E[\mathfrak{p}], v_0). \tag{3.1}$$

The objects in (3.1) are still conjectural, but we can study them in their realizations. The results of this chapter are in the Hodge realization. However, the picture is easiest to introduce in the ℓ-adic realization.

As a running example, take E to be the CM elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}[i])$ and $\mathfrak{p} \subset \mathbb{Z}[i]$ an ideal. The ℓ-adic realization of the motivic fundamental group, $\pi_1^{(\ell)}(E - E[\mathfrak{p}], v_0)$, is simply the pro-$\ell$ completion of the topological fundamental group $\pi_1(E - E[\mathfrak{p}], 0)$. It is equipped with an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by automorphisms.

The Maltsev construction ([D3], §9) makes out of $\pi_1^{(\ell)}(E - E[\mathfrak{p}], v_0)$ a pro-ℓ Lie algebra $A_{E,\mathfrak{p}}$ over $\mathbb{Q}_\ell$, generated by $H_1(E;\mathbb{Z})$ and loops around the punctures in $E[\mathfrak{p}]$. It carries two filtrations: by *weight* and by *depth*. The increasing weight filtration W (see [D1]) is invariant under the Galois action, and the geometric Frobenius element acts on $\mathrm{gr}^w A_{E,p}$ with eigenvalues of norm $\ell^{w/2}$. The decreasing depth filtration D is defined by the lower central series of the linearization of

$$\ker\left(\pi_1^{(\ell)}(E - E[\mathfrak{p}], v_0) \to \pi_1^{(\ell)}(E, v_0)\right).$$

The filtrations W and D induce filtrations on $\mathrm{End}(A_{E,\mathfrak{p}})$, and, by restriction, on the image of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Taking its associated graded Lie algebra for the weight filtration, we obtain a graded Lie algebra $\mathrm{Lie}_{(\ell)}(E, E[\mathfrak{p}])$, the *elliptic Galois Lie algebra*. We study the quotient of this Lie algebra induced by the quotient of $A_{E,\mathfrak{p}}$ by the adjoint action of $H_1(E;\mathbb{Z})$, and take the coinvariants of the translation action of $E[\mathfrak{p}]$ on E (amounting to averaging the base point).This quotient is called the *symmetric* Galois Lie algebra $\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])$.

The structure of the depth-d graded quotients of $\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])$ is well understood in depths 0 and 1. In depth 0, this algebra simply vanishes. In depth 1, it is abelian, and spanned over $\mathbb{Q}_\ell$ by the classes constructed by Beilinson [B2] and in a different way by Beilinson and Levin [BL]. These classes are parametrized by a $\mathfrak{p}$-torsion point of E and an element of the symmetric algebra of $H_1(E;\mathbb{Z})$. These constructions work in the Hodge realization as well as in the ℓ-adic one, and the mechanism of motivic correlators (described in §2) gives alternative proofs of these statements. In particular, Beilinson and Levin's elliptic polylogarithms can be expressed in terms of the depth-1 Hodge correlator integrals – Kronecker-Eisenstein series ([BL], §3).

In this chapter, we focus on the depth 2, the first case in which there is a nonzero Lie bracket. To describe the structure of the elliptic Galois Lie algebra, we can consider its standard cochain complex. Recall that the standard cochain complex of a Lie algebra L is a complex of the exterior powers of its dual $L^\vee$, where the coboundary map δ is the dualization of the Lie bracket $[\,,\,] : L \wedge L \to L$:

$$\mathrm{CE}^\bullet(L^\vee) = \left(0 \to L^\vee \xrightarrow{\delta} L^\vee \wedge L^\vee \to L^\vee \wedge L^\vee \wedge L^\vee \to \dots\right).$$

If $L^\vee$ is a graded Lie coalgebra, then $\mathrm{CE}^\bullet(L^\vee)$ is also graded. Applying the construction to the associated graded for the depth filtration of $\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])$, we obtain a cochain complex that is graded by weight and depth. In depth 1 the complex is concentrated in degree 1. However, the depth-2 part of this complex has a nontrivial coboundary map: the depth-2 elements map to wedge products of Beilinson-Levin classes:

$$\mathrm{gr}^{D=2}\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])^\vee \to \left(\mathrm{gr}^{D=1}\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])^\vee\right)^{\wedge 2}. \tag{3.2}$$

We connect (the Hodge analogue of) this complex to the geometry of Bianchi hyperbolic threefolds.

Bianchi tessellation

Now let us describe the other side of the story. The Bianchi tessellation of the upper half-space $\mathbb{H}^3$ for the ring $\mathbb{Z}[i]$ is the 3-dimensional version of the famous modular triangulation of the upper half-plane; the latter is the restriction of the Bianchi tessellation to the plane in $\mathbb{H}^3$ lying above the real line (see Figure 3.1). This beautiful construction was given by Bianchi in 1892 [B4]; see [G8] for a modern review. The fundamental domain is an octahedron with vertices at $0, 1, i, i+1, \frac{1+i}{2}, \infty)$. Through the standard action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{H}^3$, the group $\mathrm{GL}_2(\mathbb{Z}[i])$ acts transitively on the cells of the tessellation. If $\mathfrak{p}$ is a prime ideal in $\mathbb{Z}[i]$ and $\Gamma_1(\mathfrak{p}) \subset \mathrm{GL}_2(\mathbb{Z}[i])$ is the congruence subgroup

$$\Gamma_1(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod \mathfrak{p} \right\},$$

the quotient $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$ is a finite-volume hyperbolic manifold with cusps.

We build the following local system on this manifold. The group $H_1(E;\mathbb{Z})$ has the structure of a $\mathbb{Z}[i]$-module, giving $H_1(E;\mathbb{Z}) \oplus H_1(E;\mathbb{Z})$ the structure of a $\mathrm{GL}_2(\mathbb{Z}[i])$-module. We take its symmetric algebra $\mathrm{Sym}^\bullet(H_1(E;\mathbb{Z}) \oplus H_1(E;\mathbb{Z}))$. This $\mathrm{GL}_2(\mathbb{Z}[i])$-module determines an infinite-dimensional graded local system on $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$. Denote this local system by T_2.

Consider the chain complex of the Bianchi tessellation, placed in the cohomological degrees $[0, 2]$. It is generated by the octahedral cells in degree 0, by ideal triangles in degree 1, and by geodesics in degree 2. Tensoring over $\Gamma_1(\mathfrak{p})$ with $\mathrm{Sym}^\bullet(H_1(E;\mathbb{Z}) \oplus H_1(E;\mathbb{Z}))$, we get the chain complex of $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$ with coefficients in the local system T_2.

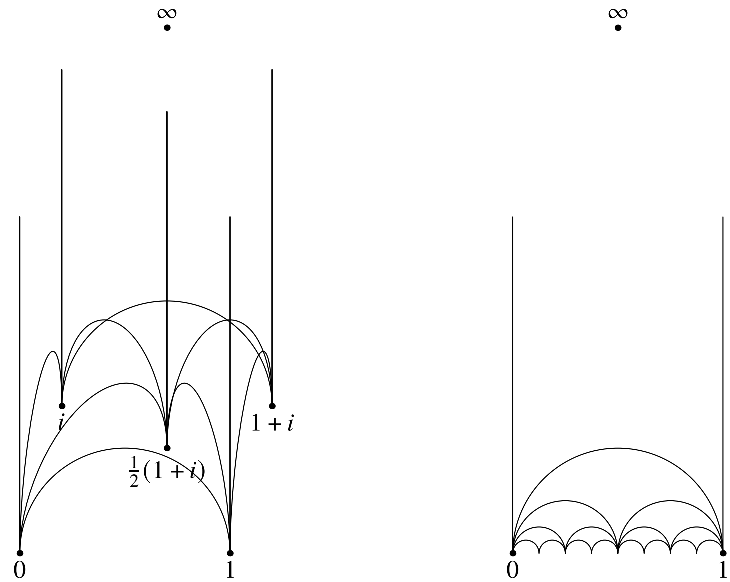


Figure 3.1. *Left:* The fundamental octahedron of the Bianchi tessellation for $\mathbb{Z}[i]$.
Right: The modular triangulation of the upper half-plane.

The main construction

There is a Hodge analogue of $\mathrm{Lie}^{\mathrm{sym}}_{(\ell)}(E, E[\mathfrak{p}])$, denoted $\mathrm{Lie}^{\mathrm{sym}}_{\mathrm{Hod}}(E, E[\mathfrak{p}])$. In §3.3.3 (Theorem 3.13), we construct a surjective morphism of complexes of graded $\mathbb{Z}[i]$-modules:

$$\left(\begin{array}{c}\text{chain complex of the Bianchi orbifold} \\ \Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3 \text{ with coefficients in } T_2\end{array}\right) \to \left(\begin{array}{c}\text{Hodge analogue of} \\ \text{the complex (3.2)}\end{array}\right) \tag{3.3}$$

In particular, we get surjective homomorphisms:

$$H^i(\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3, T_2) \to H^i(\mathrm{gr}^D \mathrm{Lie}^{\mathrm{sym}}_{\mathrm{Hod}}(E, E[\mathfrak{p}]), \mathbb{Q})_{D=2}.$$

A key idea of Goncharov [G8], which we develop further in this chapter, is to map the cusps of the Bianchi orbifold $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$ to $\mathfrak{p}$-torsion points of E. This itself generalizes a similar picture for modular curves, first described in [G2], where cusps of modular curves are identified with p-torsion points of $\mathbb{G}_m$ in the study of the double logarithm at roots of unity. When we advance to depth 2, the geodesics of the Bianchi tessellation map to wedge products of elements parametrized by $\mathfrak{p}$-torsion points, and the triangles must map to certain elements parametrized by three $\mathfrak{p}$-torsion points.

Let us elaborate the map (3.3) in each degree. In degree 2, we map a geodesic (α, β) of the Bianchi tessellation modulo $\Gamma_1(\mathfrak{p})$, with a coefficient in the described local system to a wedge product of two depth-1 classes in the Galois Lie coalgebra. The data parametrizing a geodesic with a coefficient in $\mathrm{Sym}^\bullet(H_1(E;\mathbb{Z}) \oplus$

$H_1(E;\mathbb{Z})) \cong \mathrm{Sym}^\bullet(H_1(E;\mathbb{Z}))^{\otimes 2}$ is *identical* to the data parametrizing a pair of classes in the image. In degree 1, the domain is generated by triangles in the Bianchi tessellation with a coefficient in the local system. Thus the image should be described in terms of elements depending on three $\mathfrak{p}$-torsion points and three elements of $\mathrm{Sym}^\bullet(H_1(E;\mathbb{Z}))$. These elements are *motivic correlators*, which we introduce in the remainder of the introduction and in §2. These elements can be visualized as a sequence of $\mathfrak{p}$-torsion points and 1-forms on E written around a circle, modulo some relations. Their coproduct has a simple combinatorial description, and their real Hodge periods can be explicitly computed via Feynman integrals.

In summary, the maps are constructed as follows:

$$\text{ideal triangle } (\alpha,\beta,\gamma) \mapsto \text{element in } \mathrm{gr}^{D=2}\mathrm{Lie}^{\mathrm{sym}}(E, E[\mathfrak{p}]) \text{ depending on } \mathfrak{p}\text{-torsion points } \alpha,\beta,\gamma,$$

$$\text{geodesic } (\alpha,\beta) \mapsto \text{wedge product of Beilinson-Levin elements determined by } \alpha \text{ and } \beta.$$

Remarkably, the combinatorial structure of the Bianchi tessellation is preserved in the space of motivic correlators, and thus the chain complex of the Bianchi orbifold *maps surjectively* onto the standard cochain complex of a quotient of the Galois Lie algebra of $E \setminus E[\mathfrak{p}]$.

Hodge realization

Now we will sketch this picture in the Hodge realization, which is the focus of this chapter.

Let E be a complex elliptic curve and $S \subset E$ a finite set of punctures. The pronilpotent completion of the fundamental group $\pi_1^{\mathrm{nil}}(E - S, v_0)$, with tangential base point at v_0, is a Lie algebra in the category of mixed $\mathbb{Q}$-Hodge structures. The category of mixed $\mathbb{Q}$-Hodge structures is canonically equivalent to the category of representations of a graded Lie algebra over $\mathbb{Q}$. Let us take its image in the representation defining $\pi_1^{\mathrm{nil}}(E - S, v_0)$, and consider the graded dual Lie coalgebra $\mathrm{Lie}^\vee_{\mathrm{Hod}}(E, S)$.

The Hodge correlators, introduced by Goncharov in [G9], are canonical elements

$$\mathrm{Cor}_{\mathrm{Hod}}(\Omega_0, z_0, \ldots, \Omega_n, z_n) \in \mathrm{Lie}^\vee_{\mathrm{Hod}}(E, S), \tag{3.4}$$

where $z_0, \ldots, z_n \in S$ and $\Omega_1, \ldots, \Omega_n$ are elements in the tensor algebra of $H^1(E;\mathbb{C})$. The coalgebra $\mathrm{Lie}^\vee_{\mathrm{Hod}}(E, S)$ carries a filtration by depth; the element (3.4) has depth n. These elements describe the real mixed Hodge structure on $\pi_1^{\mathrm{nil}}(E-S, v_0) \otimes \mathbb{R}$. Their canonical real periods are the Hodge correlator functions, functions of $n+1$ points on E. We find new linear relations among the elements (3.4).

At a cusp on the modular curve, as E degenerates to the nodal projective line, these relations specialize to known relations among periods of the mixed Tate motive associated with $\mathbb{P}^1$ punctured at a finite set of points. If $n = 2$, our elliptic relations specialize to the full set of *double shuffle relations*, the most general known relations, which were described in Chapter 2 using Hodge correlators.

Suppose that E is one of the CM elliptic curves $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ or $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-3}}{2}\right))$, $O = \mathrm{End}E$, and $\mathfrak{p}$ is a prime in O. The subalgebra $\mathrm{Lie}^{\mathrm{sym}}_{\mathrm{Hod}}(E, E[\mathfrak{p}])$ of $\mathrm{Lie}_{\mathrm{Hod}}(E, S)$ is constructed as in the ℓ-adic case. We construct the morphism (3.3) in this setting, where the object standing on the right is the complex $\mathrm{CE}^\bullet\left(\mathrm{gr}^D \mathrm{Lie}^{\mathrm{sym}}_{\mathrm{Hod}}(E, E[\mathfrak{p}])\right)_2$.

Our construction simultaneously generalizes several results of Goncharov:

(1) *The relation between Voronoi complexes and mixed Tate motives:* The Bianchi complexes are the higher-degree analogues of the Voronoi complexes, complexes of $\mathrm{GL}_k(\mathbb{Z})$-modules from tessellations of the upper half-plane $\mathbb{H}^2$. A map from the Voronoi complexes to motivic objects associated with rational curves punctured at roots of unity constructed for $k = 2, 3, 4$, using either multiple polylogarithms ([G3]) or motivic correlators ([G10]), which satisfy the double shuffle relations. The relations we found for elliptic motivic correlators in depth 2 are deformations of the second shuffle relations.

(2) *Euler complexes:* The map from the Bianchi complexes to a space of motivic theta functions on elliptic curves constructed by [G8] in depth 2 and weight 4. We generalize this construction to all weights: [G8]'s map is the restriction of our map to the trivial local system.

Structure

In §3.1.2 we explain our results on the level of Hodge correlator integrals.

In §3.2 we establish some properties of motivic correlators on elliptic curves. The main new result of this section is the dihedral symmetry relation for depth 2 correlators (Theorem 3.8).

In §3.3 we review the definitions of the Bianchi complexes, define the modular complexes for imaginary quadratic fields, and construct a map between the two in the Gaussian and Eisenstein cases. In §3.3.3 we combine the results of the two preceding sections to prove the main results relating Bianchi complexes and the elliptic Galois Lie algebra.

In §3.4, we show how our results generalize those of [G3, G8] and those in Chapter 2.

3.1.2 Relations for correlators

Recall from §1.1.1 that the Hodge correlators satisfy a first shuffle relation, and that in Chapter 2 we describe a family of *second shuffle relations* for correlators on $\mathbb{P}^1$, the terms of which were formed by permuting quotients of successive arguments. One of the main results of this chapter is that the Hodge correlators on an elliptic curve satisfy second shuffle relations in depth 2.

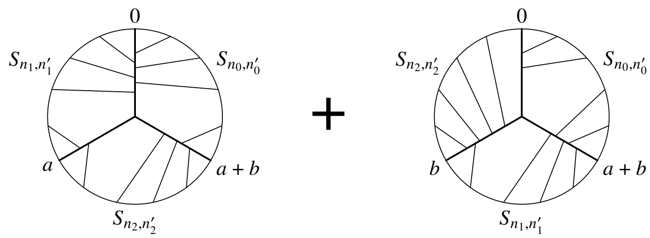


Figure 3.2. The highest-depth terms of the second shuffle relation on an elliptic curve.

For Hodge correlators of depth 2 on an elliptic curve with arbitrary base point, this new relation has the form:

$$\mathrm{Cor}_{\mathcal{H}}(S_{n_0,n_0'},0,S_{n_1,n_1'},a,S_{n_2,n_2'},a+b)+\mathrm{Cor}_{\mathcal{H}}(S_{n_0,n_0'},0,S_{n_2,n_2'},b,S_{n_1,n_1'},a+b)+\text{lower-depth terms}=0, \quad (3.5)$$

where an argument $S_{n,n'}$ indicates that we sum over all possible ways to insert in some order the arguments

$$\underbrace{\omega,\dots,\omega}_{n},\underbrace{\overline{\omega},\dots,\overline{\omega}}_{n'}.$$

The highest-depth terms in these relations arise from shuffles of the *differences* between successive arguments, $x_i - x_{i-1}$, together with the 1-forms between those arguments. For example, in (3.5) we have shuffled a (with n_1 copies of ω and n_1' of $\overline{\omega}$) with b (with n_2 ω's and n_2' $\overline{\omega}$'s). See Figure 3.2) for an illustration.

We describe the lower-depth correction terms in §3.2.2. In the simplest case – weight 4 – the full relation is:

$$\begin{aligned}
\mathrm{Cor}_{\mathcal{H}}(0,a,a+b)+\mathrm{Cor}_{\mathcal{H}}(0,b,a+b)-(\mathrm{Cor}_{\mathcal{H}}(0,\omega,\overline{\omega},a+b)+\mathrm{Cor}_{\mathcal{H}}(0,\overline{\omega},\omega,a+b))&\\
-\frac{1}{2}\Big(\mathrm{Cor}_{\mathcal{H}}(0,a,\omega,\overline{\omega})-\mathrm{Cor}_{\mathcal{H}}(0,a,\overline{\omega},\omega)&\\
+\mathrm{Cor}_{\mathcal{H}}(0,b,\omega,\overline{\omega})-\mathrm{Cor}_{\mathcal{H}}(0,b,\overline{\omega},\omega)&\\
+\mathrm{Cor}_{\mathcal{H}}(\omega,\overline{\omega},a,a+b)-\mathrm{Cor}_{\mathcal{H}}(\overline{\omega},\omega,a,a+b)&\\
+\mathrm{Cor}_{\mathcal{H}}(\omega,\overline{\omega},b,a+b)-\mathrm{Cor}_{\mathcal{H}}(\overline{\omega},\omega,b,a+b)\Big)&=0.
\end{aligned}$$

The relation (3.5) can be formulated simply as a functional equation on biperiodic functions of three complex variables. It states that a sum of several integrals over an elliptic curve is equal to 0, modulo correlators of depth 1 (which are expressed by Kronecker-Eisenstein series). However, this functional equation is difficult to prove. To understand it, we will need to upgrade it to the Hodge-theoretic or motivic setting.

The second shuffle relations have a prehistory. The first objects known to satisfy a system of first and second shuffle relations of this form were the multiple polylogarithms (see §1.2.3 and [G3]): these relations follow from two alternative expressions for multiple polylogarithms: as power series and as iterated integrals. In Chapter 2, for $X = \mathbb{P}^1$, we found second shuffle relations for Hodge correlators, in every depth, and described the lower-depth terms. In depth 2, these relations depend on integers $n_0, n_1, n_2 \geq 0$ and points $a, b \in \mathbb{G}_m \setminus \{1\}$. They state:

$$\mathrm{Cor}_{\mathcal{H}}(\underbrace{0,\ldots,0}_{n_0},1,\underbrace{0,\ldots,0}_{n_1},a,\underbrace{0,\ldots,0}_{n_2},ab)+\mathrm{Cor}_{\mathcal{H}}(\underbrace{0,\ldots,0}_{n_0},1,\underbrace{0,\ldots,0}_{n_2},b,\underbrace{0,\ldots,0}_{n_1},ab)$$
$$+\text{lower-depth terms} = 0. \qquad (3.6)$$

The highest-depth terms in these relations arise from shuffles of the quotients between successive arguments, $\frac{x_i}{x_{i-1}}$, together with the 0s between those arguments. For example, in (3.6) we have shuffled a (with n_1 0s) with b (with n_2 0s).

Conjecturally, the first and second shuffle relations give all linear relations among the Hodge correlators on $\mathbb{P}^1$. While the first shuffle relations emerge from the trivalent tree construction – they hold on the level of the *integrands* in (1.4) – the proof of the second shuffle relations is difficult, requiring motivic or Hodge-theoretic arguments even in depth 2.

Note the similarity between (3.6) and (3.5). In fact, as an elliptic curve degenerates to a nodal projective line, a variant of the second shuffle relation (3.5) specializes to (3.6).

In §3.2.3 (Theorem 3.8) we will prove these relations, as well as their upgrades to the Hodge correlator coalgebra. Assuming the motivic formalism, all results of this chapter should hold in the ℓ-adic realization as well.

3.2 Motivic correlators on elliptic curves

3.2.1 Main properties

Definitions

We work with a complex elliptic curve E. Recall $S \subset E$ is a finite set of punctures. Let $\mathcal{O} = \mathrm{End}(E)$, so either $\mathcal{O} = \mathbb{Z}$ or a lattice in an imaginary quadratic field.

Let $\omega, \overline{\omega}$ be a symplectic basis for $H^1(E;\mathbb{C})$. $\mathcal{CL}^\vee_{E,S,v_0}$ is generated by elements

$$\begin{aligned} C_s(\Omega_0, s_0, \ldots, \Omega_n, s_n) = &\underbrace{\omega_{0,1} \otimes \cdots \otimes \omega_{0,k_0}}_{\Omega_0} \otimes \{s_0\} \\ &\otimes \underbrace{\omega_{1,1} \otimes \cdots \otimes \omega_{1,k_1}}_{\Omega_1} \otimes \{s_1\} \\ &\otimes \cdots \\ &\otimes \underbrace{\omega_{n,1} \otimes \cdots \otimes \omega_{n,k_n}}_{\Omega_n} \otimes \{s_n\} \end{aligned}$$

$s_i \in S$ and Ω_i range over the basis of $T_{\mathbb{Z}}(H^1(E,\mathbb{C}))$ consisting of elements $\bigotimes_{j=1}^{k_i} \omega_{i,j}$ with $\omega_{i,j} \in \{\omega, \overline{\omega}\}$. This generator lies in the component of $\mathcal{CL}^\vee_{E,S,s}$ of depth n and weight $2n + \sum_{i=0}^n k_i$.

Suppose a tangent vector v_s has been chosen at each $s \in S$. We assemble the $\mathcal{CL}^\vee_{E,S,v_s}$ as the base point s ranges over S into a Lie coalgebra

$$\widetilde{\mathcal{CL}}^\vee_{E,S} := \bigoplus_{s \in S} \mathcal{CL}^\vee_{E,S,v_s}.$$

All direct summands are isomorphic, but the maps $\mathrm{Cor}_{\mathrm{Hod}}$ on different components do not coincide. We will write Cor_s as a short notation for the map $\mathrm{Cor}_{\mathrm{Hod}}$ on the component corresponding to s, extended so that $\mathrm{Cor}_s(s, \ldots) = 0$, i.e., the correlator of an element that contains the base point vanishes.

Generating series

We will package the correlators of depth n into generating series in $2(n+1)$ commuting formal variables $t_0, \overline{t}_0, t_1, \overline{t}_1, \ldots, t_n, \overline{t}_n$. We identify $t_i, \overline{t}_i$ with generators of $H_1(E,\mathbb{Z})$ dual to $\omega, \overline{\omega}$. That is, the monomials in the $t_i, \overline{t}_i$ are identified with the generators of $\bigotimes_{i=0}^n \mathrm{Sym}(H_1(E,\mathbb{Z}))$.

For $x_0, \ldots, x_n \in S$ and $s \in S$, define the generating series

$$\Theta_s\big(x_0 : x_1 : \cdots : x_n \mid t_0 : t_1 : \cdots : t_n\big) = \sum_{\Omega_0,\ldots,\Omega_n} \mathrm{Cor}_s\left(\Omega_0, x_0, \ldots, \Omega_n, x_n)\right)(\Omega_0^* \otimes \cdots \otimes \Omega_n^*), \tag{3.7}$$

where the sum is taken over the basis of $T_{\mathbb{Z}}(H^1(E,\mathbb{C}))$ as above. The coefficient of $\prod_i t_i^{m_i} \overline{t}_i^{m'_i}$ is the sum of all generators where m_i copies of ω and m'_i copies of $\overline{\omega}$ appear between s_i and s_{i+1}. Letting $S_{m,m'}$ be the sum of generators of the degree-(m,m') component of $T_{\mathbb{Z}}(H^1(E,\mathbb{C}))$, i.e., the sum of all permutations of $\omega^{\otimes m} \otimes \overline{\omega}^{\otimes m'}$, this sum can be written

$$\mathrm{Cor}_s\left(S_{m_0,m'_0} \otimes (x_0) \otimes S_{m_1,m'_1} \otimes (x_1) \otimes \cdots \otimes S_{m_n,m'_n} \otimes (x_n)\right). \tag{3.8}$$

These coefficients are called the *symmetric* Hodge correlators.

We also define, for $w_0, \ldots, w_n \in E$ with $w_0 + \cdots + w_n = 0_E$,

$$\Theta_s^*\big(w_0, w_1, \ldots, w_n \mid t_0 : t_1 : \cdots : t_n\big) = \Theta_s\big(0 : w_1 : w_1 + w_2 : \cdots : w_1 + \cdots + w_n \mid t_0 : t_1 : \cdots : t_n\big),$$

and, for $u_0 + \cdots + u_n = 0$,

$$\Theta_s\big(x_0 : x_1 : \cdots : x_n \mid u_0, u_1, \ldots, u_n\big) = \Theta\big(x_0 : x_1 : \cdots : x_n \mid 0 : u_1 : u_1 + u_2 : \ldots, u_1 + \cdots + u_n\big).$$

The subspace generated by the elements of $C\mathcal{L}^\vee_{E,S}$ having the form of the argument of (3.8) is dual to a certain quotient of the Lie algebra $\mathrm{Der}^S(\mathrm{gr}^W \pi_1^{\mathrm{nil}}(E - S, v_0))$. This is the quotient by the image of the adjoint action of $H_1(E; \mathbb{Z})$ mentioned in the introduction. In depth 0 and weight > 1, the elements (3.8) vanish, by the shuffle relations. In depth 0 and weight 1 – i.e., elements $\mathrm{Cor}(\omega_1, s_0)$ – the elements are identified with elements $[s_0] - [s]$ in the Jacobian of E (see [G9], §10.5), and, in particular, vanish if s and s_0 are torsion points. As we will see below, modulo the depth filtration, the symmetric correlators form a subcoalgebra, as the terms δ_{Cas} of the coproduct vanish.

Now let us establish some basic properties of the generating series.

Lemma 3.1. *(a) For $n > 0$, the generating series $\Theta(\ :|:\ )$ are homogeneous in the t_i and satisfy the dihedral symmetry relations:*

$$\begin{aligned}
&\Theta_s\big(x_0 : \cdots : x_n \mid t_0 : \cdots : t_n\big) \\
&= \Theta_s\big(x_0 + x : \cdots : x_n + x \mid t_0 + t : \cdots : t_n + t\big) && \textit{(homogeneity)} \\
&= \Theta_s\big(x_1 : \cdots : x_n : x_0 \mid t_1 : \cdots : t_n : t_0\big) && \textit{(cyclic symmetry)} \\
&= (-1)^{n+1}\Theta_s\big(x_n : \cdots : x_1 : x_0 \mid t_n : \cdots : t_1 : t_0\big). && \textit{(reflection)}
\end{aligned}$$

(b) For an automorphism $\phi \in \mathrm{Aut}(E)$,

$$\Theta_s\big(x_0, \ldots, x_n \mid t_0 : \cdots : t_n\big) = \Theta_s\big(\phi(x_0), \ldots, \phi(x_n) \mid \phi \cdot t_0 : \cdots : \phi \cdot t_n\big),$$

where ϕ acts on the t_i by the adjoint action on $H_1(E, \mathbb{Z})$.

(c) The elements $\Theta_s\big(x_0 : x_1 : \cdots : x_n \mid u_0, u_1, \ldots, u_n\big)$ *satisfy the first shuffle relations:*

$$\sum_{\sigma\in\Sigma_{i,j}} \Theta_s\big(x_{\sigma^{-1}(1)} : \cdots : x_{\sigma^{-1}(i+j)} : x_0 \mid u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(i+j)}, u_0\big) = 0. \tag{3.9}$$

Proof. The dihedral symmetry relations in (a) and the relation (b) are clear from the definition of Hodge correlators.

The difficult part is homogeneity in t_i and the first shuffle relation. For the former, it is enough to show

$$\Theta_s\big(x_0 : \cdots : x_n \mid 0 : t_1 : \cdots : t_n\big) = \Theta_s\big(x_0 : \cdots : x_n \mid t_0 : t_0 + t_1 : \cdots : t_0 + t_n\big).$$

Consider the coefficient of $\prod_i t_i^{m'_i} \overline{t_i}^{m'_i}$ in the sum defining each side (3.7). For each i, fix an an ordering $\omega_{i,1} \ldots \omega_{i,m_i+m'_i}$ of the word $\omega^{m_i}\overline{\omega}^{m'_i}$ and look at the terms in this coefficient in which the elements indexed by t_i appear in the order specified by the word.

If $m_0 = m'_0$, then both sides have exactly one such term

$$\mathrm{Cor}_{\mathrm{Hod}}(x_0, 1, x_1, \bigotimes_i \omega_{1,i} \ldots, x_n, \bigotimes_i \omega_{n,i})),$$

and they coincide. Otherwise, the coefficient on the left side is 0, while the terms on the right side are exactly the first shuffle relation on

$$\mathrm{Cor}_{\mathrm{Hod}}(x_0, \underbrace{\bigotimes \omega_{0,i}}, \underbrace{x_1, \bigotimes \omega_{1,i} \ldots, x_n, \bigotimes \omega_{n,i}}),$$

which is 0. This proves homogeneity in the t_i.

Finally, (c) also follows from the first shuffle relation on the coefficients. To obtain the relation where $\{1, \ldots, i\}$ are shuffled with $\{i+1, \ldots, i+j\}$, we keep x_0 fixed and shuffle the $x_1, \ldots, x_i$ and the forms indexed by $u_1, \ldots, u_i$ with the other elements. (The proofs are identical for those for correlators on $\mathbb{P}^1$; see Lemma 2.12.) □

Coproduct

The coproduct of the generating function Θ_s is in general difficult to write down. However, we can describe the terms of highest depth, which come from the δ_S component of the coproduct.

Lemma 3.2. *The coproduct of the generating functions Θ_s is given by*

$$\begin{aligned}
&\delta\Theta_s\big(x_0 : \cdots : x_n \mid t_0 : \cdots : t_n\big) = \\
&= \sum_{\text{cyc}} \sum_{k=0}^{n} \Theta_s\big(x_0 : \cdots : x_k \mid t_0 : \cdots : t_k\big) \wedge \Theta_s\big(x_k : x_{k+1} : \cdots : x_n \mid t_0 : t_{k+1} : \cdots : t_n\big) \\
&\quad + \text{lower depth terms.}
\end{aligned}$$

The coproduct of the generating functions Θ_s^* is given by

$$\begin{aligned}
&\delta\Theta_s^*\big(x_0, \ldots, x_n \mid t_0 : \cdots : t_n\big) = \\
&= \sum_{\text{cyc}} \sum_{k=0}^{n} \Theta_s^*\big(-(x_1 + \cdots + x_k), x_1 : \ldots, x_k \mid t_0 : t_1 : \cdots : t_k\big) \wedge \Theta_s^*\big(x_0, x_{k+1}, \ldots, x_n \mid t_0 : t_{k+1} : \cdots : t_n\big) \\
&\quad + \text{lower depth terms.}
\end{aligned} \tag{3.10}$$

The lower-depth terms are Hodge correlators of elements that do not depend on s.

Proof. The formula for the coproduct of Θ_s arises from the definition of the δ_S term of the coproduct. The formula for the coproduct of Θ_s^* would follow immediately from that for Θ_s if the Θ_s were invariant under an additive shift of the arguments x_i. This is Theorem 3.5 below, which is independent of (3.10). □

These formulas for the coproduct formally coincide with those for the dihedral Lie coalgebra, defined by Goncharov in [G4] in order to study multiple polylogarithms, as well as in the quasidihedral Lie coalgebra modulo the depth filtration, defined in Chapter 2 to study Hodge correlators on $\mathbb{P}^1$.

3.2.2 Symmetric correlators modulo depth

In this section, $H^1(X)$ always refers to $H^1(X;\mathbb{C})$.

Change of base point formula

Fix $p \in S$. Let us define a map $\rho_p : T(H^1(X)) \to T(H^1(X) \oplus \mathbb{Q}[S])$ as follows.

For a word $\omega_1 \otimes \cdots \otimes \omega_n \in T(H^1(X))$,

$$\rho_p(\omega_1 \otimes \cdots \otimes \omega_n) = \sum_k (-1)^k \sum_{\substack{i_1 < i_2, \cdots < i_k < n \\ i_{j+1} > i_j + 1}} \omega_1 \otimes \cdots \otimes (\langle \omega_{i_j}, \omega_{i_j+1} \rangle (p)) \otimes \cdots \otimes \omega_n,$$

where $\langle , \rangle$ is the skew-symmetric pairing: $\langle \omega, \overline{\omega} \rangle = -\langle \overline{\omega}, \omega \rangle = 1$. That is, it is the sum over all possible replacements of pairs $(\omega \otimes \overline{\omega})$ and $(\overline{\omega} \otimes \omega)$ by the puncture p, taken with appropriate sign. For example, we have:

$$
\begin{aligned}
\rho_p(1) &= 1, \\
\rho_p(\omega) &= \omega, \\
\rho_p(\omega \otimes \omega) &= \omega \otimes \omega, \\
\rho_p(\omega \otimes \overline{\omega}) &= (\omega \otimes \overline{\omega}) - (p), \\
\rho_p(\omega \otimes \overline{\omega} \otimes \omega) &= (\omega \otimes \overline{\omega} \otimes \omega) - (p \otimes \omega) + (\omega \otimes p), \\
\rho_p(\omega \otimes \overline{\omega} \otimes \omega \otimes \overline{\omega}) &= (\omega \otimes \overline{\omega} \otimes \omega \otimes \overline{\omega}) - (p \otimes \omega \otimes \overline{\omega}) - (\omega \otimes \overline{\omega} \otimes p) + (\omega \otimes p \otimes \overline{\omega}) + (p \otimes p).
\end{aligned}
$$

For $a \in S$, define $\rho_p(a) = (a) - (p)$, extended by linearity to $\mathbb{Q}[S]$. Then, extend ρ_p to $CT(H^1(X) \oplus \mathbb{Q}[S])$: if $x_0, \ldots, x_k \in \mathbb{Q}[S]$, and $\Omega_0, \ldots, \Omega_k \in T(H^1(X))$, then

$$
\rho_p(\Omega_0 \otimes x_0 \otimes \cdots \otimes \Omega_k \otimes x_k) = \rho_p(\Omega_0) \otimes \rho_p(x_0) \otimes \cdots \otimes \rho_p(\Omega_k) \otimes \rho_p(x_k).
$$

Lemma 3.3 (Change of base point formula)**.** *Suppose that $p \neq q$. Then the following relation holds for Hodge correlators in weight > 2:*

$$
\mathrm{Cor}_p(x) = \mathrm{Cor}_q\left(\rho_p(x)\right). \tag{3.11}
$$

On the right side stands a sum of correlators obtained from the one on the left by taking all possible replacements of punctures and pairs of adjacent cohomology classes with (p), taken with the appropriate sign.

Before proceeding to the proof, let us illustrate the formula on some examples. In weight 4,

$$
\begin{aligned}
\mathrm{Cor}_p(a, b, c) &= \mathrm{Cor}_q(a, b, c) - \mathrm{Cor}_q(p, b, c) - \mathrm{Cor}_q(a, p, c) - \mathrm{Cor}_q(a, b, p), \\
\mathrm{Cor}_p(a, b, \omega, \overline{\omega}) &= \mathrm{Cor}_q(a, b, \omega, \overline{\omega}) \\
&\quad - \mathrm{Cor}_q(p, b, \omega, \overline{\omega}) - \mathrm{Cor}_q(a, p, \omega, \overline{\omega}) - \mathrm{Cor}_q(a, b, p) + \mathrm{Cor}_q(p, p, \omega, \overline{\omega}), \\
\mathrm{Cor}_p(a, \omega, \overline{\omega}, \omega, \overline{\omega}) &= \mathrm{Cor}_q(a, \omega, \overline{\omega}, \omega, \overline{\omega}) \\
&\quad - \mathrm{Cor}_q(p, \omega, \overline{\omega}, \omega, \overline{\omega}) - \mathrm{Cor}_q(a, p, \omega, \overline{\omega}) + \mathrm{Cor}_q(a, \omega, p, \overline{\omega}) - \mathrm{Cor}_q(a, \omega, \overline{\omega}, p) \\
&\quad + \mathrm{Cor}_q(p, p, \omega, \overline{\omega}) - \mathrm{Cor}_q(p, \omega, p, \overline{\omega}) + \mathrm{Cor}_q(p, \omega, \overline{\omega}, p).
\end{aligned}
$$

If the left side of the expression only contains punctures, we recover a formula identical to the one found by [GR], Theorem 2.6, for Hodge correlators on the punctured $\mathbb{P}^1$. More generally, for symmetric correlators, we have:

Corollary 3.4. *Suppose that $p \neq q$. Then we have the relation in weight > 2:*

$$\begin{aligned}
&\mathrm{Cor}_p(S_{m_0,m_0'} \otimes x_0 \otimes S_{m_1,m_1'} \otimes x_1 \otimes \cdots \otimes S_{m_n,m_n'} \otimes x_n) \\
&= \mathrm{Cor}_q(S_{m_0,m_0'} \otimes ((x_0) - (p)) \otimes \cdots \otimes S_{m_n,m_n'}((x_n) - (p))) \\
&= \sum_k (-1)^k \sum_{i_1 < \cdots < i_k} \mathrm{Cor}_q(S_{m_0,m_0'} \otimes x_0 \otimes \cdots \otimes p \otimes \cdots \otimes p \otimes \cdots \otimes S_{m_n,m_n'} \otimes x_n),
\end{aligned}$$

where on the right the punctures $x_{i_1}, \ldots, x_{i_k}$ are replaced with q.

Proof. For all $m, m' \geq 0$, $\rho_p(S_{m,m'}) = S_{m,m'}$. □

Proof of Lemma 3.3. We first prove the change of base point formula in the real Hodge realization, i.e, that it holds on the level of the Hodge correlator functions $\mathrm{Cor}_{\mathcal{H}}$.

The Green's functions associated to the points p and q are related by

$$G_p(x, y) = G_q(x, y) - G_q(x, p) - G_q(y, p) + C,$$

where C is a constant that depends on the choices of tangent vectors at p and q. Now consider any tree contributing to the Hodge correlator of $\Omega_0 \otimes (x_0) \otimes \cdots \otimes \Omega_k \otimes x_k$. Write the Green's function $G_p(x, y)$ assigned to each edge in terms of the G_q, and examine the contribution of the three terms in $G_p(x, y) - G_q(x, y)$: $G_q(x, p)$, $G_q(y, p)$, and C for a given edge F. There are three cases:

(1) An external edge F decorated by a puncture a, assigned the function $G_p(x, a)$. Assigning the form $-G_q(x, p)$ to F gives the correlator where a has been replaced by $-(p)$. The terms C and $G_q(a, p)$ are constants. Because the Hodge correlator has weight > 2, there is at least one internal edge in the tree, so the correlator where a constant has been placed on F is the integral of an exact form $d^{\mathbb{C}}(\ldots)$.

(2) An internal edge F that splits the tree into two parts, one of which is decorated by two 1-forms. Suppose that in $G_p(x, y)$, the vertex assigned the variable x is adjacent to external vertices labeled ω_1 and ω_2. Then the terms $G_q(x, p)$ and C are independent of y, and the integral splits into a product; the integrand for the subtree growing from y is an exact form, so we get 0. For the term $-G_q(y, p)$, the integral also splits into a product of $\int_E \omega_1 \wedge \omega_2$ and the correlator with x replaced by an external vertex $-(p)$.

(3) An internal edge F that splits the tree into two parts, each of which is decorated by at least one puncture. Then, as in the previous case, each term in the expression for $G_p(x, y)$ is independent of either x or y. The integral splits into a product of two factors, one of which is 0.

We conclude that the change of base point is computed by adding all possible replacements of external punctures a by $-(p)$ and pairs $\omega_1 \otimes \omega_2$ by $-\langle \omega_1, \omega_2 \rangle\,(p)$. This implies the lemma.

(Note that the assumption of weight > 2 was crucial to all arguments involving integration of the exact form.)

One easily checks by induction that the coproducts of the two sides of (3.11) are equal. This implies the result on the level of the Hodge correlators $\mathrm{Cor}_{\mathrm{Hod}}$. ◻

Independence of base point

In this part, we prove the following important result.

Theorem 3.5. *The symmetric Hodge correlators in weight* > 2 *are independent of the base point modulo the depth filtration.*

Formally, let $x \in CT(H^1(X) \oplus \mathbb{Q}[S])$. *Then there exists* $\widetilde{x}$, *equal to* x *modulo lower-depth terms, such that* $\mathrm{Cor}_p(\widetilde{x})$ *is independent of* p.

In terms of generating functions, this theorem implies:

Corollary 3.6. *The generating functions* Θ^* *satisfy the dihedral symmetry relations of Lemma 3.1:*

$$\begin{aligned}
&\Theta_s^*\big(w_0, \ldots, w_n \mid t_0 : \cdots : t_n\big)\\
&= \Theta_s^*\big(w_1, \ldots, w_n, w_0 \mid t_1 : \cdots : t_n : t_0\big)\\
&= (-1)^{n+1}\Theta_s^*\big(w_n, \ldots, w_1, w_0 \mid t_n : \cdots : t_1 : t_0\big)
\end{aligned}$$

modulo lower-depth terms that are independent of s.

Proof. By the cyclic symmetry and dihedral relations on correlators, these expressions are equal up to an additive shift in the correlators' arguments, equivalently, a change in base point. ◻

Notice that all terms on the right side of (3.11) have *higher* or equal depth to the left side. It will be necessary to find correction terms of lower depth to obtain a formula of the form

$$\mathrm{Cor}_p(h_0 \otimes x_0 \otimes \cdots \otimes h_k \otimes x_k) + \mathrm{Cor}_p(\text{lower depth}) = \mathrm{Cor}_q(h_0 \otimes x_0 \otimes \cdots \otimes h_k \otimes x_k) + \mathrm{Cor}_q(\text{l.d.})$$

when each h_i a symmetric expression $S_{m,m'}$.

The proof of the theorem relies on a key construction. We will find elements:

$$S_{m_0,m_0'} * S_{m_1,m_1'} * \cdots * S_{m_n,m_n'} \in T(H^1(X))$$

such that

$$\begin{aligned} &\rho_p(S_{m_0,m_0'} * S_{m_1,m_1'} * \cdots * S_{m_n,m_n'}) = \\ &= \sum_k \sum_{i_1<\cdots<i_k} \left(S_{m_0,m_0'} * \cdots * S_{m_{i_1-1},m'_{i_1-1}}\right) \otimes (p) \otimes \left(S_{m_{i_1},m'_{i_1}} * \cdots * S_{m_{i_2-1},m'_{i_2-1}}\right) \otimes (p) \otimes \ldots . \end{aligned} \tag{3.12}$$

Before showing how to construct these elements, let us prove the theorem, assuming these elements exist.

Proof of Theorem 3.5. Consider an element

$$x = S_{m_0,m_0'} \otimes x_0 \otimes S_{m_1,m_1'} \otimes x_1 \otimes \cdots \otimes S_{m_n,m_n'} \otimes x_n .$$

Let I be a proper subset of $\{0, \ldots, n\}$. Write I as the union of its cyclically contiguous subsets, each of the form $\{i, i+1, \ldots, i+k\}$ (indices modulo $n+1$). Let $x_{/I}$ be the element formed by replacing each

$$S_{m_i,m_i'} \otimes x_i \otimes \cdots \otimes x_{i+k} \otimes S_{m_{i+k},m'_{i+k}}$$

by $S_{m_i,m_i'} * \cdots * S_{m_{i+k},m'_{i+k}}$.

Now consider the corrected element:

$$\widetilde{x} = \sum_I (-1)^{|I|} x_{/I} .$$

It is equal to x modulo the depth filtration. Also, let

$$y_q = S_{m_0,m_0'} \otimes q \otimes S_{m_1,m_1'} \otimes q \otimes \cdots \otimes S_{m_n,m_n'} \otimes q,$$

and define $\widetilde{y}_q$ in the same way. By a standard inclusion-exclusion argument, the property (3.12) implies that

$$\rho_p(\widetilde{x} + \widetilde{y}_q) = \widetilde{x} + (\text{terms containing } q).$$

Because the correlator with base point q is zero for the terms containing q, this gives

$$\mathrm{Cor}_p(\widetilde{x}+\widetilde{y}_q)=\mathrm{Cor}_q(\widetilde{x}).$$

On the other hand, the Hodge correlator $\mathrm{Cor}_p(\widetilde{y}_q)$ depends only on $p-q$, and thus $p\mapsto \mathrm{Cor}_p(\widetilde{x})-\mathrm{Cor}_0(\widetilde{x})$ provides a group homomorphism $E\to\mathbb{R}$, and must be 0. Therefore, $\mathrm{Cor}_p(\widetilde{x})$ is independent of p. $\square$

Lemma 3.7. *There exist elements, independent of choice of symplectic basis of $H^1(X)$, satisfying (3.12).*

Proof. We produce such elements explicitly:

$$S_{m_0,m_0'}*\cdots*S_{m_k,m_k'}=\frac{1}{2^k}\sum_{n_0,n_0',\ldots,n_k,n_k'}\pm S_{n_0,n_0'}\otimes S_{n_1,n_1'}\otimes\cdots\otimes S_{n_k,n_k'},$$

where the sum is taken over the $n_i,n_i'\geq 0$ such that:

$$n_i+n_i'=\begin{cases} m_i+m_i'+1 & i=0,k\\ m_i+m_i'+2 & 0<i<k\end{cases},$$

$$(n_0-n_0')+\cdots+(n_k-n_k')=(m_0-m_0')+\cdots+(m_k-m_k').$$

A term is taken with the sign $-$ if there is an odd number of i ($i=0,\ldots,k-1$) such that

$$(n_0-n_0')+\cdots+(n_i-n_i')<(m_0-m_0')+\cdots+(m_i-m_i'),$$

otherwise with the sign $+$.

Examples:

$$\begin{aligned}
S_{0,0}*S_{0,0}&=\frac{1}{2}\left(\omega\overline{\omega}-\overline{\omega}\omega\right),\\
S_{0,0}*S_{1,0}&=\frac{1}{2}\left(\omega\omega\overline{\omega}+\omega\overline{\omega}\omega-\overline{\omega}\omega\omega\right),\\
S_{0,0}*S_{0,0}*S_{0,0}&=\frac{1}{4}\left(\omega\omega\overline{\omega}\overline{\omega}+\omega\overline{\omega}\omega\overline{\omega}-\omega\overline{\omega}\overline{\omega}\omega-\overline{\omega}\omega\omega\overline{\omega}+\overline{\omega}\omega\overline{\omega}\omega+\overline{\omega}\overline{\omega}\omega\omega\right).
\end{aligned}$$

We explain the construction by picture. The basis elements of $T(H^1(X))$ of a given weight are in bijection with lattice paths: a word $\omega_1\otimes\cdots\otimes\omega_n$ corresponds to the path whose i-th step is $(1,0)$ if $\omega_i=\omega$ and $(0,1)$ if $\omega_i=\overline{\omega}$. The elements of $T(H^1(X)\oplus\mathbb{Q}[p])$ are lattice paths that also allow the diagonal step $(1,1)$, corresponding to (p). (The points of the lattice path are simply the Hodge bidegrees of the initial subwords.)

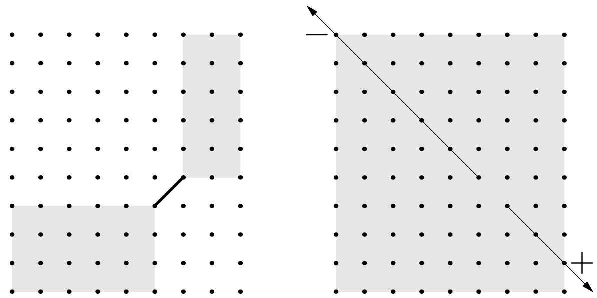


Figure 3.3. Construction of the element $S_{m_0,m'_0} * S_{m_1,m'_1}$: the paths crossing the rays marked $-$ and $+$ are taken with the corresponding sign.

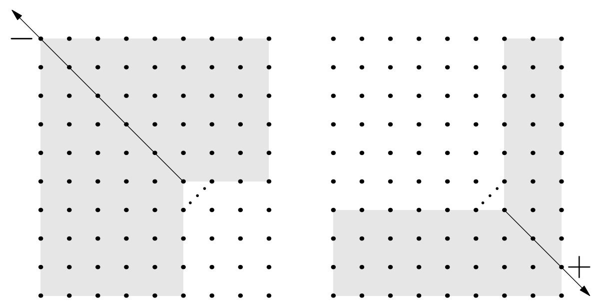


Figure 3.4. The point of nonconcavity contributing a term to the right side of (3.12).

The map ρ_p replaces a path by the sum of all paths obtained by replacing steps (up, right) or (right, up) with diagonal steps, in the latter case changing the sign.

To construct the element, we first consider the concatenation of paths in $S_{m_0,m'_0}, \ldots, S_{m_k,m'_k}$, with a step $(1,1)$ inserted between successive pairs. Draw a diagonal line ℓ_i bisecting the step that was inserted between $S_{m_{i-1},m'_{i-1}}$ and S_{m_i,m'_i}. Any path of the Hodge bidegree $\left(\sum m_i + k, \sum m'_i + k\right)$ appears in a unique term $S_{n_0,n'_0} \otimes \cdots \otimes S_{n_k,n'_k}$, and each S_{n_i,n'_i} is the sum of paths between the lines ℓ_i and ℓ_{i+1}. The sign of a path is a product of factors determined by the rays on which it crosses the diagonal lines: $+1$ if below the step, -1 if above. (See Figure 3.3.)

Now fix a choice of a ray of each such diagonal, and consider the terms coming from lattice paths crossing these rays. We claim that any such term satisfies (3.12) modified by a factor of $\frac{1}{2^k}$. Indeed, these are the lattice paths lying in a certain rectilinear region (right part of the figure). Most terms in ρ_p are canceled; the only terms remaining are those with segments $(1,1)$ at the points of nonconcavity of this region. This is precisely the expression on the right of (3.12). (See Figure 3.4.) $\square$

The simplest example of the corrected correlator, for $(a) \otimes (b) \otimes (c)$:

$$
\begin{aligned}
(a) \otimes (b) \otimes (c) - \frac{1}{2}\left(\omega \otimes \overline{\omega} - \overline{\omega} \otimes \omega\right) \otimes (b) \otimes (c) \\
- \frac{1}{2}(a) \otimes \left(\omega \otimes \overline{\omega} - \overline{\omega} \otimes \omega\right) \otimes (c) \\
- \frac{1}{2}(a) \otimes (b) \otimes \left(\omega \otimes \overline{\omega} - \overline{\omega} \otimes \omega\right).
\end{aligned}
$$

(The terms where two points were replaced are 0, because of the reflection relations.)

3.2.3 Second shuffle relations

The depth 2 case: Dihedral symmetry

Theorem 3.8. *The corrected symmetric Hodge correlators in depth 2 satisfy the second shuffle (dihedral symmetry) relations modulo terms of lower depth that are independent of the base point.*

The corrected element for

$$
\begin{aligned}
& S_{m_0,m_0'} \otimes (0) \otimes S_{m_1,m_1'} \otimes (x_1) \otimes S_{m_2,m_2'} \otimes (x_1 + x_2) \\
+ & S_{m_0,m_0'} \otimes (0) \otimes S_{m_2,m_2'} \otimes (x_2) \otimes S_{m_1,m_1'} \otimes (x_1 + x_2)
\end{aligned}
\tag{3.13}
$$

lies in the kernel of the map Cor_s *for every* s.

Proof. The corrected element for (3.13 changes sign under the map $x \mapsto (x_1 + x_2 - x)$ and reflection. On the other hand, it is invariant under this operation up to an additive shift (i.e., change in base point). □

Relations in higher depth

The *second shuffle relations* are relations of the form

$$
\sum_{\sigma \in \Sigma_{i,j}} \Theta^*\left(x_0, x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(i+j)}\right) \mid t_0, t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, \ldots, t_{\sigma^{-1}(i+j)}\big) + \ldots,
$$

perhaps with additional terms of lower depth. The Hodge correlators on $\mathbb{P}^1$ are known to obey such relations, in addition to the first shuffle relations, the structural relations in $\mathcal{CL}^\vee_{X,S,v_0}$; the lower-depth terms were described precisely in Chapter 2.

The relation of Theorem 3.8 is a special case of a second shuffle relation. In depth > 2, the second shuffle relations are not equivalent to dihedral symmetry. However, one hopes for a generalization.

Conjecture 3.9. *The second shuffle relations for symmetric elliptic Hodge correlators hold modulo the depth filtration. The lower-depth terms are independent of the base point s.*

The lower-depth correction terms in depth > 2 are not known. In particular, the corrected correlators do not satisfy the second shuffle relations in higher depth. However, calculations in low weight support this conjecture. We may expect the elliptic relations to be deformations of the relations for $\mathbb{P}^1$ (see §3.4.2).

3.3 Bianchi hyperbolic threefolds and modular complexes

3.3.1 Bianchi tessellations and orbifolds

Definition

Let $K = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic field with lattice of integers $\mathcal{O}$. The *Bianchi tessellation* ([B4]) is an ideal polyhedral tessellation of the upper half-space $\mathbb{H}^3$ associated with $\mathcal{O}$, whose cell complex has a natural structure of a complex of $\mathrm{GL}_2(\mathcal{O})$-modules. We define it now.

Let $\overline{\mathcal{F}}$ be the space of positive semidefinite Hermitian forms on $(\mathcal{O}^2 \otimes_{\mathcal{O}} \mathbb{C})^*$. The subset $\mathcal{F}$ of positive definite forms is a dense open subset of $\overline{\mathcal{F}}$. We identify $\mathbb{H}^3$ and its compactification $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ with the real projectivizations of $\mathcal{F}$ and $\overline{\mathcal{F}}$, respectively. The action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ provides an action on $\overline{\mathcal{F}}$ that descends to an action on $\overline{\mathbb{H}}^3$.

Every $v \in \mathcal{O}^2$ provides a positive semidefinite form $|\langle -, v\rangle|^2 \in \partial\overline{\mathcal{F}}$. The convex hull of the set

$$\left\{|\langle -, v\rangle|^2 : v \text{ a primitive vector in } \mathcal{O}^2\right\}$$

is a polyhedron in $\overline{\mathcal{F}}$ with vertices on the boundary. The polyhedron projects to an ideal tessellation of $\mathbb{H}^3$ with vertices on $\mathbb{P}^1(\mathcal{O}) \subset \mathbb{P}^1(\mathbb{C})$. Let $B^\bullet$ be the polyhedral cell complex over $\mathbb{Z}$ of this ideal tessellation. We will shift this complex in degree so that the space of i-dimensional cells it in degree $3 - i$ ($i = 0, 1, 2, 3$). We get a cohomological complex

$$B^0 \xrightarrow{\partial} B^1 \xrightarrow{\partial} B^2 \xrightarrow{\partial} B^3.$$

The group $\mathrm{GL}_2(\mathcal{O})$ acts on the Bianchi tessellation, giving $B^\bullet$ the structure of a complex of left $\mathrm{GL}_2(\mathcal{O})$ modules.

The quotient $\mathrm{GL}_2(\mathcal{O}) \setminus \mathbb{H}^3$ is a finite-volume hyperbolic threefold with cusps in bijection with the ideal class group of $\mathcal{O}$. If Γ is a finite-index subgroup of $\mathrm{GL}_2(\mathcal{O})$, the quotient $\Gamma \setminus \mathbb{H}^3$ is also a finite-volume hyperbolic threefold with a finite map to $\mathrm{GL}_2(\mathcal{O}) \setminus \mathbb{H}^3$.

A right $\mathrm{GL}_2(O)$-module T provides a local system on $\Gamma \setminus \mathbb{H}^3$, which we also denote by T. Then the chain complex of $\mathrm{GL}_2(O) \setminus \mathbb{H}^3$ with coefficients in T is

$$T \otimes_\Gamma B^\bullet \cong (\mathbb{Z}[\Gamma \setminus \mathrm{GL}_2(O)] \otimes T) \otimes_{\mathrm{GL}_2(O)} B^\bullet. \tag{3.14}$$

The Gaussian and Eisenstein cases

Following [G8], for $d = 1$ ($O = \mathbb{Z}[i]$) and $d = 3$ ($O = \mathbb{Z}[\rho]$) we have the following description of the Bianchi complexes in degrees 1 and 2.

The action of $\mathrm{GL}_2(O)$ is transitive on the i-dimensional cells for each i. Choose $\mathrm{GL}_2(O)$-generators $G_i \in B^i$: we may take

$$G_1 = (\text{the ideal triangle } (1, 0, \infty))$$
$$G_2 = (\text{the geodesic } (0, \infty))$$

where $(v_1, \ldots, v_n)$, $v_i \in \mathbb{P}^1(O) = \mathbb{P}(V^2(O))$, denotes the oriented cell with ideal vertices at $v_1, \ldots, v_n$ under the identification of $\mathbb{P}^1(\mathbb{C})$ with the boundary of $\overline{\mathbb{H}}^3$. Let D_i be the subgroup of $\mathrm{GL}_2(O)$ stabilizing G_i.

The group D_1 stabilizing the triangle $(0, 1, \infty)$ is isomorphic to

$$S_3 \times O^\times.$$

The first component S_3 acts on $(v, w) \in O \oplus O$ by permutations of the triple $(v, w, -v - w)$, i.e., the generators of S_3 are represented by

$$(123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The second component acts by scalars. There is a sign homomorphism $\chi_1 : D_1 \to \mathbb{Z}$ keeping track of the action of D_1 on the orientation, with $\chi_1((123)) = 1$ and $\chi_1((12)) = -1$. So the space of 2-cells is

$$B^1 = \mathbb{Z}[GL_2(O)] \otimes_{D_1} \chi_1.$$

The group D_2 stabilizing the geodesic $(0, \infty)$ is isomorphic to

$$S_2 \ltimes (O^\times \times O^\times),$$

with S_2 acting on $O^\times \times O^\times$ by permutation of the factors. The nontrivial element of S_2 acts by $(v, w) \mapsto (w, v)$

and $O^\times \times O^\times$ acts diagonally. There is a sign homomorphism $\chi_2 : D_2 \to \mathbb{Z}$, and the space of 1-cells is

$$B^2 = \mathbb{Z}[GL_2(\mathbb{Z}[i])] \otimes_{D_2} \chi_2.$$

Let $\mathfrak{p}$ be a prime ideal in O. The group $\mathrm{GL}_2(O)$ acts on the quotient $(\mathbb{Z}[i]/\mathfrak{p})^2$. Let $\Gamma_1(\mathfrak{p})$ be the stabilizer in $\mathrm{GL}_2(O)$ of the vector $(0,1) \in (\mathbb{Z}[i]/\mathfrak{p})^2$. The action on the vector $(0,1)$ provides an isomorphism of $\mathrm{GL}_2(O)$-modules

$$\mathbb{Z}[\Gamma_1(\mathfrak{p}) \setminus \mathrm{GL}_2(O)] \cong \mathbb{Z}[\mathbb{F}_{\mathfrak{p}}^2 - 0], \quad \mathbb{F}_{\mathfrak{p}} = O/\mathfrak{p}.$$

The chain complex (3.14) of $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$ with coefficients in a local system T is then identified in degrees 1 and 2 with

$$\begin{aligned} T \otimes_{\Gamma_1(\mathfrak{p})} B^\bullet &\cong (\mathbb{Z}[\Gamma_1(\mathfrak{p}) \setminus \mathrm{GL}_2(O)] \otimes T) \otimes_{\mathrm{GL}_2(O)} B^\bullet \\ &\cong \left(\mathbb{Z}[\mathbb{F}_{\mathfrak{p}}^2 - 0] \otimes T\right) \otimes_{\mathrm{GL}_2(O)} \left(\mathbb{Z}[\mathrm{GL}_2(O)] \otimes_{D_\bullet} \chi_\bullet\right). \end{aligned}$$

This space is generated in degree i by elements

$$((\alpha,\beta) \otimes t) \otimes (G_i), \quad (\alpha,\beta) \in \mathbb{F}_{\mathfrak{p}}^2 - 0, \quad t \in T.$$

3.3.2 Modular complexes

Definition

Let $O = \mathbb{Z}$ or the lattice of integers in an imaginary quadratic field. We are going to define the modular complexes $M_k^\bullet$, complexes of left $\mathrm{GL}_k(O)$-modules that generalize the complexes defined by [G3] for $\mathrm{GL}_k(\mathbb{Z})$.

Fix a k-dimensional O-vector space V. An *extended basis* of V is a sequence of vectors $\langle v_0, v_1, \ldots, v_k \rangle$, $v_i \in V$, such that $v_0 + \cdots + v_k = 0$ and $v_1, \ldots, v_k$ form a basis of V. (Consequently, any other set of k vectors in this sequence form a basis.) We also use the notation

$$\begin{aligned} [v_1, \ldots, v_k] &= \langle -v_1 - \cdots - v_k, v_1, v_2, \ldots, v_k \rangle, \\ [v_1 : \cdots : v_k] &= [v_2 - v_1, v_3 - v_2, \ldots, v_k - v_{k-1}, -v_k]. \end{aligned}$$

The set B_V of extended bases of V is a principal homogeneous space for $\mathrm{GL}(V)$.

The complex of left $\mathrm{GL}_k(O)$-modules $M_k^\bullet$ lies in the degrees $1, \ldots, n$. The module M_k^1 is the quotient of

$\mathbb{Z}[B_V]$ by the double shuffle relations

$$\sum_{\sigma \in \Sigma_{i,j}} \left[v_{\sigma^{-1}(1)} : \cdots : v_{\sigma^{-1}(i+j)}\right] = 0, \qquad \textit{(first shuffle)} \tag{3.15}$$

$$\sum_{\sigma \in \Sigma_{i,j}} \left[v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(i+j)}\right] = 0. \qquad \textit{(second shuffle)} \tag{3.16}$$

Lemma 3.10 ([G3], Theorem 4.1)**.** *The double shuffle relations imply the dihedral symmetry relations:*

$$\langle v_0, v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_k, v_0 \rangle = (-1)^{k+1} \langle v_k, \ldots, v_1, v_0 \rangle = \langle -v_0, -v_1, \ldots, -v_k \rangle . \tag{3.17}$$

The module M_k^n is generated by elements

$$[v_1, \ldots, v_{k_1}] \wedge \cdots \wedge [v_{k_{n-1}+1}, \ldots, v_{k_n}],$$

where each block $\left[v_{k_{i-1}+1}, \ldots, v_{k_i}\right]$ is an extended basis of a sublattice V_i in V, and $V = V_1 \oplus \cdots \oplus V_n$ (from which is follows that $k_1 + \cdots + k_n = k$). The double shuffle relations are imposed on each of the blocks, and the blocks anticommute.

The coproduct $\delta : M_k^1 \to M_k^2$ is defined by

$$\delta \langle v_0, v_1, \ldots, v_k \rangle = \sum_{\text{cyc}} \sum_{i=1}^{k} [v_0, \ldots, v_{i-1}] \wedge [v_{i+1}, \ldots, v_k]$$

with the outer cyclic sum is over $\{0, 1, \ldots, k\}$. The coproduct is extended by the Leibniz rule to the higher degrees, i.e.,

$$\delta(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_1 \wedge \cdots \wedge \delta(x_i) \wedge \cdots \wedge x_n.$$

We will also consider the *relaxed modular complex* $\widetilde{M}_k^n$, in which impose only the first shuffle relations (3.15) and the dihedral symmetry relations (3.17). By the lemma, the modular complex is the quotient of the relaxed modular complex by the second shuffle relations (3.16).

Relating the Gaussian and Eisenstein Bianchi and modular complexes for $k = 2$

In this section, suppose $\mathcal{O} = \mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$. We will construct an isomorphism between the modular complex $M_2^\bullet$ and the Bianchi complex $B^\bullet$ in degrees 1 and 2.

Recall that $B^\bullet$ is generated by the ideal triangle $(1, 0, \infty)$ in degree 1 and the geodesic $(0, \infty)$ in degree 2, with the boundary map given by

$$(1,0,\infty) \mapsto (1,0) + (0,\infty) + (\infty,1).$$

The modular complex $M_2^\bullet$ is generated in degree 1 by the extended basis $[e_1, e_2]$, with the coproduct

$$[e_1, e_2] \mapsto [-e_1 - e_2] \wedge [e_2] + [e_1] \wedge [-e_1 - e_2] + [e_2] \wedge [e_1].$$

Making as before the identification of $\mathbb{P}^1(O)$ with $\mathbb{P}^1(V)$, define the map $\psi : M_2^\bullet \to B^\bullet$ by

$$\psi\,(\langle v_1, v_2, v_3\rangle) = \text{the triangle } (v_1, v_2, v_3), \quad \psi\,([v_1] \wedge [v_2]) = \text{the geodesic } (v_1, v_2).$$

Lemma 3.11. *The map ψ is an isomorphism of complexes of* $\mathrm{GL}_2(O)$*-modules.*

Proof. By construction, ψ is a surjective map of abelian groups. We must verify (1) ψ commutes with the action of $\mathrm{GL}_2(O)$, (2) ψ commutes with the coproduct, (3) ψ respects the double shuffle relations, and the images of the double shuffle and anticommutation relations are all relations in $B^\bullet$.

(1) holds by construction. For (2), notice that

$$\begin{aligned}\delta\,[e_1, e_2] &= [-e_1 - e_2] \wedge [e_2] + [e_1] \wedge [-e_1 - e_2] + [e_2] \wedge [e_1]. \\ &= [e_2] \wedge [e_1] + \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} [e_2] \wedge [e_1] + \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^2 [e_2] \wedge [e_1]\end{aligned}$$

and that $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ acts by cyclic permutation on $(0,1,\infty)$.

For (3), double shuffle relation in M_2^1 is just equivalent to dihedral symmetry, which is precisely the relation imposed by $\otimes_{D_1}\chi_1$. The only relations in M_2^2 are the anticommutation relation and the relation $[v_1] = [-v_1]$, whose images are the only relations among the 1-cells in B^2. $\square$

As a consequence, the chain complex of $\Gamma_1(\mathfrak{p}) \setminus \mathbb{H}^3$ with coefficients in a local system T is identified with

$$\left(\mathbb{Z}[\mathbb{F}_{\mathfrak{p}}^2 - 0] \otimes T\right) \otimes_{\mathrm{GL}_2(O)} M_2^\bullet$$

and generated in degree i by

$$((\alpha,\beta) \otimes t) \otimes [v_1, v_2], \quad (\alpha,\beta) \in \mathbb{F}_{\mathfrak{p}}^2 - 0, \quad t \in T.$$

3.3.3 Relating the modular and Bianchi complexes to the Galois Lie coalgebra

Motivic correlators at torsion points and averaged base point Hodge correlators

Let E be an elliptic curve, p a prime, $\mathfrak{p} \subset \mathrm{End}E$ a prime over p, and $S = E[\mathfrak{p}]$. There is an canonical up to root of unity choice of tangent vector v_0 at $0 \in E$, given by the Dedekind eta function. Extend it to a translation-invariant vector field on E and take v_s to be its fiber at s.

Recall that we packaged the Lie coalgebras $\mathcal{CL}^\vee_{E,S,v_s}$ into a coalgebra $\mathcal{CL}^\vee_{E,S} = \bigoplus_s \mathcal{CL}^\vee_{E,S,v_s}$. The $\mathcal{CL}^\vee_{E,S,v_s}$ for different s are canonically isomorphic, so there is a natural diagonal $D \subset \mathcal{CL}^\vee_{E,S}$. The image of D under $\mathrm{Cor}_{\mathrm{Hod}}$ is the space of *averaged base point correlators*. Equivalently, it is the image of the *averaged base point correlator map* $\mathrm{Cor}_{\mathrm{av}} = \frac{1}{|E[\mathfrak{p}]|}\sum_s \mathrm{Cor}_s$. The image of the restriction to the space of symmetric correlators is called the coalgebra of *symmetric averaged base point Hodge correlators* and denoted $\mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[\mathfrak{p}])$. It is the dual to the quotient of $\mathrm{Lie}_{\mathrm{Hod}}(E, E[\mathfrak{p}])$ induced by the quotient of $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(E - E[\mathfrak{p}], v_0)$ by the adjoint action of $H_1(E;\mathbb{Z})$ and the translation action of $E[\mathfrak{p}]$ on E.

Relaxed modular complexes and Hodge correlators

Suppose that E is an elliptic curve, and $\mathcal{O}$ its endomorphism ring, and suppose $\mathfrak{p}$ is a prime in $\mathcal{O}.\mathfrak{p}$.

Let T_k denote the graded right $\mathrm{GL}_k(\mathcal{O})$-module $\mathrm{Sym}\left(H_1(E;\mathbb{Z})^{\oplus k}\right) \otimes \mathbb{Q}$, identified with the algebra of polynomials in the variables $t_1, \bar{t}_1, \dots, t_k, \bar{t}_k$, and let $\Gamma_1(\mathfrak{p}) \subset \mathrm{GL}_k(\mathcal{O})$ be the stabilizer of the vector $(0, \dots, 0, 1) \in (\mathcal{O}/\mathfrak{p})^k$. We will define a map θ from the relaxed modular complex with coefficients in T_k to the depth k component of the standard cochain complex of the Lie coalgebra $\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[\mathfrak{p}])$:

$$\theta : T_k \otimes_{\Gamma_1(\mathfrak{p})} \widetilde{M}_k^\bullet \to \mathrm{CE}^\bullet\left(\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[p])\right)_{D=k}.$$

Fix an extended basis $\langle v_1, \dots, v_k, v_0\rangle$ of $V_k(\mathcal{O})$. Also fix an identification of $\mathbb{F}_\mathfrak{p}$ with $E[\mathfrak{p}]$.We will abuse notation and identify $\alpha \in \mathbb{F}_\mathfrak{p}$ with $\alpha \in E[p]$. Last, we identify the domain of θ with

$$\left(\mathbb{Z}[\mathbb{F}_\mathfrak{p}^k - 0] \otimes T_k\right) \otimes_{\mathrm{GL}_2(\mathcal{O})} M_k^\bullet.$$

In the degree 1 component, define the map on the level of generating series by

$$\begin{aligned}&\sum_{n_1, n_1', \dots, n_k, n_k'} \left((\alpha_1, \dots, \alpha_k) \otimes t_1^{n_1}\bar{t}_1^{n_1'} \dots t_k^{n_k}\bar{t}_k^{n_k'}\right) \otimes [v_1, \dots, v_k] \\ &\mapsto \frac{1}{|E[\mathfrak{p}]|}\sum_{s \in E[\mathfrak{p}]} \Theta_s^*\big(\alpha_1, \dots, \alpha_k, -(\alpha_1 + \dots + \alpha_k) \mid t_1 : \dots : t_k : 0\big). \qquad (3.18)\end{aligned}$$

The maps in higher degrees are given by

$$\left(\mathbb{Z}[\mathbb{F}_{\mathfrak{p}} - 0] \otimes T_k\right) \otimes_{\mathrm{GL}_2(O)} M_k^n \to \left(\bigwedge^n \mathrm{gr}^D \mathrm{Lie}^{\vee}_{\mathrm{sym}}(E, E[p])\right)_{D=k},$$

$$\sum_{n_1, n_1', \ldots, n_k, n_k'} \left((\alpha_1, \ldots, \alpha_k) \otimes t_1^{n_1} \bar{t}_1^{n_1'} \ldots t_k^{n_k} \bar{t}_k^{n_k'}\right) \otimes \left(\left[v_1, \ldots, v_{k_1}\right] \wedge \cdots \wedge \left[v_{k_{n-1}+1}, \ldots, v_{k_n}\right]\right)$$

$$\mapsto \frac{1}{|E[\mathfrak{p}]|} \sum_{s \in E[\mathfrak{p}]} \Theta_s^*\left(\alpha_1, \ldots, \alpha_{k_1}, -(\alpha_1 + \cdots + \alpha_{k_1}) \mid t_1 : \cdots : t_{k_1} : 0\right) \wedge \cdots \wedge$$

$$\wedge\, \Theta_s^*\left(\alpha_{k_{n-1}+1}, \ldots, \alpha_{k_n}, -(\alpha_{k_{n-1}+1} + \cdots + \alpha_{k_n}) \mid t_{k_{n-1}+1} : \cdots : t_{k_n} : 0\right).$$

Theorem 3.12. *The map θ is a well-defined surjective morphism of complexes of graded O-modules.*

Proof. The map θ is a morphism of graded $\mathrm{GL}_k(O)$-modules by construction (recall the $t_i, \bar{t}_i$ are dual to the cohomology generators $\omega, \overline{\omega}$), and is surjective by construction. We need to verify that the map θ respects (1) the first shuffle relations, (2) the dihedral symmetry relations, (3) the coproduct. We show the three in order.

The first shuffle relation on the image holds termwise – for each $s \in E[p]$ – and is equivalent to the relation on the dual generating series (Lemma 3.1(c)):

$$\sum_{\sigma \in \Sigma_{i,j}} \Theta_s\left(\beta_{\sigma^{-1}(1)} : \cdots : \beta_{\sigma^{-1}(k)} : 0 \mid u_{\sigma^{-1}(1)}, u_{\sigma^{-1}(2)}, \ldots, u_{\sigma^{-1}(i+j)}, -(u_1 + \cdots + u_{i+j})\right),$$

where $t_i = u_1 + \cdots + u_i$, $\alpha_i = \beta_i - \beta_{i-1}$. This is the first shuffle relation on the generating series $\Theta_s(\ :|,)$, which holds a priori.

The images of the dihedral symmetry relations are exactly the relations of Corollary 3.6, which hold modulo the correlators of elements that are independent of s.

Finally, the map θ intertwines the coproduct. The general case follows from the degree 1. Set $t_0 = 0$. By (3.10), we have

$$\delta\theta\left(\sum_{n_1, n_1', \ldots, n_k, n_k'} \left((\alpha_1, \ldots, \alpha_k) \otimes t_1^{n_1} \bar{t}_1^{n_1'} \ldots t_k^{n_k} \bar{t}_k^{n_k'}\right) \otimes [v_1, \ldots, v_k]\right) =$$

$$= \frac{1}{|E_{\mathfrak{p}}|} \sum_{s \in E[\mathfrak{p}]} \left(\sum_{\mathrm{cyc}} \sum_{i=0}^{k} \Theta_s^*\left(\alpha_1, \ldots, \alpha_i, -(\alpha_1 + \cdots + \alpha_i) \mid t_1 : \cdots : t_i : t_0\right) \wedge \right.$$

$$\left. \wedge\, \Theta_s^*\left(\alpha_{i+1}, \ldots, \alpha_k, -(\alpha_{i+1} + \cdots + \alpha_k) \mid t_{i+1} : \cdots : t_k : t_0\right) + \text{lower depth terms}\right), \tag{3.19}$$

where the lower-depth terms are correlators of elements independent of s, and the cyclic sum is over indices modulo $k+1$. On the other hand, we have

$$\delta\Bigg(\sum_{n_1,n_1',\dots,n_k,n_k'}\Big((\alpha_1,\dots,\alpha_k)\otimes t_1^{n_1}\bar{t}_1^{n_1'}\dots t_k^{n_k}\bar{t}_k^{n_k'}\Big)\otimes[v_1,\dots,v_k]\Bigg)$$

$$=\Bigg(\sum_{n_1,n_1',\dots,n_k,n_k'}\Big((\alpha_1,\dots,\alpha_k)\otimes t_1^{n_1}\bar{t}_1^{n_1'}\dots t_k^{n_k}\bar{t}_k^{n_k'}\Big)\otimes\sum_{\mathrm{cyc}}\sum_{i=0}^{k}-[v_1,\dots,v_i]\wedge[v_{i+1},\dots,v_k]\Bigg). \tag{3.20}$$

The cyclic shift in $\mathrm{GL}_k(O)$, which maps $v_0\mapsto v_1\mapsto v_2\mapsto\cdots\mapsto v_k\mapsto v_0$, acts by the transpose action on the t_i by $t_i\mapsto t_{i+1}-t_1$ (indices modulo $k+1$; recall $t_{k+1}=t_0=0$). Thus the image of (3.20) under θ agrees with (3.19) in each summand of the cyclic sum, except with an additive shift of the arguments. It remains to apply the homogeneity of the Θ_s^*. □

Remark

Why do we define the map θ using averaged base point correlators? It would have been possible to define the map to $\mathrm{Lie}^\vee_{\mathrm{Hod}}(E,E[\mathfrak{p}])$ using the correlators with fixed base point, Cor_s. However, this map would be zero. Indeed, any correlator with base point s vanishes modulo the depth filtration in $\mathrm{Lie}^\vee_{\mathrm{Hod}}(E,E[\mathfrak{p}])$ induced by Cor_s, since any correlator can be written modulo those of lower depth by the change of base point formula (3.11). Those correlators of lower depth depend on s, so this does not imply the image of $\mathrm{Cor}_{\mathrm{av}}$ is zero.

On the other hand, the map θ can be modified, replacing $E[\mathfrak{p}]$ by its subgroup of order p (if $|E[\mathfrak{p}]|=p^2$). We will use this when we specialize θ to the nodal projective line.

Biachi complexes and Hodge correlators in depth 2

Let $k=2$ and E one of the CM elliptic curves with endomorphism ring $O=\mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$. According to Lemma 3.11, there is an isomorphism $\psi: B^\bullet\to M_2^\bullet$ from the Bianchi complex to the modular complex. The relaxed modular complex $\widetilde{M}_2^\bullet$ is canonically isomorphic to the modular complex $M_2^\bullet$, since the second shuffle relations are equivalent to the dihedral symmetry relations. Thus we have a map

$$\theta\circ\psi: T_2\otimes_{\Gamma_1(\mathfrak{p})}B^\bullet\to\mathrm{CE}^\bullet\left(\mathrm{gr}^D\mathrm{Lie}^\vee_{\mathrm{sym}}(E,E[\mathfrak{p}])\right)_{D=2}.$$

The complex of the left side is the chain complex with coefficients in the local system T_2 on the orbifold $\Gamma_1(\mathfrak{p})\setminus\mathbb{H}^3$. We arrive at the following important result:

Theorem 3.13. *Let E be one of the CM elliptic curves $E = \mathbb{C}/\mathbb{Z}[i]$ or $E = \mathbb{C}/\mathbb{Z}[\rho]$. Then*

$$\theta \circ \psi : \mathrm{CE}^\bullet \left(\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{av}}(E, E[\mathfrak{p}])\right)_{D=2}$$

is a surjective morphism of complexes.

It is tempting to extend Theorem 3.12 to higher depth by showing the map θ descends to the modular complex $M_k^\bullet$. This requires showing the second shuffle relations for the averaged base point Hodge correlators modulo the depth filtration. The following would follow from Conjecture 3.9:

Conjecture 3.14. *The map θ descends to a morphism of complexes*

$$\theta : T_k \otimes_{\Gamma_1(\mathfrak{p})} M_k^\bullet \to \mathrm{CE}^\bullet \left(\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[\mathfrak{p}])\right)_{D=k}.$$

3.4 Applications

3.4.1 The weight 4 case: Euler complexes

Let us show how the map in Theorem 3.13 generalizes those constructed by [GL, G8]. To be consistent with those sources, we use the motivic language in this section, but the same results hold in the Hodge realization as well.

The elements θ_E

For torsion points $a, b, c \in E$ with $a + b + c = 0$, elements $\theta_E(a, b, c)$ are constructed by [G8] as follows.

For E an elliptic curve over a field k, $\mathfrak{p} \subset \mathrm{End}E$ a prime over p, and z a nonzero $\mathfrak{p}$-torsion point of E, there are elements $\theta_E(z)$, which are p-torsion elements in $\overline{k}_z^* \otimes \mathbb{Z}\left[\frac{1}{p}\right]$, where k_z is the extension generated by the coordinates of z. They are identified with weight-2 elements in the mixed Tate Lie coalgebra $\mathrm{Lie}^\vee_{\mathrm{MT}/\overline{k}}$. The real period of the motive $\theta_E(z)$ is $-\log|\theta_E(z)|$.

The elements $\theta_E(a : b : c)$ lie in the Bloch group of $\overline{k}$, which is identified with the weight-4 part of $\mathrm{Lie}^\vee_{\mathrm{MT}/\overline{k}}$. We also use the notation $\theta_E(a, b, c) = \theta_E(a : a + b : a + b + c)$, which is unambiguous when $a+b+c = 0$ because the $\theta_E(a : b : c)$ are invariant under translation. They are characterized by the following properties:

(1) The coproduct is given by

$$\delta\theta_E(a, b, c) = \theta_E(a) \wedge \theta_E(b) + \theta_E(b) \wedge \theta_E(c) + \theta_E(c) \wedge \theta_E(a).$$

(2) The real period of $\theta_E(a : b : c)$ is given up to a constant multiple by the averaged Chow dilogarithm ([G7]). The latter can be rewritten as

$$\frac{1}{p^5} \sum_{x \in E[p]} \int_{E(C)} \log|f_{a,x}| \, d^{\mathbb{C}} \log|f_{b,x}| \wedge d^{\mathbb{C}} \log|f_{c,x}|, \tag{3.21}$$

where $f_{a,b}$ is a function on E with $\operatorname{div} f_{a,b} = p(\{a\} - \{b\})$.

The elements θ_E and motivic correlators

According to [G9], §10.5.5, for $a, b \in E[p]$, the elements $\theta_E(a - b)$ are equal up to a constant multiple to $\mathrm{Cor}_{\mathrm{av}}(a, b)$. There is a version for the depth 2 elements.

Lemma 3.15. *Let E be an elliptic curve over a number field. Then, for $a, b, c \in E[\mathfrak{p}] \setminus \{0\}$ with $a + b + c = 0$, the elements $\theta_E(a : b : c)$ are equal up to a constant multiple to* $\mathrm{Cor}_{\mathrm{av}}(a, b, c)$.

Proof. The coproduct formulas for the θ_E and the $\mathrm{Cor}_{\mathrm{av}}$ coincide ([G9], Lemma 10.9). It remains to see the periods are equal. Indeed, we take $f_{a,x}$ such that $\log|f_{a,x}(z)| = pG_x(a, z)$, and likewise for b and c. Then the formula (3.21) is evidently a constant multiple of the Hodge correlator

$$\frac{1}{|E[\mathfrak{p}]|} \sum_{x \in E[\mathfrak{p}]} \mathrm{Cor}_{\mathcal{H},x}(a, b, c),$$

where $\mathrm{Cor}_{\mathcal{H},x}$ denotes the Hodge correlator function computed using the Green's function with base point x, as desired. □

The map constructed by [G8], for $\mathcal{O} = \mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$ is:

$$\begin{aligned} \theta' : \mathbb{Z}[\Gamma_1(\mathfrak{p}) \setminus \mathrm{GL}_2(\mathcal{O})] \otimes_{\Gamma_1(\mathfrak{p})} M_2^{\bullet} &\to \mathrm{Lie}^{\vee}_{\mathrm{Mot}}(E, E[\mathfrak{p}]), \\ (\alpha_1, \alpha_2) \otimes [v_1, v_2] &\mapsto \theta_E(\alpha_1, \alpha_2, -(\alpha_1 + \alpha_2)), \\ (\alpha_1, \alpha_2) \otimes ([v_1] \wedge [v_2]) &\mapsto \theta_E(\alpha_1) \wedge \theta_E(\alpha_2). \end{aligned}$$

Theorem 3.16. *The map θ' is a constant multiple of the component of $\theta \circ \psi$ corresponding to the constant term of the local system T_2.*

Proof. After unraveling the definitions, in degree 1, this is exactly Lemma 3.15, while in degree 2 it amounts to showing that

$$\mathrm{Cor}_{\mathrm{av}}(0, a) \wedge \mathrm{Cor}_{\mathrm{av}}(0, b) = \frac{1}{|E[\mathfrak{p}]|} \sum_{s \in E[\mathfrak{p}]} \mathrm{Cor}_s(0, a) \wedge \mathrm{Cor}_s(0, b).$$

Expanding the sums and using that $\mathrm{Cor}_s(x,y) \sim \theta_E(x-y) - \theta_E(x-s) - \theta_E(y-s)$ (where we set $\theta_E(0) = 0$), this simplifies to

$$\sum_s \theta_E(s) \wedge \theta_E(b-s) + \sum_s \theta_E(a-s) \wedge \theta_E(s) + \sum_s \theta_E(a-s) \wedge \theta_E(b-s) = 0.$$

The three sums are both symmetric and antisymmetric under the involutions $s \mapsto b-s$, $s \mapsto a-s$, and $s \mapsto a+b-s$, respectively, so the sum is 0. □

A slight abuse of notation has taken place: θ maps to $\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[\mathfrak{p}])$ and θ' to $\mathrm{Lie}^\vee_{\mathrm{MT}}(E, E[\mathfrak{p}])$. However, there is no discrepancy, as the second shuffle relation in weight 4 and depth 2 holds without the lower-depth correction terms, and so the constant term of θ can be viewed as a map to $\mathrm{Lie}^\vee_{\mathrm{sym}}(E, E[\mathfrak{p}])$, by the following lemma.

Lemma 3.17. *For E any elliptic curve and $a, b \in E[\mathfrak{p}]$,*

$$\mathrm{Cor}_{\mathrm{av}}(a, b, S_{0,0} * S_{0,0}) = 0,$$
$$\mathrm{Cor}_{\mathrm{av}}(a, S_{0,0} * S_{0,0} * S_{0,0}) = 0.$$

Proof. For the first equality, recall that

$$S_{0,0} * S_{0,0} = -\frac{1}{2}(\omega \otimes \overline{\omega} - \overline{\omega} \otimes \omega).$$

It is easily verified that the coproduct is 0. For the periods, there are two trees contributing to the integral expansion of the correlator. For the tree where $\omega, \overline{\omega}$ are not incident to a common interior vertex, the terms with $\omega \otimes \overline{\omega}$ and with $\overline{\omega} \otimes \omega$ sum to 0. The other tree contributes a constant multiple of

$$\begin{aligned}
&\sum_{s \in E[\mathfrak{p}]} \int_E G_s(z,w)\, d^{\mathbb{C}} G_s(w,a) \wedge d^{\mathbb{C}} G_s(w,b) \wedge \omega(z) \wedge \overline{\omega}(z) \\
&= \sum_{s \in E[\mathfrak{p}]} \int_E G_{\mathrm{Ar}}(s-w)\, d^{\mathbb{C}} G_s(w,a) \wedge d^{\mathbb{C}} G_s(w,b) \\
&= \sum_{s \in E[\mathfrak{p}]} \int_E G_{\mathrm{Ar}}(s-w)\, d^{\mathbb{C}}(G_{\mathrm{Ar}}(a-w) - G_{\mathrm{Ar}}(s-w)) \wedge d^{\mathbb{C}}(G_{\mathrm{Ar}}(b-w) - G_{\mathrm{Ar}}(s-w)) \qquad = 0.
\end{aligned}$$

This follows from the distribution relations for the function G_{Ar}, which state that

$$\sum_{s \in E[\mathfrak{p}]-0} G_{\mathrm{Ar}}(s) = 0.$$

The second equality follows simply from dihedral symmetry. □

3.4.2 Degeneration to rational curves: Voronoi complexes and multiple ζ-values

In this section we study the behavior of the motivic correlators at the boundary of the moduli space $\mathcal{M}'_{1,n}$ of elliptic curves with n marked points and a distinguished tangent vector. The results here are a new case of the specialization theorem for correlators on rational curves (§2.3), and the definitions and proof are analogous. (We envision a similar picture for other boundary strata and for higher-genus curves, which can be regarded as the higher-weight version of the results of Wentworth [W] about degeneration of Green's functions. We do not expand this subject here.)

Setup

It will be enough for us to consider the top boundary stratum in $\overline{\mathcal{M}}'_{1,n}$ in which the elliptic curve E degenerates to nodal $\mathbb{P}^1$. On an open subset of this stratum, all marked points remain distinct. Furthermore, we will consider degeneration along the direction $\tau = it$, $t \to \infty$ on the modular curve.

Consider an elliptic curve E over $\omega B \to \overline{\mathcal{M}}'_{1,n}$, with an open subset $B \to \mathcal{M}'_{1,n}$, whose complement $D = \overline{B} \setminus B$ is a normal crossings divisor. A Hodge correlator on E determines a variation of mixed Hodge structures over B, which has a canonical extension along every normal vector to D. We will describe this canonical extension in the aforementioned case.

The curve $E_\tau \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ has canonical coordinate z, and the nonsingular locus of nodal $\mathbb{P}^1$ has coordinate z (with the node at $z = 0, \infty$) such that

$$s_a = \begin{cases} z = a & \tau \neq 0 \\ z = e^{2\pi i a} & \tau = 0 \end{cases},$$

for $a \in \mathbb{C}$, is a smooth section over $t \in (0, \infty]$. Also fix the relative 1-forms $\omega = \frac{1}{\sqrt{\operatorname{Im}\tau}} dz, \overline{\omega} = \frac{1}{\sqrt{\operatorname{Im}\tau}} \overline{dz}$ on E, which have limit 0 on $\mathbb{P}^1$.

Let v_0 be a relative tangent vector at s_0 and $Z_S \subset \mathbb{C}^*$, $S = \{s_a : a \in Z_S\}$. Let $\mathcal{D} \subset \mathcal{CL}^\vee_{E/B,S,v_0}$ be the subcoalgebra generated by the sections s_a, where all the s_a factors are distinct, and the relative 1-forms $\omega, \overline{\omega}$. Also fix the tangent vector $v_0 = \frac{\partial}{\partial z}$ at $1 \in \mathbb{P}^1$.

Let us define a degeneration map

$$\pi_D : \mathcal{D} \to \mathcal{CL}^\vee_{\mathbb{P}^1, Z_S \cup \{0\}, v_0} \oplus \mathcal{CL}^\vee_{\mathbb{P}^1, Z_S \cup \{\infty\}, v_0}.$$

Let $\mathcal{D}_T$ be the subspace of $\mathcal{D}$ generated by the s_a and elements $\omega \otimes \overline{\omega}$ and $\overline{\omega} \otimes \omega$, which we call the elements of Tate type. If $x \in \mathcal{D}$ is not of Tate type, we set $\pi_D(x) = 0$. Otherwise, we set

$$\pi_D(s_a) = e^{2\pi i a},$$
$$\pi_D(\omega \otimes \overline{\omega}) = -\pi_D(\overline{\omega} \otimes \omega) = \frac{1}{2}\left((0) + (\infty)\right),$$

extended to preserve the tensor product.

(One can verify, using straightforward but cumbersome combinatorics, that this map is well-defined, i.e., respects the first shuffle relations. This is not required for the results below, since we only require the composition of π_D with the Hodge correlator map, a fortiori well-defined.)

Lemma 3.18. *The map π_D is a morphism of coalgebras.*

Proof. Each term in the coproduct of a generator not of Tate type clearly has a factor that is not of Tate type, because the coproduct preserves the weight. So it is enough to see the map respects the coproduct on the generators of Tate type, considering only the terms of the coproduct where both generators are of Tate type.

Let us do this for the first component of the map, to $\mathcal{CL}^\vee_{\mathbb{P}^1, Z_s \cup \{0\}, v_0}$; the other is analogous. Let x be a generator in $\mathbb{D}_T$. The coproduct of x has two types of terms:

(1) the cuts with vertex at some s_a (the term δ_S;

(2) the cuts that give the terms δ_{Cas}.

The coproduct of $\pi_D(x)$ has two types of terms:

(1′) the cuts with vertex at some $a \neq 0$;

(2′) the cuts with vertex at a 0.

The terms (1) that have both factors of Tate type are in obvious bijection with the (1’): observe that the segment that is cut must have the same number of ω and $\overline{\omega}$ factors on each side – see Figure 3.5, left. Similarly, the terms (2) that have both factors of Tate type are in bijection with the (2’) – see Figure 3.5, right. □

Suppose now that $\overline{B}$, D are as above and D maps to the boundary stratum in $\overline{\mathcal{M}}'_{1,n}$. Let $\mathrm{Spec}_\infty \mathrm{Cor}_{\mathrm{Hod}}(x)$ denote the canonical extension of the variation $\mathrm{Cor}_{\mathrm{Hod}}(x)$ on B to a normal vector to D. We then have the following result.

Theorem 3.19. *Suppose $x \in \mathcal{D}$ is of weight $n > 2$.*

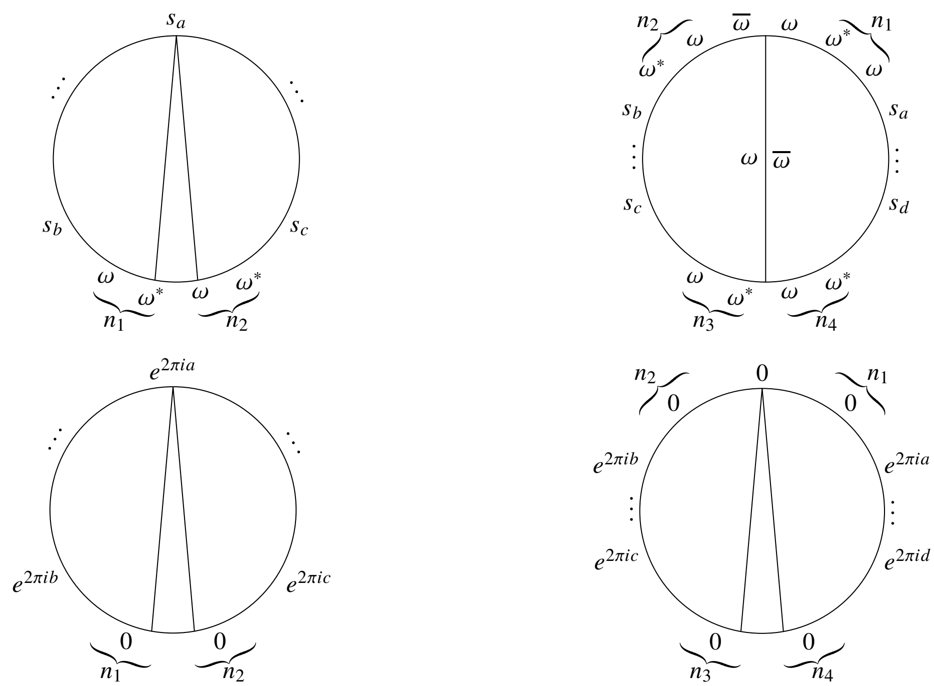


Figure 3.5. Bijection used in the proof of Lemma 3.18

(a) This specialization of the Hodge correlator $\mathrm{Cor}_{\mathrm{Hod}}(x)$ *coincides with the Hodge correlator of the degeneration:*

$$\begin{tikzcd}
\mathcal{D}_{w>2} \arrow[r, "\pi_D"] \arrow[d, "\mathrm{Cor}_{\mathrm{Hod}}"'] & \left(\mathcal{CL}^{\vee}_{\mathbb{P}^1, Z_S\cup\{0\}, v_0} \oplus \mathcal{CL}^{\vee}_{\mathbb{P}^1, Z_S\cup\{\infty\}, v_0}\right)_{w>2} \arrow[d, "\mathrm{Cor}_{\mathrm{Hod}}"] \\
\mathrm{Lie}^{\vee}_{\mathrm{Hod}/B} \arrow[r, "\mathrm{Spec}_D"] & \mathrm{Lie}^{\vee}_{\mathrm{Hod}}
\end{tikzcd}.$$

(b) The Hodge correlator functions on E specialize to the Hodge correlators on $\mathbb{P}^1$*. That is, if* $x \in \mathcal{D}$ *and* $\tau = it$*, then*

$$\lim_{t\to\infty} \mathrm{Cor}^{(E_\tau)}_{\mathcal{H}}(x) \sim \mathrm{Cor}^{\mathbb{P}^1}_{\mathcal{H}}(\pi_D(x)).$$

With an appropriate choice of tangent vector on E_τ*, this also holds in weight 2.*

Proof. We may let $\mathcal{M}$ be the moduli space of sets Z_S of n ordered points in $\mathbb{C}^*$ and $\overline{B} = (0, \infty] \times \mathcal{M}$. We then simultaneously show the following:

(1) The periods of $\mathrm{Cor}_{\mathrm{Hod}}(x)$ extend continuously to D.

(2) The periods of the specialization of $\mathrm{Cor}_{\mathrm{Hod}}(x)$ (i.e., the limits of the periods at D) coincide with the periods of the degenerations $d(x)$.

The proof is by induction on w. Let us see how these imply the result.

Because the coproduct commutes with specialization, the mixed Hodge structure $\mathrm{Spec}_D \mathrm{Cor}_{\mathrm{Hod}}(x) - \mathrm{Cor}_{\mathrm{Hod}}(\pi_D(x))$ lies in $\mathrm{Ext}^1_D(\mathbb{R}(0), \mathbb{R}(p,q))$, which is one-dimensional and controlled by the period. By (2) it coincides with the period of the degeneration, which immediately gives (1). This implies (a) and (b) in weight w.

To show (1), let $q = e^{2\pi i\tau} = e^{-2\pi t}$ be a parameter at the cusp. We show that for $x \in \mathcal{D}$, $\mathrm{Cor}_{\mathcal{H}}(x)$ can be represented as a polynomial in $\log q$, where the coefficient of $\log q$ appearing in positive degree has coefficients vanishing at $q = 0$ (*tame logarithmic singularities*). This is shown by induction: if x is of weight $w > 2$, then $d^{\mathbb{C}}\mathrm{Cor}_{\mathcal{H}}(x)$ is expressed in terms of periods of δx. The latter has logarithmic singularities, by the inductive hypothesis and the fact that the Hodge correlators in weight 1 have logarithmic singularities (see the lemma that follows). Therefore, $\mathrm{Cor}_{\mathcal{H}}(x)$ has tame logarithmic singularities.

By rigidity of Ext^1, we conclude that the difference $\lim_{t\to\infty} \mathrm{Cor}^{(E_\tau)}_{\mathcal{H}}(x) - \mathrm{Cor}^{(\mathbb{P}^1)}_{\mathcal{H}}(\pi_D(x))$ is independent of the point on D, that is, of the choice of Z_S. The following lemma implies (3). $\square$

This lemma comprises the analytic ingredients in the preceding proof:

Lemma 3.20. *(a) Let x be an element of $\mathcal{D}$ of weight 2. Then $\mathrm{Cor}_{\mathcal{H}}(x)$ has a logarithmic singularity at $q = 0$, and there are constants c, C such that $\lim_{t\to\infty} \mathrm{Cor}^{(E_\tau)}_{\mathcal{H}}(x) - \frac{C}{t} - c\mathrm{Cor}^{(\mathbb{P}^1)}_{\mathcal{H}}(\pi_D(x)) = 0$.*

(b) Let $x = x_0 \otimes \cdots \otimes x_n$, where each $x_i \in \{s_a, \omega, \overline{\omega}\}$, be a generator in $\mathcal{D}$ of weight $w > 2$. Suppose $\lim_{t\to\infty} \mathrm{Cor}^{(E_\tau)}_{\mathcal{H}}(x) - \mathrm{Cor}^{(\mathbb{P}^1)}_{\mathcal{H}}(\pi_D(x))$ is independent of the choice of Z_S. Then this difference is 0.

Proof. (a) There are three main cases to consider (the remaining ones are symmetric): $x = (s_a) \otimes (s_b)$, $x = (s_a) \otimes \omega \otimes \overline{\omega}$, $x = (s_a) \otimes \omega \otimes \omega$. The last of those is trivial. For the first two, we use the fact that there is a constant C such that

$$\lim_{t\to\infty} \left(G^{(E_{it})}_{\mathrm{Ar}}(a) - \frac{C}{t} \right) = \log \left| \left(1 - e^{2\pi i(a)}\right) \left(1 - e^{-2\pi i(a)}\right) \right|.$$

Therefore, for an appropriate choice of tangent vector v_t at $0 \in E_{it}$, we have

$$\begin{aligned} \lim_{t\to\infty} G^{E_{it}}_{v_t}(a,b) &= \log \left| \frac{\left(1 - e^{2\pi i(a-b)}\right)\left(1 - e^{-2\pi i(a-b)}\right)}{\left(1 - e^{2\pi ia}\right)\left(1 - e^{-2\pi ia}\right)\left(1 - e^{2\pi ib}\right)\left(1 - e^{-2\pi ib}\right)} \right| \\ &= 2\log \left| \frac{e^{2\pi ia} - e^{2\pi ib}}{\left(1 - e^{2\pi ia}\right)\left(1 - e^{2\pi ib}\right)} \right| \qquad = cG^{(\mathbb{P}^1)}_{v_0}(a,b), \end{aligned}$$

where $v_0 = \frac{\partial}{\partial z}$ is a tangent vector at $1 \in \mathbb{P}^1$. This completes the case $x = (s_a) \otimes (s_b)$.

For the case $x = (s_a) \otimes \omega \otimes \overline{\omega}$, notice that $\mathrm{Cor}_{\mathcal{H}}^{(E_{it})}(x) = -G_{\mathrm{Ar}}^{(E_{it})}(a)$, so

$$\lim_{t\to\infty} \mathrm{Cor}_{\mathcal{H}}^{(E_{it})}(x) = -\log\left|(1 - e^{2\pi i a})(1^{-2\pi i a})\right|.$$

On the other hand, we also have

$$\begin{aligned} G_{v_0}^{(\mathbb{P}^1)}(z, 0) &= \log\left|\frac{z}{1-z}\right|, \\ G_{v_0}^{(\mathbb{P}^1)}(z, \infty) &= \log\left|\frac{1}{1-z}\right|, \\ G_{v_0}^{(\mathbb{P}^1)}(z, 0) + G_{v_0}^{(\mathbb{P}^1)}(z, \infty) &= \log\left|\frac{z}{(1-z)^2}\right| \qquad = -\log\left|(1-z)\,(1-1/z)\right|. \end{aligned}$$

So we have shown that

$$\lim_{t\to\infty} \mathrm{Cor}_{\mathcal{H}}^{(E_{it})}(x) = \frac{c}{2}\left(G_{v_0}^{(\mathbb{P}^1)}(e^{2\pi i a}, 0) + G_{v_0}^{(\mathbb{P}^1)}(e^{2\pi i a}, \infty)\right) = c\mathrm{Cor}_{\mathcal{H}}^{(\mathbb{P}^1)}(\pi_D(x)).$$

(b) Let s_a be one of the factors in x (without loss of generality, $x_0 = s_a$). We will integrate over a on the segment $[0, 1]$. For arbitrary τ, we have

$$\int_{a=0}^{1} \mathrm{Cor}_{\mathcal{H}}^{(E_\tau)}(x)\, da = 0,$$

since $\int_{a=0}^{1} G^{(E_\tau)}(a, z)\, da = 0$ for all z by the properties of the Arakelov Green's function, and

$$\int_{a=0}^{1} \mathrm{Cor}_{\mathcal{H}}^{(\mathbb{P}^1)}(\pi_D(x))\, da = 0,$$

since $\int_{|z|=1} \mathrm{Cor}_{\mathcal{H}}^{(\mathbb{P}^1)}(z, b, c) = \mathcal{L}_2\left(\frac{z-b}{z-c}\right) = 0$ by the properties of the dilogarithm. Therefore,

$$\int_{a=0}^{1} \left(\lim_{t\to\infty} \mathrm{Cor}_{\mathcal{H}}^{(E_\tau)}(x) - \mathrm{Cor}_{\mathcal{H}}^{(\mathbb{P}^1)}(\pi_D(x))\right) da = 0.$$

The integrand is independent of a, so it is 0.

□

Shuffle relations in depth 2

Let $x_0, \ldots, x_k \in E$ and $m_0, \ldots, m_k \geq 0$. Define

$$C_{m_0,\ldots,m_k}(x_0, \ldots, x_k) := \underbrace{S_{0,0} * \cdots * S_{0,0}}_{m_0} \otimes (x_0) \otimes \cdots \otimes \underbrace{S_{0,0} * \cdots * S_{0,0}}_{m_k} \otimes (x_k).$$

There is a version of the corrected correlator for this element, where subsets of $\{x_0, \ldots, x_k\}$ are replaced by "$*$". We write it in depth 2:

$$\begin{aligned}\overline{C}_{m_0,m_1,m_2}(x_0, x_1, x_2) := C_{m_0,m_1,m_2}(x_0, x_1, x_2) - C_{m_0,m_1+m_2}(x_0, x_2) - C_{m_1,m_2+m_0}(x_1, x_0) - C_{m_2,m_0+m_1}(x_2, x_1)\\ + C_{m_0+m_1+m_2}(x_0) + C_{m_0+m_1+m_2}(x_1) + C_{m_0+m_1+m_2}(x_2).\end{aligned}$$

We have the following variant of Theorem 3.8:

Lemma 3.21. *For $m_0, m_1, m_2 \geq 0$ and $a, b, c \in E$ with $a + b + c = 0$,*

$$\mathrm{Cor}\left(\overline{C}_{m_0,m_1,m_2}(a, a + b, a + b + c) + \overline{C}_{m_0,m_2,m_1}(a, a + c, a + b + c)\right) = 0.$$

This is a different version of a second shuffle relation. The proof is identical to that of Theorem 3.8 (we may suppose $a = 0$).

Now suppose $a, a + b, a + c, a + b + c$ are distinct and take the correlator with base point 0 over a family with varying τ:

$$\mathrm{Cor}\left(\overline{C}_{m_0,m_1,m_2}(s_a, s_{a+b}, s_{a+b+c}) + \overline{C}_{m_0,m_2,m_1}(s_a, s_{a+c}, s_{a+b+c})\right) = 0. \tag{3.22}$$

An abuse of notation has taken place: the definition of $\overline{C}$ is assumed to use the relative 1-forms $\omega, \overline{\omega}$ as in the previous section.

The specialization of the correlator of $\overline{C}_{m_0,m_1,m_2}(s_a, s_{a+b}, s_{a+b+c})$ as $\tau \longrightarrow i\infty$ is easily seen to be

$$\begin{aligned}\mathrm{Cor}(\pi_D(\overline{C}_{m_0,m_1,m_2}(s_{x_0}, s_{x_1}, s_{x_2}))) = &\ \mathrm{Cor}^{(v_0)}_{m_0,m_1,m_2}(e^{2\pi i x_0}, e^{2\pi i x_1}, e^{2\pi i x_2}) \\ &- \frac{1}{2}\Big(\mathrm{Cor}^{(v_0)}_{m_0,m_1+m_2}(e^{2\pi i x_0}, e^{2\pi i x_2}) \\ &+ \mathrm{Cor}^{(v_0)}_{m_1,m_2+m_0}(e^{2\pi i x_1}, e^{2\pi i x_0}) \\ &+ \mathrm{Cor}^{(v_0)}_{m_2,m_0+m_1}(e^{2\pi i x_2}, e^{2\pi i x_1})\Big) \\ &+ \mathrm{Cor}^{(v_0)}(\text{terms with } \infty).\end{aligned}$$

(recall v_0 is the tangent vector at $1 \in \mathbb{P}^1$). In particular, by varying a and applying an automorphism of $\mathbb{P}^1$, we find that the specialization of the relation (3.22) holds for any choice of base point at $\mathbb{P}^1 \setminus \{0, \infty\}$. When it is specialized to ∞, the terms with ∞ in the specialized correlator vanish. We obtain the relation:

$$\begin{aligned}\mathrm{Cor}_{m_0,m_1,m_2}(e^{2\pi i a}, e^{2\pi i(a+b)}, e^{2\pi i(a+b+c)}) - \frac{1}{2}\Big(&\mathrm{Cor}_{m_0,m_1+m_2}(e^{2\pi i a}, e^{2\pi i(a+b+c)}) \\ &+ \mathrm{Cor}_{m_1,m_2+m_0}(e^{2\pi i(a+b)}, e^{2\pi i a}) \\ &+ \mathrm{Cor}_{m_2,m_0+m_1}(e^{2\pi i(a+b+c)}, e^{2\pi i(a+b)})\Big)\end{aligned}$$

$$\begin{aligned}+ \mathrm{Cor}_{m_0,m_1,m_2}(e^{2\pi i a}, e^{2\pi i(a+c)}, e^{2\pi i(a+b+c)}) - \frac{1}{2}\Big(&\mathrm{Cor}_{m_0,m_1+m_2}(e^{2\pi i a}, e^{2\pi i(a+b+c)}) \\ &+ \mathrm{Cor}_{m_1,m_2+m_0}(e^{2\pi i(a+c)}, e^{2\pi i a}) \\ &+ \mathrm{Cor}_{m_2,m_0+m_1}(e^{2\pi i(a+b+c)}, e^{2\pi i(a+c)})\Big)\end{aligned}$$

correlators now taken with base point at ∞. Finally, fix $1 = e^{2\pi i a}$ and let $\alpha = e^{2\pi i b}$, $\beta = e^{2\pi i c}$. Rescaling, we arrive at

$$\mathrm{Cor}_{m_0,m_1,m_2}(1, \alpha, \alpha\beta) + \mathrm{Cor}_{m_0,m_2,m_1}(1, \beta, \alpha\beta) - \mathrm{Cor}_{m_0,m_1+m_2}(1, \alpha\beta) - \mathrm{Cor}_{m_2+m_0,m_1}(1, \alpha) - \mathrm{Cor}_{m_1+m_0,m_2}(1, \beta).$$

This is precisely the relation (3.6) in depth 2.

Remark

Let $\mu_p \subset \mathbb{G}_m$ denote the p-th roots of unity. In [G10] (§2.7), a map from the modular complex for $\mathrm{GL}_2(\mathbb{Z})$ to the standard cochain complex of $\mathrm{gr}^D \mathrm{Lie}^\vee_{\mathrm{Hod}}(\mathbb{G}_m, \mathbb{G}_m - \mu_p)$ is defined using motivic correlators, by a formula similar to (3.18):

$$\gamma_2^\bullet : M_2^\bullet \otimes_{\Gamma_1(\mathfrak{p})} \mathbb{Q} \to \mathrm{CE}^\bullet\left(\mathrm{gr}^D \mathrm{Lie}_{\mathrm{Hod}}^\vee\right)_2 .$$

Alternatively, we can obtain such a map by specializing the map θ. For a generic elliptic curve $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, we have $\mathcal{O} = \mathbb{Z}$. We use the variant of the map θ (3.18) defined using an order-p subgroup of $E[\mathfrak{p}]$ (see the remark at the end of §3.3.3). For a family of elliptic curves E_τ degenerating to nodal $\mathbb{P}^1$ as $\tau \to +i\infty$, make a continuous choice of identification of an order-p subgroup of $E[\mathfrak{p}]$ with the set of points with real coordinates $\left\{0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}\right\}$. Thus we get a family of sections s_i ($i \in \mathbb{F}_\mathfrak{p}$) that specialize to the p-th roots of unity on $\mathbb{P}^1$. We recover the map $\gamma_2^\bullet$ by specializing (3.18).

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