6-1-2019

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June 2019

COWLES FOUNDATION DISCUSSION PAPER NO. 2180

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Abstract

We propose a multiple-prior model of preferences under ambiguity that provides a unified lens through which to understand different formalizations of ambiguity aversion, as well as context-dependent negative and positive ambiguity attitudes documented in experiments. This model, *Boolean expected utility (BEU)*, represents the belief the decision-maker uses to evaluate any uncertain prospect as the outcome of a game between two conflicting forces, Pessimism and Optimism. We prove, first, that BEU provides a novel representation of the class of invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004). Second, BEU accommodates rich patterns of ambiguity attitudes, which we characterize in terms of the relative power allocated to each force in the game.

1 Introduction

A central approach to modeling preferences under ambiguity is based on the idea that the decision-maker (DM) quantifies uncertainty with a set of relevant beliefs (i.e., probability measures) and may use a different belief to evaluate each uncertain prospect. A well-known limitation underlying many such multiple-prior models—notably Gilboa and Schmeidler's (1989) maxmin expected utility model and several of its generalizations—is a restrictive mechanism of belief selection, whereby the DM evaluates each prospect according to the worst possible relevant belief. Behaviorally, this restriction is reflected by Schmeidler's (1989) uncertainty aversion axiom, which captures a negative attitude to ambiguity through a strong form of preference for hedging. Subsequent work has questioned this formalization of ambiguity aversion and proposed several alternative definitions and measures.[1] The experimental

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[1] E.g., Epstein (1999); Ghirardato and Marinacci (2002); Baillon, L'Haridon, and Placido (2011); Dow and Werlang (1992); Baillon, Huang, Selim, and Wakker (2018).

literature documents yet more nuanced patterns of ambiguity attitudes, with the same subjects appearing ambiguity-averse in some decision problems but ambiguity-seeking in others, depending on contextual features of each problem (for a survey, see Trautmann and van de Kuilen, 2015).

In this note, we propose a multiple-prior model that provides a unified lens through which to understand different formalizations of ambiguity aversion, as well as the context-dependent negative and positive ambiguity attitudes documented in experiments. To capture a flexible mechanism of belief selection, our model adopts a "dual self" perspective on ambiguity, by representing the belief the DM uses to evaluate any given prospect as the outcome of a game between two conflicting forces or selves (henceforth, Pessimism and Optimism).[2]

The baseline version of our model is a parsimonious generalization of Gilboa and Schmeidler's (1989) maxmin expected utility representation. Under *Boolean expected utility (BEU)*,[3] there is a compact collection $\mathbb{P}$ of closed and convex sets of beliefs and an affine utility u such that the DM evaluates each act f according to

$$W_{\mathrm{BEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)]. \tag{1}$$

That is, the belief used to evaluate f is the outcome of a sequential zero-sum game: First, Optimism chooses a set of beliefs P from the collection $\mathbb{P}$ with the goal of maximizing the DM's expected utility to f; then Pessimism chooses a belief μ from P with the goal of minimizing expected utility. As we show (Remark 1), the specific structure of the selves' action sets in (1) and the fact that Optimism moves first is without loss of generality. Maxmin expected utility corresponds to the extreme special case where Optimism has no choice, while the opposite extreme case, maxmax expected utility, provides Pessimism with no choice. Other special cases include Choquet expected utility (Schmeidler, 1989) and α-maxmin.

Our first main result is that BEU represents the class of preferences over Anscombe-Aumann acts that satisfy all of Gilboa and Schmeidler's (1989) axioms except for uncertainty aversion (Theorem 1). Equivalently, the presence of ambiguity is captured solely by relaxing independence to certainty independence, without additionally restricting the DM's ambiguity attitude to be negative (or positive). Obtaining an easy-to-interpret representation for this class of preferences—which are known as *invariant biseparable*—has been considered an

[2]The idea that the DM consists of multiple strategic selves with conflicting motives is employed frequently in behavioral economics, for example to model risk preferences and intertemporal choices (e.g., Thaler and Shefrin, 1981; Fudenberg and Levine, 2006; Brocas and Carrillo, 2008).

[3]We borrow this terminology from set theory: A Boolean representation of a set X consists of a family $\{X_i\}_{i \in I}$ of sets X_i and a family $\{S_j\}_{j \in J}$ of subsets S_j of I such that $X = \bigcup_{j \in J} \bigcap_{i \in S_j} X_i$ (e.g., Ovchinnikov, 2001). By analogy, denoting the max and min operator by $\vee$ and $\wedge$, respectively, (1) can be written as $W(f) = \bigvee_{P \in \mathbb{P}} \bigwedge_{\mu \in P} \mathbb{E}_\mu[u(f)]$; that is, as a max-min Boolean polynomial in disjunctive normal form.

important question in the ambiguity literature. Section 4.2 contrasts BEU with existing representations due to Ghirardato, Maccheroni, and Marinacci (2004) (generalized α-maxmin) and Amarante (2009) (Choquet integration over beliefs).

Proposition 1 shows that any BEU preference $\succsim$ uniquely reveals a set of *relevant priors* $C = \bigcup_{P \in \mathbb{P}} P$, which represents the possible outcomes of the belief-selection game.[4] Moreover, C admits a behavioral characterization in terms of the extent to which $\succsim$ departs from independence, as it coincides with Ghirardato, Maccheroni, and Marinacci's (2004) unanimity representation of the largest independent subrelation of $\succsim$.

The second main contribution of our model is to provide a unified framework through which to represent and contrast a wide range of theoretically and experimentally appealing ambiguity attitudes. We begin by showing that the standard comparative notion of ambiguity aversion is represented by a natural preorder over BEU representations, which captures the relative *power allocated to the second mover* (Pessimism) in the belief-selection game (Proposition 2). While for a given set of relevant priors, maxmin and maxmax expected utility are maximal and minimal in this order, the result highlights how less extreme allocations of power across the selves can generate a rich hierarchy of intermediate ambiguity attitudes, which we proceed to characterize in Sections 3.2 and 3.3:

First, Theorem 2 shows that several different shades of ambiguity aversion—as captured by varying degrees of preference for hedging—are characterized by the extent of overlap of sets in $\mathbb{P}$. Specifically, Ghirardato and Marinacci's (2002) notion of *absolute ambiguity aversion* (i.e., being more ambiguity-averse than some subjective expected utility preference), which corresponds to a preference for *complete* hedges that fully eliminate uncertainty, is characterized by the intersection of all sets in $\mathbb{P}$ being nonempty. This requires that at least one prior is always available to Pessimism irrespective of Optimism's choice, and is strictly weaker than uncertainty aversion (i.e., a preference for all hedges), which requires that all relevant priors are always available to Pessimism. While absolute ambiguity aversion is inconsistent with experimental evidence that subjects are often ambiguity-averse for bets involving moderate odds but *ambiguity-seeking for small odds* (e.g., Dimmock, Kouwenberg, Mitchell, and Peijnenburg, 2015; Kocher, Lahno, and Trautmann, 2018), we introduce the even weaker notion of *k-ambiguity aversion* (for some $k = 2, 3, \ldots$) that can accommodate this evidence. This notion imposes a preference for complete hedges only among any k acts and is characterized by the requirement that the intersection of any k sets in $\mathbb{P}$ is nonempty.

Second, motivated by experimental findings on *source dependence*, whereby subjects' ambiguity attitudes may be negative or positive depending on their familiarity with the domain of payoff-relevant uncertainty (e.g., Heath and Tversky, 1991), we further relax k-ambiguity

[4]Uniqueness holds up to convex closure and elimination of redundant (never selected) beliefs.

aversion to a "local" analog, which characterizes the sign of an event-based ambiguity aversion index commonly used in experimental work. While BEU can accommodate source dependence by allowing the sign of this index to vary across events (Proposition 3), we show that this is not the case for the special case of α-maxmin that is widely used in applications.

As we discuss in Section 4.1, our model can be generalized to classes of preferences that further relax certainty independence. The resulting Boolean representations feature more general game payoffs and can capture additional experimental findings.

2 Boolean Expected Utility

2.1 Setup

Let Z be a finite set of prizes and let $\Delta(Z)$ denote the space of probability measures over Z. We refer to typical elements $p, q \in \Delta(Z)$ as lotteries. Let S be a finite set of states. An *(Anscombe-Aumann) act* is a mapping $f : S \to \Delta(Z)$. Let $\mathcal{F}$ be the space of all acts, with typical elements f, g, h. For any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, define the mixture $\alpha f + (1-\alpha)g \in \mathcal{F}$ to be the act that in each state $s \in S$ yields lottery $\alpha f(s) + (1-\alpha)g(s) \in \Delta(Z)$. Slightly abusing notation, we identify each lottery $p \in \Delta(Z)$ with the constant act that yields lottery p in each state $s \in S$.

Let $\Delta(S)$ denote the set of all probability measures over S, which we embed in $\mathbb{R}^S$ and endow with the Euclidean topology. We refer to typical elements $\mu, \nu \in \Delta(S)$ as beliefs. Given any act $f \in \mathcal{F}$ and map $u : \Delta(Z) \to \mathbb{R}$, let $u(f)$ denote the element of $\mathbb{R}^S$ given by $u(f)(s) = u(f(s))$ for all $s \in S$, and let $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$.

The DM's *preference* over $\mathcal{F}$ is given by a binary relation $\succsim$ on $\mathcal{F}$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$.

2.2 Representation

We now introduce our baseline model, Boolean expected utility. Let $2^{\Delta(S)}$ denote the set of all nonempty sets of beliefs, endowed with the Hausdorff topology. A ***belief-set collection*** is a nonempty compact collection $\mathbb{P} \subseteq 2^{\Delta(S)}$ of sets of beliefs such that each $P \in \mathbb{P}$ is closed and convex.

Definition 1. A ***Boolean expected utility (BEU)*** representation of preference $\succsim$ consists of a belief-set collection $\mathbb{P}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that

$$W_{\text{BEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)] \tag{2}$$

represents $\succsim$.[5]

Just as Gilboa and Schmeidler's (1989) maxmin expected utility model, BEU is a multiple-prior model of ambiguity: The DM has in mind a set of relevant beliefs $\bigcup_{P\in\mathbb{P}} P$, and might use a different belief to evaluate each act. But unlike maxmin expected utility, the belief μ used to evaluate any given act f is not necessarily worst-case among all relevant beliefs. Instead, μ is the outcome of a sequential zero-sum game between two conflicting forces or "selves:" First, self 1 ("Optimism") chooses an action $P \in \mathbb{P}$ with the goal of maximizing expected utility to act f; then self 2 ("Pessimism") chooses an action $\mu \in P$ with the goal of minimizing expected utility to f.

As Remark 1 below shows, both the specific form of action sets and the order of moves in (2) are without loss of generality. Note that maxmin expected utility corresponds to the extreme special case of BEU where Optimism's action set is trivial (i.e., $\mathbb{P} = \{P\}$ is a singleton), as in this case (2) reduces to $W(f) = \min_{\mu\in P} \mathbb{E}_\mu[u(f)]$. Likewise, maxmax expected utility, $W(f) = \max_{\mu\in P} \mathbb{E}_\mu[u(f)]$, corresponds to the opposite extreme where Pessimism's action set is always trivial (i.e., $\mathbb{P} = \{\{\mu\} : \mu \in P\}$ is a collection of singletons).

Our first main result is that BEU represents the class of preferences that satisfy all subjective expected utility axioms, except that independence is relaxed to certainty independence:

Axiom 1 (Weak Order)**.** $\succsim$ is complete and transitive.

Axiom 2 (Monotonicity)**.** If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom 3 (Nondegeneracy)**.** There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom 4 (Archimedean)**.** For all $f, g, h \in \mathcal{F}$ with $f \succ g \succ h$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h.$$

Axiom 5 (Certainty Independence)**.** For all $f, g \in \mathcal{F}$, $p \in \Delta(Z)$, and $\alpha \in (0, 1]$,

$$f \succsim g \iff \alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p.$$

Theorem 1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a BEU representation.*

Thus, like maxmin expected utility, BEU captures the possible *presence* of ambiguity by imposing independence only for mixtures with constant acts, i.e., mixtures that apply

[5] The functional (2) is well-defined since $\mathbb{P}$ is nonempty and compact.

equally to all states.[6] However, unlike maxmin expected utility, BEU does not additionally impose uncertainty aversion, which reflects a *negative attitude* toward ambiguity through a preference for hedging (see Axiom 6).

Theorem 1 shows that BEU provides a novel, easy-to-interpret representation of the class of preferences that Ghirardato, Maccheroni, and Marinacci (2004) (henceforth GMM) term *invariant biseparable*. In Section 4.2, we contrast BEU with existing representations due to GMM and Amarante (2009). In addition, Section 4.1 shows that natural generalizations of BEU represent classes of preferences that further relax certainty independence.

Our proof of Theorem 1 (Appendix B.1) first invokes the well-known fact that $\succsim$ satisfies Axioms 1–5 if and only if there exists a constant-linear and monotonic functional $I : [-1,1]^S \to \mathbb{R}$ and a nonconstant affine utility $u : \Delta(Z) \to [-1,1]$ such that $\succsim$ is represented by $I \circ u$. For sufficiency, consider the belief-set collection $\mathbb{P}^*$ given by

$$\mathbb{P}^* := \{P^*_\phi : \phi \in \mathbb{R}^S\} \ \text{ with } \ P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}, \tag{3}$$

where $\partial I(\underline{0}) \subseteq \Delta(S)$ denotes the Clarke differential of I at $\underline{0}$ (Clarke, 1990, see Appendix A.2). We prove that $\mathbb{P}^*$ yields a BEU representation of I, i.e., for all $\phi \in [-1,1]^S$, $I(\phi) = \max_{P^*_\psi \in \mathbb{P}^*} \min_{\mu \in P^*_\psi} \mathbb{E}_\mu[\phi]$. A key step is to show that I can be expressed as a Boolean representation of affine functionals (building on Ovchinnikov, 2001), where the slope of each functional is given by its Clarke differential.

Remark 1. General action sets. The specific form of action sets for Optimism and Pessimism in (2) is without loss of generality. Indeed, $\succsim$ admits a BEU representation with utility u if and only if there exist *arbitrary* action sets A_1, A_2 and a mapping $\mu : A_1 \times A_2 \to \Delta(S)$ from action profiles to beliefs such that

$$W(f) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} \mathbb{E}_{\mu(a_1,a_2)}[u(f)] \tag{4}$$

is well-defined and represents $\succsim$.[7]

Min-max form. While BEU takes the max-min form in which Optimism is the first mover, it is equivalent to consider representations of the min-max form. That is, $\succsim$ admits a BEU representation if and only if it can be represented by the functional $W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$ for some belief-set collection $\mathbb{Q}$. However, the collection $\mathbb{Q}$ need not

[6] See Ghirardato, Maccheroni, and Marinacci (2005), who argue why certainty independence is important for achieving a separation of tastes and beliefs.

[7] To see this, suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then (4) represents $\succsim$ with $A_1 := \mathbb{P}$, $A_2 := \prod_{P \in \mathbb{P}} P$, and $\mu(P, \sigma) := \sigma(P)$ for all $P \in A_1$, $\sigma \in A_2$. Conversely, suppose (4) represents $\succsim$ for some (A_1, A_2, μ, u). Then setting $\mathbb{P} := \{\overline{\text{co}}(\mu(a_1, A_2)) : a_1 \in A_1\}$ yields a BEU representation of $\succsim$.

coincide with $\mathbb{P}$ in general. See Appendix S.2 for more details.

Single-self interpretation. In addition to the dual-self interpretation above, BEU admits a single-self interpretation, whereby the DM optimally selects her own ambiguity preference from a feasible set.[8] Specifically, feasible ambiguity preferences take the maxmin expected utility form $\min_{\mu \in P} \mathbb{E}_\mu[u(f)]$ and depending on f, the DM optimally controls the parameter P, where $\mathbb{P}$ represents the constraints of the subjective optimization. ▲

2.3 Relevant Priors

A natural way to identify the DM's set of ***relevant priors*** under BEU is to consider the union $\bigcup_{P \in \mathbb{P}} P$ of all sets in the belief-set collection. This captures all possible outcomes of the belief-selection game between Optimism and Pessimism. To eliminate redundant beliefs that are never selected, we focus on the *smallest* closed, convex set of beliefs that can arise under any BEU representation. Proposition 1 shows that this set is uniquely identified:

Proposition 1. *If $\succsim$ satisfies Axioms 1–5, then there exists a unique closed, convex set $C \subseteq \Delta(S)$ such that*

$$C \subseteq \overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P \tag{5}$$

for all BEU representations $(\mathbb{P}, u)$ of $\succsim$, with equality for some $(\mathbb{P}, u)$.

We call a BEU representation ***tight*** if (5) holds with equality. To prove Proposition 1 (Appendix B.2), we show that for any BEU representation, $\overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P$ contains the Clarke differential $\partial I(\underline{0})$ at $\underline{0}$ of the functional I from the proof of Theorem 1. Since the representation $\mathbb{P}^*$ in (3) satisfies $\overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}^*} P = \partial I(\underline{0})$, this implies that the set of relevant priors C is precisely $\partial I(\underline{0})$ and that $\mathbb{P}^*$ is a tight representation.

An implication of this Clarke-differential characterization of C is that our definition of the DM's relevant priors as the possible outcomes of the belief-selection game is equivalent to the following behavioral definition due to GMM, which quantifies departures from independence. For any preference $\succsim$ satisfying Axioms 1–5, GMM define the ***unambiguous preference*** $\succsim^*$ as the largest independent subrelation of $\succsim$. Equivalently, $\succsim^*$ is defined by $f \succsim^* g$ if $\alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h$ holds for all $\alpha \in (0, 1]$ and $h \in \mathcal{F}$.

Note that $\succsim^*$ is incomplete whenever $\succsim$ violates independence. GMM show that $\succsim^*$ admits a unanimity representation à la Bewley (2002) and identify the unique closed, convex set of priors in the unanimity representation as the DM's relevant set of priors.[9] Since GMM show that the latter set again coincides with $\partial I(\underline{0})$, we obtain the following corollary:

[8] See Sarver (2018) for an analogous model in the context of risk preferences.

[9] Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) take an alternative approach by including $\succsim^*$ as part of the primitive. Ghirardato and Siniscalchi (2012) extend GMM's characterization of relevant

Corollary 1. *If $\succsim$ admits a BEU representation with utility u, then the set of relevant priors C is the unique closed, convex set such that*

$$f \succsim^* g \Longleftrightarrow \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in C. \tag{6}$$

Remark 2 (Uniqueness)**.** Our results in the remainder of this paper apply to *all* BEU representations of a given preference, and thus do not require unique identification of a particular representation.

Nevertheless, standard arguments imply that the utility u under BEU is unique up to positive affine transformation. Moreover, Supplementary Appendix S.1 shows that the belief-set collection $\mathbb{P}$ is unique up to "half-space closure," analogous to recent representations featuring collections of sets of utilities (e.g., Hara, Ok, and Riella, 2019). ▲

3 Ambiguity Attitude

In this section, we highlight that BEU provides a unified framework through which to represent and contrast different attitudes toward ambiguity.

3.1 Comparative Ambiguity Attitude

We begin by noting that while the set of relevant priors C captures all possible outcomes of the belief-selection game between Optimism and Pessimism, the same set C can correspond to many distinct BEU preferences, as the DM's ambiguity attitude depends on the selves' relative "power" to determine which belief from C is used to evaluate any given act. This is exemplified by maxmin expected utility ($\mathbb{P} = \{C\}$) and maxmax expected utility ($\mathbb{P} = \{\{\mu\} : \mu \in C\}$), which respectively allocate all power or no power to Pessimism and capture extreme negative and positive ambiguity attitudes.

To formalize this connection in general, we define a preorder $\sqsupseteq$ over belief-set collections by $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$ if

$$\text{for all } P_1 \in \mathbb{P}_1 \text{ there exists } P_2 \in \mathbb{P}_2 \text{ with } P_1 \supseteq P_2.$$

This captures a natural sense in which collection $\mathbb{P}_1$ ***allocates more power to the second mover*** than $\mathbb{P}_2$: Indeed, for any potential move P_1 of Optimism under $\mathbb{P}_1$, Optimism has a move $P_2 \subseteq P_1$ in $\mathbb{P}_2$ that restricts Pessimism's action set more. Thus, Pessimism's relative power to influence the DM's belief is weaker under $\mathbb{P}_2$ than under $\mathbb{P}_1$.

priors beyond the invariant biseparable class. See Klibanoff, Mukerji, and Seo (2014) for a discussion of the interpretation of C.

The following result shows that this order represents the standard comparative notion of ambiguity aversion (Ghirardato and Marinacci, 2002), whereby $\succsim_1$ is ***more ambiguity-averse*** than $\succsim_2$ if whenever $f \succsim_1 p$ for some $f \in \mathcal{F}$ and $p \in \Delta(Z)$, then $f \succsim_2 p$.

Proposition 2. *Suppose $\succsim_1$, $\succsim_2$ admit BEU representations. The following are equivalent:*

1. *$\succsim_1$ is more ambiguity-averse than $\succsim_2$.*
2. *$\succsim_1$ admits a BEU representation $(\mathbb{P}_1, u_1)$ such that every BEU representation $(\mathbb{P}_2, u_2)$ of $\succsim_2$ satisfies $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$ and $u_1 \approx u_2$.*

Note that Proposition 2 does not assume any relationship between the sets of relevant priors C_1 and C_2 associated with $\succsim_1$ and $\succsim_2$.[10] The proof exhibits a representation $\overline{\mathbb{P}}_i$ of $\succsim_i$ that allocates more power to the second mover than any other representation, and shows that $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if $\overline{\mathbb{P}}_1 \sqsupseteq \overline{\mathbb{P}}_2$.

While maxmin and maxmax expected utility represent the most and least ambiguity-averse BEU representations for a given set of relevant priors, the following two subsections proceed to characterize a hierarchy of intermediate ambiguity attitudes that BEU can accommodate.

3.2 Shades of Ambiguity Aversion

Existing decision-theoretic definitions of ambiguity aversion postulate a preference for hedging, or randomization, but vary in the degree to which they impose this attitude. The seminal axiom in this literature, Schmeidler's (1989) *uncertainty aversion*, postulates that the DM *always* takes up an opportunity to hedge between two equally valued prospects.

Axiom 6 (Uncertainty Aversion)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \succsim f$.

The second standard definition is the notion of absolute ambiguity aversion introduced by Ghirardato and Marinacci (2002), which relies on the comparative definition considered in the previous section. Analogous to the definition of absolute risk aversion as more risk-averse than a risk-neutral preference, we say that $\succsim$ is ***absolutely ambiguity-averse*** if it is more ambiguity-averse than some nondegenerate subjective expected utility preference.[11] The following axiom can be used to provide a behavioral characterization:

[10] This is in contrast with GMM's characterization of comparative ambiguity aversion, which assumes that $C_1 = C_2$ (Proposition 12 in GMM).

[11] See Epstein (1999) for another approach that takes as its benchmark probabilistic sophistication instead of subjective expected utility.

Axiom 7 (k-Ambiguity Aversion)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Longrightarrow p \succsim f_1.$$

Axiom 7 only imposes a preference for *complete* hedging between k equally valued prospects, that is, for hedges that eliminate subjective uncertainty entirely. We say that $\succsim$ satisfies ***∞-ambiguity aversion*** if it satisfies k-ambiguity aversion for all k. This corresponds to the notion of preference for sure diversification used by Chateauneuf and Tallon (2002) to characterize absolute ambiguity aversion under Choquet expected utility. We show that this characterization extends to BEU:[12]

Lemma 1. *Suppose $\succsim$ admits a BEU representation. Then $\succsim$ is absolutely ambiguity-averse if and only if $\succsim$ satisfies ∞-ambiguity aversion.*

Theorem 2 provides an intuitive representation of the degrees of ambiguity aversion captured above, by clarifying how under BEU, these notions allocate successively less power to Pessimism:

Theorem 2. *Suppose that $\succsim$ admits a BEU representation $(\mathbb{P}, u)$.*

1. *$\succsim$ satisfies uncertainty aversion if and only if $\bigcap_{P \in \mathbb{P}} P = C$.*
2. *$\succsim$ is absolutely ambiguity-averse if and only if $\bigcap_{P \in \mathbb{P}} P \neq \emptyset$.*
3. *$\succsim$ satisfies k-ambiguity aversion if and only if $\bigcap_{i=1,...,k} P \neq \emptyset$ for all $P_1, ..., P_k \in \mathbb{P}$.*

Uncertainty aversion corresponds to the maximal allocation of power to Pessimism, in the sense that *all* relevant priors $\mu \in C$ are available to Pessimism irrespective of Optimism's choice. The game thus boils down to Pessimism choosing a belief $\mu \in C$, yielding maxmin expected utility; indeed, note that if $(\mathbb{P}, u)$ is tight, then $\succsim$ satisfies uncertainty aversion iff $\mathbb{P} = \{C\}$.

Absolute ambiguity aversion allocates less power to Pessimism, requiring only that there is *some* prior $\mu \in \bigcap_{P \in \mathbb{P}} P$ that is always available to Pessimism regardless of Optimism's choice. Thus, the DM's valuation of any act f is bounded above by the expected utility

[12] Lemma 1 can also be obtained as a consequence of Grant and Polak (2013), who extend Chateauneuf and Tallon's (2002) characterization to the class of monotonic, continuous, translation-invariant, and unbounded representations. BEU representations satisfy the first three properties, and can be extended to an unbounded domain by positive homogeneity (i.e., scale invariance) of the functional I.

$\mathbb{E}_\mu[u(f)]$ of f under prior μ, which implies that $\succsim$ is more ambiguity-averse than the expected utility preference with belief μ and utility u.

Finally, while absolute ambiguity aversion requires the intersection of *all* sets in $\mathbb{P}$ to be non-empty, k-ambiguity aversion imposes this only for any k sets in $\mathbb{P}$. Thus, k-ambiguity aversion further decreases the power allocated to Pessimism, and more so the smaller k. Indeed, whenever k-ambiguity aversion holds at $\mathbb{P}_1$, then any representation $\mathbb{P}_2 \sqsupseteq \mathbb{P}_1$ displays a weakly higher degree of k-ambiguity aversion.

The relevance of further relaxing the DM's negative ambiguity attitude in this manner is underscored by experimental evidence. Indeed, one notable pattern suggesting that subjects' preferences might be better described by k-ambiguity aversion for small k than for large k is *ambiguity seeking for small odds*, which was originally conjectured by Ellsberg (e.g., footnote 4 in Becker and Brownson, 1964; Ellsberg, 2011) and subsequently confirmed in laboratory experiments:

Example 1 (Ellsberg urn with many colors)**.** Consider an urn with 10 balls with unknown composition from up to 10 different colors. A ball is drawn from the urn and its color observed. State space $S = \{1, \cdots, 10\}$ represents the observed color. For each event $E \subseteq S$, let f_E denote the uncertain bet that pays \$10 if the color of the ball belongs to E and \$0 otherwise, and let p_α denote the objective lottery that pays \$10 with probability α and \$0 otherwise.

When the cardinality of E is 5, this setting is similar to Ellsberg's two-color urn experiment, suggesting a preference for the objective lottery $p_{0.5}$ over the uncertain bet f_E, consistent with 2-ambiguity aversion. However, when E is a singleton event, many subjects prefer f_E to the corresponding objective lottery $p_{0.1}$ (e.g., Dimmock, Kouwenberg, Mitchell, and Peijnenburg, 2015; Kocher, Lahno, and Trautmann, 2018). Assuming that $f_{\{1\}} \sim \ldots \sim f_{\{10\}}$ by symmetry, this contradicts 10-ambiguity aversion as $p_{0.1} = \frac{1}{10} f_{\{1\}} + \cdots \frac{1}{10} f_{\{10\}}$. ▲

The following simple example illustrates that BEU allows for flexible degrees of k-ambiguity aversion and hence can accommodate the aforementioned experimental evidence. This is a notable difference with Siniscalchi's (2009) vector expected utility model, which also relaxes uncertainty aversion, but for which 2-ambiguity aversion and ∞-ambiguity aversion are equivalent.[13] The next section discusses the distinction with a special case of BEU, α-maxmin, that can also accommodate flexible k-ambiguity aversion.

Example 2. Consider a BEU representation $(\mathbb{P}, u)$ of the form $\mathbb{P} = \{P_s : s \in S\}$ where for

[13]Note that 2-ambiguity aversion is equivalent to Siniscalchi's (2009) Axiom 11, which he shows is equivalent to absolute ambiguity aversion (provided utilities are unbounded).

some fixed $\epsilon \geq 0$,

$$P_s := \{\mu \in \Delta(S) : \mu(s) \geq \epsilon\}$$

for each s. For each $k \leq |S|$, Theorem 2 implies that k-ambiguity aversion is satisfied if and only if $\epsilon \leq \frac{1}{k}$.[14] ▲

3.3 Ambiguity Aversion Index and Source Dependence

While the preceding notions of ambiguity aversion are "global," capturing the DM's attitude towards *any* uncertainty that can be generated in S, the experimental literature commonly takes a "local" approach, measuring the DM's ambiguity attitude relative to specific events or sources of uncertainty.

A primary local measure of ambiguity attitudes is based on the following idea originally proposed by Schmeidler (1989) and subsequently employed in both theoretical work (Dow and Werlang, 1992) and in experiments (Baillon and Bleichrodt, 2015; Baillon, Huang, Selim, and Wakker, 2018). Given any event $E \subseteq S$, we first define its ***matching probability*** $m(E) \in [0, 1]$ by the indifference condition

$$x_E y \sim m(E)\delta_x + (1 - m(E))\delta_y,$$

where $x, y \in Z$ are two outcomes such that $\delta_x \succ \delta_y$ and $x_E y$ denotes the binary act that yields x for all $s \in E$ and y otherwise.[15] Based on this, define the ***ambiguity aversion index*** associated with E by

$$AA(E) := 1 - m(E) - m(E^c). \tag{7}$$

Whereas subjective expected utility implies $AA(E) = 0$ for all E, $AA(E) > 0$ (resp. $AA(E) < 0$) is interpreted as a negative (resp. positive) attitude to ambiguity associated with E. Note that this index can be defined for any event, without imposing symmetry on the state space as is common in urn experiments.

Under BEU, the sign of $AA(E)$ is characterized by the following local analog of the binary intersection condition for 2-ambiguity aversion in Theorem 2:

[14] To see this, suppose $\epsilon \leq \frac{1}{k}$. Then for any distinct $s_1, ..., s_k$, we have $\frac{1}{k}\delta_{s_1} + \cdots \frac{1}{k}\delta_{s_k} \in \cap_{i=1}^k P_{s_i}$, so that k-ambiguity aversion holds. Conversely, if $\epsilon > \frac{1}{k}$, take any distinct $s_1, ..., s_k$. If $\mu \in \bigcap_{i=1}^k P_{s_i}$, then $\mu(s_i) > \frac{1}{k}$ for all $i = 1, ..., k$, contradicting $\mu \in \Delta(S)$. Thus, $\bigcap_{i=1}^k P_{s_i} = \emptyset$, so that k-ambiguity aversion fails.

[15] Under Axioms 1–5, $m(\cdot)$ is well-defined independent of the choice of x, y.

Lemma 2. *Suppose $\succsim$ admits a BEU representation $(\mathbb{P}, u)$. Then for any $E \subseteq S$,*

$$AA(E) \geq 0 \iff \{\mu(E) : \mu \in P\} \cap \{\mu'(E) : \mu' \in P'\} \neq \emptyset, \quad \textit{for all } P, P' \in \mathbb{P}.$$

As a result, 2-ambiguity aversion implies $AA(E) \geq 0$ for *all* events E. As such, 2-ambiguity aversion may still be too restrictive to accommodate the well-documented phenomenon of *source dependence* (e.g., Heath and Tversky, 1991): While subjects are found to display negative ambiguity attitudes for "unfamiliar" events (i.e., when they feel less competent about the relevant domain of uncertainty), attitudes are less negative, or even positive, for familiar events. The following example illustrates this in the context of home bias (French and Poterba, 1991):

Example 3 (Home bias)**.** Let $S_H = \{U, D\}$ be a state space specifying whether the domestic stock market goes up ("U") or down ("D"). Similarly, let $S_F = \{U, D\}$ describe the state of the stock market in a foreign country. Consider the product state space $S = S_H \times S_F$, and let $E_H = \{UU, UD\}$ be the event that the domestic stock market goes up, and $E_F = \{UU, DU\}$ be the corresponding event for the foreign stock market. Due to source dependence, typical subjects are more ambiguity-averse for foreign country stock than home country stock. Indeed, some subjects even reverse the sign, $AA(E_F) > 0 > AA(E_H)$, i.e., are ambiguity-seeking for E_H but ambiguity-averse for E_F (e.g. Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg, 2019).[16] ▲

The following result shows that BEU can accommodate the home bias in Example 3; indeed, it can capture source-dependent ambiguity attitudes with respect to any families $\mathcal{E}$ and $\mathcal{F}$ of unfamiliar and familiar events:

Proposition 3. *Fix any disjoint collections $\mathcal{E}$ and $\mathcal{F}$ of events, both of which are closed under complements and do not contain S. There exists a preference $\succsim$ satisfying Axioms 1–5 such that $AA(E) > 0 > AA(F)$ for all $E \in \mathcal{E}, F \in \mathcal{F}$.*

Proposition 3 highlights an important distinction with a special case of BEU, α-maxmin expected utility (α-MEU), which represents preferences by the functional

$$W_{\alpha\text{-MEU}}(f) = \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\nu \in P} \mathbb{E}_\nu[u(f)] \tag{8}$$

[16] See Figure 5 in Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019), where H and F correspond to local stock market index and foreign stock index, respectively. (As a caveat, we also note that the authors mention that the *population average* of AA does not vary much across different sources.) A related finding is Keppe and Weber (1995), in which the average AA of German subjects is positive for bets concerning US geography but negative for bets concerning German geography.

for some nonempty closed, convex set of beliefs P, $\alpha \in [0,1]$, and nonconstant affine u.

Due to its tractability, the α-MEU model is often used in applied theoretical work or for analyzing experimental data.[17] However, while α-MEU can accommodate flexible degrees of k-ambiguity aversion (based on the same idea as Example 2), Lemma 2 implies that this model is inconsistent with *any* form of source dependence. Indeed, the sign of the ambiguity index is the same for all events and is determined by the value of α:

Corollary 2. *Suppose $\succsim$ admits an α-MEU representation where P is not a singleton. Then $\alpha \geq 1/2$ (resp. $\alpha \leq 1/2$) if and only if $AA(E) \geq 0$ (resp. $AA(E) \leq 0$) for all E.*

More strongly, one can show that $\alpha \geq \frac{1}{2}$ implies 2-ambiguity aversion while $\alpha \leq \frac{1}{2}$ implies 2-ambiguity seeking (as defined in Appendix S.2).

At the same time, it is worth highlighting another special case of BEU that retains much of the tractability of α-MEU, but is enough to accommodate many forms of source dependence, including Example 3. Specifically, consider a simple generalization of α-MEU that allows for different sets of beliefs P_1 and P_2 for the max and min operator, i.e.,

$$W(f) = \alpha \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\nu \in P_2} \mathbb{E}_\nu[u(f)]. \tag{9}$$

To capture Example 3, we can set $P_1 := \{\mu : \mu(E_H) = \frac{1}{2}\}$ and $P_2 := \{\mu : \mu(E_F) = \frac{1}{2}\}$ in (9). This implies $AA(E_H) = \alpha - 1 < 0$ and $AA(E_F) = \alpha > 0$, thereby generating negative ambiguity attitudes for foreign events and positive attitudes for home events.

Remark 3. While in practice index (7) is typically defined using matching probabilities on binary partitions $\{E, E^c\}$, it can be generalized to arbitrary partitions $\mathcal{E}$ of S by setting

$$AA(\mathcal{E}) = 1 - \sum_{E \in \mathcal{E}} m(E).$$

Given this, k-ambiguity aversion implies that $AA(\mathcal{E}) \geq 0$ for all $\mathcal{E}$ with $|\mathcal{E}| \leq k$.[18] Thus, the aforementioned evidence on ambiguity seeking for small odds suggests the need to allow the sign of $AA(\mathcal{E})$ to depend on the number of events in partition $\mathcal{E}$. Example 2 can accommodate this, as the index satisfies $AA(\mathcal{E}) = 1 - \epsilon|\mathcal{E}|$ for any non-trivial partition $\mathcal{E}$. ▲

[17] See, e.g., Cherbonnier and Gollier (2015); Chen, Katuščák, and Ozdenoren (2007); Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010), and Ahn, Choi, Gale, and Kariv (2014).

[18] The proof follows from Lemma C.2 in the appendix.

4 Discussion

4.1 Generalizations

As we have seen, our baseline model, BEU, corresponds to a relaxation of subjective expected utility where independence is weakened to certainty independence and, equivalently, to dropping uncertainty aversion from Gilboa and Schmeidler's (1989) axioms. The representation adds a maximization stage to Gilboa and Schmeidler (1989), admitting an interpretation in terms of a game between Optimism and Pessimism.

We highlight that this approach generalizes beyond certainty independence, yielding intuitive representations that further relax independence but still do not impose uncertainty aversion. To illustrate, Appendix S.3 shows that replacing certainty independence with weak certainty independence (Maccheroni, Marinacci, and Rustichini, 2006) yields a representation of the form

$$W(f) = \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu), \tag{10}$$

where Optimism first chooses a cost function $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ from a collection $\mathbb{C}$ and Pessimism then chooses a belief subject to this cost. This adds a maximization stage into Maccheroni, Marinacci, and Rustichini's (2006) variational model, which corresponds to the special case that additionally satisfies uncertainty aversion. An even weaker form of independence, which applies only to objective lotteries, leads to a representation with general game payoffs, extending Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) uncertainty-averse representation (see Appendix S.4).

Further relaxing independence in this manner is motivated by additional experimental evidence. For instance, representation (10), which relaxes the scale invariance property implied by certainty independence while preserving translation invariance, can accommodate Machina's (2009) paradoxes (see also Baillon, L'Haridon, and Placido, 2011).[19] Another important finding is that ambiguity attitudes can differ for gains and losses, e.g., in urn experiments subjects who are ambiguity-averse for bets with positive payoffs are often ambiguity-seeking when the sign of the bet is reversed (Trautmann and Wakker, 2018). The latter finding is inconsistent with any representation that displays translation invariance, but can be accommodated by our most general model in Appendix S.4.

[19] This follows from the fact that Siniscalchi's (2009) vector expected utility model can accommodate these paradoxes and is a special case of (10).

4.2 Related Literature

Our paper builds on the approach of modeling preferences under ambiguity through a set of priors, from which the DM may select a different belief depending on each act. Many important multiple-prior models impose uncertainty aversion (e.g., Gilboa and Schmeidler, 1989; Maccheroni, Marinacci, and Rustichini, 2006; Chateauneuf and Faro, 2009; Strzalecki, 2011; Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011; Skiadas, 2013), which captures a worst-case mechanism of belief selection and is the central axiom that we relax in this paper.

An important exception are GMM, who propose the first representation of invariant biseparable preferences (i.e., preferences satisfying Axioms 1–5).[20] Their representation takes an act-dependent α-maxmin form,

$$W_{\text{GMM}}(f) = \alpha(f) \min_{\mu \in C} \mathbb{E}_\mu[u(f)] + (1 - \alpha(f)) \max_{\mu \in C} \mathbb{E}_\mu[u(f)], \tag{11}$$

where C is the set of priors in the Bewley representation (6) of the unambiguous preference $\succsim^*$ and $\alpha(\cdot)$ is a function from acts to $[0,1]$ that must satisfy several restrictions to ensure necessity of the axioms: Specifically, $\alpha(\cdot)$ must be measurable with respect to a particular derived equivalence relation $\asymp$ over acts and $\alpha(\cdot)$ must be such that the preference represented by (11) is monotonic (see Remark 2 in GMM).[21]

We highlight two key differences between BEU and GMM: First, while (11) imposes act dependence exogenously on ingredients of the representation, the act-dependent belief selection under BEU can be interpreted endogenously, as the outcome of a game between Optimism and Pessimism; moreover, under BEU, necessity of the axioms requires no additional restrictions on the belief-set collection $\mathbb{P}$. Second, our results in Sections 3.2–3.3, which have no counterpart in GMM, highlight the value of our game-theoretic interpretation, by representing a rich hierarchy of ambiguity attitudes in terms of the relative power of each self. At the same time, Corollary 1 shows that a common feature of BEU and GMM's representation is that the set of relevant priors C admits a behavioral characterization that captures the extent of departure from independence.

Beyond models based on act-dependent belief selection, there are several complementary approaches to modeling ambiguity (for a recent survey, see Gilboa and Marinacci, 2016). Im-

[20] Siniscalchi (2006) axiomatizes a special case of invariant biseparable preferences that have a piecewise subjective expected utility form.

[21] GMM also characterize the special case of (11) where $\alpha(\cdot)$ is constant, i.e., the subclass of α-maxmin representations (8) whose set of priors P coincides with the induced Bewley set C. Eichberger, Grant, Kelsey, and Koshevoy (2011) show that if the state space is finite, this representation reduces to maxmin or maxmax. Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) generalize (11) by relaxing certainty independence to risk independence, which entails weaker restrictions on $\alpha(\cdot)$.

portant approaches that likewise do not impose uncertainty aversion include (i) non-additive probabilities, as in Schmeidler's (1989) Choquet expected utility model and subsequent work (e.g., Chateauneuf, Eichberger, and Grant, 2007); (ii) second-order beliefs over sets of priors, as in Klibanoff, Marinacci, and Mukerji's (2005) smooth model (see also Segal, 1987; Seo, 2009); and (iii) preferences over utility dispersion (e.g., Siniscalchi, 2009; Grant and Polak, 2013). Amarante (2009) marries (i) and (ii), by providing an alternative representation of invariant biseparable preferences via the functional

$$W_{\text{Amarante}}(f) = \int_{\Delta(S)} \mathbb{E}_\mu[u(f)]\, d\nu(\mu),$$

which captures a DM who holds first-order beliefs $\mu \in \Delta(S)$ that are probability measures, but faces second-order uncertainty over first-order beliefs that takes the form of a Choquet capacity ν.

One important difference with the aforementioned papers is our focus on characterizing rich patterns of intermediate ambiguity attitudes, which is motivated in part by experimental evidence. While some of these papers provide representations of absolute ambiguity aversion, none use their models to characterize weaker degrees of ambiguity aversion.[22] We also note that all aforementioned models are special cases of either BEU or its generalizations in Section 4.1, suggesting that they could potentially be interpreted in terms of specific games between Optimism and Pessimism.

Related to the structure of BEU, several recent papers employ belief-set or utility-set collections in other contexts. While we maintain the weak order axiom and focus on relaxing independence, Lehrer and Teper (2011) and Nascimento and Riella (2011) (resp. Hara, Ok, and Riella, 2019) represent preferences over acts (resp. lotteries) that violate completeness and/or transitivity. Whereas BEU is a utility representation, these papers provide generalized unanimity representations à la Bewley (2002) and Dubra, Maccheroni, and Ok (2004), and the resulting proof methods are quite different. In the context of attitudes to randomization under ambiguity, Ke and Zhang (2019) consider preferences over lotteries over acts and propose a representation that adds *minimization* over belief-set collections to maxmin expected utility. When restricted to acts (i.e., degenerate lotteries), their representation is equivalent to Gilboa and Schmeidler (1989).

Beyond the ambiguity literature, BEU is related to Hart, Modica, and Schmeidler (1994), who provide a preference foundation for maxmin values in zero-sum games. They consider a

[22] In Klibanoff, Marinacci, and Mukerji's (2005) smooth model, absolute ambiguity aversion is equivalent to concavity of the function ϕ that aggregates expected utilities across different priors. Absolute ambiguity aversion is also equivalent to non-emptiness of the capacity's core under Choquet expected utility (Schmeidler, 1989) and to non-positivity of the adjustment function under vector expected utility (Siniscalchi, 2009).

product state space $S = S_1 \times S_2$, where S_1 and S_2 are interpreted as the DM's and opponent's action sets. They characterize when preferences over acts can be represented as the maxmin value of a simultaneous-move zero-sum game,

$$W(f) = \max_{\mu_1 \in \Delta(S_1)} \min_{\mu_2 \in \Delta(S_2)} \sum_{s_1, s_2} \mu_1(s_1)\mu_2(s_2)u(f(s_1, s_2)),$$

which is formally a strict special case of BEU.

Appendix: Proofs

A Preliminaries

Throughout this section, we fix any interval[23] $\Gamma \subseteq \mathbb{R}$ and let $U := \Gamma^S$. For any $a \in \mathbb{R}$, let $\underline{a}$ denote the vector in $\mathbb{R}^S$ with $\underline{a}(s) = a$ for all $s \in S$. For any $\phi, \psi \in \mathbb{R}^S$, write $\phi \geq \psi$ if $\phi(s) \geq \psi(s)$ for all s.

A.1 Properties of functionals

Fix any functional $I : U \to \mathbb{R}$. We call I:

- *monotonic* if $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in U$ with $\phi \geq \psi$;
- *normalized* if $I(\underline{a}) = a$ for all $a \in \Gamma$;
- *constant-additive* if $I(\phi + \underline{a}) = I(\phi) + a$ for all $\phi \in U$ and $a \in \Gamma$ with $\phi + \underline{a} \in U$;
- a *niveloid* if $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$ for all $\phi, \psi \in U$;
- *positively homogeneous* if $I(a\phi) = aI(\phi)$ for all $\phi \in U$ and $a \in \mathbb{R}_+$ with $a\phi \in U$;
- *constant-linear* if I is constant-additive and positively homogeneous.

It is easy to see that if $\underline{0} \in \Gamma$, then any constant-linear functional I is normalized. Moreover, I is a niveloid if and only if it is monotonic and constant-additive (Lemma 25 in Maccheroni, Marinacci, and Rustichini, 2006).

[23] That is, $\Gamma \subseteq \mathbb{R}$ is one of $[a, b], [a, b), (a, b], or (a, b)$, where we allow for $a = -\infty$ and $b = \infty$.

A.2 Clarke derivative and differential

Consider a locally Lipschitz[24] functional $I : U \to \mathbb{R}$. For every $\phi \in \text{int}U$ and $\xi \in \mathbb{R}^S$, the *Clarke (upper) derivative* of I in ϕ in the direction of ξ is

$$I^\circ(\phi;\xi) := \limsup_{\psi \to \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The *Clarke (sub)differential* of I at ϕ is the set

$$\partial I(\phi) := \{\chi \in \mathbb{R}^S : \chi \cdot \xi \leq I^\circ(\phi;\xi), \forall \xi \in \mathbb{R}^S\}.$$

We will frequently invoke the following properties of the Clarke differential. First, if I is locally Lipschitz continuous, then Rademacher's theorem yields a subset $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Combining this with Theorem 2.5.1 in Clarke (1990), we obtain the following approximation of the Clarke differential:

Lemma A.1 (Theorem 2.5.1 in Clarke (1990))**.** *Suppose $I : U \to \mathbb{R}$ is locally Lipschitz continuous. Then there exists $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero, I is differentiable at each $\psi \in \hat{U}$, and for every $\phi \in \text{int}U$, we have*

$$\partial I(\phi) = \text{co}\{\lim_n \nabla I(\phi_n) : \phi_n \to \phi, \phi_n \in \hat{U}\}. \tag{12}$$

The next result is an "envelope theorem" for Clarke differentials:

Lemma A.2 (Theorem 2.8.6 in Clarke (1990))**.** *Suppose functional $I : U \to \mathbb{R}$ is given by*

$$I(\cdot) = \sup_{t \in T} I_t(\cdot)$$

for some indexed family of functionals $(I_t)_{t \in T}$ with domain U. Assume that there exists some $K > 0$ such that $|I_t(\psi) - I_t(\xi)| \leq K\|\psi - \xi\|$ for every $t \in T$ and $\psi, \xi \in \text{int}U$. Then for every $\phi \in \text{int}U$,

$$\partial I(\phi) \subseteq \text{co}\{\lim_{i \to \infty} \nabla I_{t_i}(\phi_i) : \phi_i \to \phi, t_i \in T, I_{t_i}(\phi) \to I(\phi)\}.$$

Finally, we have the following relationship between properties of I and its Clarke differential:

Lemma A.3 (Part 1 of Proposition A.3 in GMM)**.** *If $I : U \to \mathbb{R}$ is locally Lipschitz, positively homogeneous, and $\underline{0} \in \text{int}U$, then $\partial I(\phi) \subseteq \partial I(\underline{0})$ for all $\phi \in \text{int}U$.*

[24]Slightly abusing terminology, we say I is locally Lipschitz on U if it is locally Lipschitz on $\text{int}U$.

Lemma A.4 (Parts 2–3 of Proposition A.3 in GMM)**.** *If $I : U \to \mathbb{R}$ is locally Lipschitz, monotonic, and constant-additive, then $\partial I(\phi) \subseteq \Delta(S)$ for all $\phi \in \text{int}U$.*

A.3 Boolean representation of I

Throughout this subsection, we assume that $I : U \to \mathbb{R}$ is monotonic, normalized, and locally Lipschitz continuous. Let $\hat{U}$ be the generic subset given by Lemma A.1.

Lemma A.6 below shows that, restricted to $\hat{U}$, we can express I as a Boolean representation of affine functionals, where the slope of each functional is given by the gradient of I. The proofs build on ideas in Ovchinnikov (2001).

We begin with a preliminary result:

Lemma A.5. *For every $\phi, \psi \in \hat{U}$ and $\epsilon > 0$, there exists $\xi \in \hat{U}$ such that*

$$I(\xi) - I(\psi) + \nabla I(\xi) \cdot (\psi - \xi) \geq 0, \quad I(\xi) - I(\phi) + \nabla I(\xi) \cdot (\phi - \xi) \leq \epsilon.$$

Proof. Take any $\phi, \psi \in \hat{U}$ and $\epsilon > 0$. Let $m := I(\psi) - I(\phi)$. If $\nabla I(\phi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \phi$. Likewise if $\nabla I(\psi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \psi$. It remains to consider the case

$$\nabla I(\phi) \cdot (\psi - \phi), \nabla I(\psi) \cdot (\psi - \phi) < m. \tag{13}$$

Define

$$H(\lambda) := I(\phi + \lambda(\psi - \phi)) - \lambda m - I(\phi)$$

for each $\lambda \in \mathbb{R}$ with $\phi + \lambda(\psi - \phi) \in U$. Since $\phi, \psi \in \hat{U}$, H is differentiable at $\lambda \in \{0, 1\}$, with $H(0) = H(1) = 0$ and $H'(0), H'(1) < 0$ by assumption (13). Hence, H is negative for small enough $\lambda > 0$ and positive for $\lambda < 1$ close enough to 1. Thus, the set $\{\lambda \in (0, 1) : H(\lambda) = 0\}$ is non-empty and closed; let λ^* denote its supremum.

Since H is locally Lipschitz continuous, we have $H(\lambda) = \int_{\lambda^*}^{\lambda} H'(\lambda')d\lambda'$ for all λ. As $H(\lambda) > 0$ for all $\lambda \in (\lambda^*, 1)$, we can choose $\lambda^{**} \in (\lambda^*, 1)$ at which H is differentiable with $H'(\lambda^{**}) > 0$ and $H(\lambda^{**}) \in (0, \epsilon)$. But then

$$H'(\lambda^{**}) = \lim_{t \to 0} \frac{I(\phi + (\lambda^{**} + t)(\psi - \phi)) - I(\phi + \lambda^{**}(\psi - \phi))}{t} - m > 0,$$

which implies that

$$I^\circ(\phi + \lambda^{**}(\psi - \phi); \psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

Since $I^\circ(\xi; \zeta) = \max_{\mu \in \partial I(\xi)} \mu \cdot \zeta$ for any ζ, ξ (e.g., Proposition 2.1.2 in Clarke, 1990), this yields some $\mu \in \partial I(\phi + \lambda^{**}(\psi - \phi))$ such that

$$\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

By (12), there exists a sequence $\xi_n \to \phi + \lambda^{**}(\psi - \phi)$ such that $\xi_n \in \hat{U}$ for each n and $\lim_n \nabla I(\xi_n) = \mu$. Then

$$\begin{aligned}
\lim_n (I(\xi_n) - I(\psi) + \nabla I(\xi_n) \cdot (\psi - \xi_n)) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\psi) + (1 - \lambda^{**})\mu \cdot (\psi - \phi) \\
&= H(\lambda^{**}) - (1 - \lambda^{**})m + (1 - \lambda^{**})\mu \cdot (\psi - \phi) \\
&> (1 - \lambda^{**})H'(\lambda^{**}) > 0,
\end{aligned}$$

where the inequalities use the fact that $H(\lambda^{**}) > 0$ and that $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Similarly,

$$\begin{aligned}
\lim_n (I(\xi_n) - I(\phi) + \nabla I(\xi_n) \cdot (\phi - \xi_n)) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\phi) - \lambda^{**}\mu \cdot (\psi - \phi) \\
&= H(\lambda^{**}) + \lambda^{**}m - \lambda^{**}\mu \cdot (\psi - \phi) \\
&< \epsilon - \lambda^{**}H'(\lambda^{**}) < \epsilon
\end{aligned}$$

where the inequalities use $H(\lambda^{**}) < \epsilon$ and $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Thus, for any large enough n, $\xi_n \in \hat{U}$ is as desired. □

We now establish the Boolean representation of I:

Lemma A.6. *For each $\phi \in \hat{U}$, we have*

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi),$$

where $K_\psi := \{\xi \in \hat{U} : I(\xi) + \nabla I(\xi) \cdot (\psi - \xi) \geq I(\psi)\}$ for all $\psi \in \hat{U}$.

Proof. For each $\phi, \psi \in \hat{U}$ and $\epsilon > 0$, Lemma A.5 yields some $\xi \in K_\psi$ such that $I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi) + \epsilon$. Thus, $\inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi)$. Moreover, by definition of K_ϕ, $\inf_{\xi \in K_\phi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \geq I(\phi)$. Hence, $I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi)$, as required. □

B Proofs for Section 2

B.1 Proof of Theorem 1

We invoke the following standard result:

Lemma B.1 (Lemma 1 in GMM)**.** *Preference $\succsim$ satisfies Axioms 1–5 if and only if there exists a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \Longleftrightarrow I(u(f)) \geq I(u(g)). \tag{14}$$

Moreover, I is unique and u is unique up to positive affine transformation.

The necessity proof for Theorem 1 is standard and we omit it. To prove sufficiency, suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. Consider the collection $\mathbb{P}^*$ given by (3), i.e.,

$$\mathbb{P}^* := \{P^*_\phi : \phi \in \mathbb{R}^S\} \text{ with } P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}.$$

Note that since I is monotonic and constant-linear, it is 1-Lipschitz. Thus, $\partial I(\underline{0}) \subseteq \Delta(S)$ by Lemma A.4, so that $\mathbb{P}^* \subseteq 2^{\Delta(S)}$. We will show that $\mathbb{P}^*$ is compact and that for all $\phi \in \mathbb{R}^S$,

$$I(\phi) = \max_{P^*_\psi \in \mathbb{P}^*} \min_{\mu \in P^*_\psi} \mu \cdot \phi, \tag{15}$$

which by (14) ensures that $(\mathbb{P}^*, u)$ is a BEU representation of $\succsim$.

Lemma A.1 yields a set $\hat{U} \subseteq \mathbb{R}^S$ such that $\mathbb{R}^S \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Moreover, since I is positively homogeneous, Lemma A.3 implies that $\partial I(\phi) \subseteq \partial I(\underline{0})$ for all $\phi \in \mathbb{R}^S$, so that for all $\phi \in \hat{U}$, we have $\mu_\phi := \nabla I(\phi) \in \partial I(\underline{0})$. The proof proceeds by establishing two lemmas:

Lemma B.2. *For each $\phi \in \hat{U}$, $I(\phi) = \mu_\phi \cdot \phi$.*

Proof. Take any $\phi \in \hat{U}$. By positive homogeneity of I, $\alpha\phi \in \hat{U}$ and $\nabla I(\phi) = \nabla I(\alpha\phi)$ for any $\alpha \in (0, 1)$. Thus, the function $h : [0, 1] \to \mathbb{R}$ defined by $h(\alpha) = I(\alpha\phi)$ is differentiable at every $\alpha \in (0, 1)$ and Lipschitz continuous. Hence, $I(\phi) = h(1) - h(0) = \int_0^1 h'(\alpha')d\alpha' = \int_0^1 (\nabla I(\alpha\phi) \cdot \phi)d\alpha' = \phi \cdot \mu_\phi$. $\square$

Lemma B.3. *P^*_ψ is continuous in ψ under the Hausdorff topology.*

Proof. Fix any ψ and sequence $\psi_n \to \psi$, and take any $\epsilon > 0$. Note that the distance between two hyperplanes $\{\mu \in \mathbb{R}^S : \mu \cdot \psi = w\}$ and $\{\mu \in \mathbb{R}^S : \mu \cdot \psi = w'\}$ is given by $\frac{|w-w'|}{\|\psi\|}$. Thus, $\mu' \in C$ is in the ϵ-neighborhood of P^*_ψ if $\mu' \cdot \psi \geq I(\psi) - \epsilon\|\psi\|$. Take N such that $|I(\psi) - I(\psi_n)|, \|\psi - \psi_n\| < \frac{\epsilon \min\{\|\psi\|, \|\psi_n\|\}}{2}$ for all $n \geq N$. Then for any $n \geq N$ and $\mu' \in P^*_{\psi_n}$, $\mu' \cdot \psi_n \geq I(\psi_n)$ implies $\mu' \cdot \psi \geq I(\psi) - \epsilon\|\psi\|$, so that μ' is in the ϵ-neighborhood of P^*_ψ. Likewise any $\mu \in P^*_\psi$ is in the ϵ-neighborhood of $P^*_{\psi_n}$ for all $n \geq N$. $\square$

To complete the proof of (15), first take any $\phi, \psi \in \hat{U}$ and let $K_\psi := \{\xi \in \hat{U} : I(\xi) + \mu_\xi \cdot (\psi - \xi) \geq I(\psi)\}$ be as in Lemma A.6. Then

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \mu_\xi \cdot (\phi - \xi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} \mu_\xi \cdot \phi, \tag{16}$$

where the first equality holds by Lemma A.6 and the second by Lemma B.2. Letting $P_\psi := \{\mu_\xi : \xi \in \hat{U}, \mu_\xi \cdot \psi \geq I(\psi)\}$, Lemma B.2 ensures that $\xi \in K_\psi$ if and only if $\mu_\xi \in P_\psi$. Moreover, (12) implies that $\overline{\mathrm{co}} P_\psi = P^*_\psi$. Combining these two observations with (16) yields

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\mu \in P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in \overline{\mathrm{co}} P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in P^*_\psi} \mu \cdot \phi. \tag{17}$$

Next, take any $\phi, \psi \in U$. Then there exist sequences $\phi_n \to \phi, \psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$. By (17), $\min_{\mu \in P^*_{\psi_n}} \mu \cdot \phi_n \leq I(\phi_n)$ for each n. By Lemma B.3 and the continuity of I, this implies

$$\min_{\mu \in P^*_\psi} \mu \cdot \phi \leq I(\phi).$$

In particular, for $\psi = \phi$, we have $\min_{\mu \in P^*_\phi} \mu \cdot \phi \leq I(\phi) \leq \min_{\mu \in P^*_\phi} \mu \cdot \phi$, where the final inequality holds by definition of P^*_ϕ. Hence, as required, we have

$$I(\phi) = \min_{\mu \in P^*_\phi} \phi \cdot \mu = \max_{P^*_\psi \in \mathbb{P}^*} \min_{\mu \in P^*_\psi} \mu \cdot \phi.$$

Finally, we verify that $\mathbb{P}^*$ is compact. Note first that by positive homogeneity of I, each set P^*_ϕ is scale-invariant, i.e., $P^*_\phi = P^*_{a\phi}$ for every $a \in \mathbb{R}_+$. Thus, $\mathbb{P}^*$ satisfies $\mathbb{P}^* = \{P^*_\phi : \phi \in [-1,1]^S\}$. Given this, Lemma B.3 implies that $\mathbb{P}^* = \{P^*_\phi : \phi \in [-1,1]^S\}$ is closed and hence compact. □

B.2 Proof of Proposition 1

We begin with the following lemma:

Lemma B.4. *Consider any functional $I : \mathbb{R}^S \to \mathbb{R}$ and belief-set collection $\mathbb{P}$ such that $I(\phi) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi$ for all $\phi \in \mathbb{R}^S$. Then*

$$\partial I(\underline{0}) \subseteq \overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P.$$

Proof. For each $P \in \mathbb{P}$, let $I_P(\phi) := \min_{\mu \in P} \mu \cdot \phi$ for each ϕ. Thus, $I(\phi) = \max_{P \in \mathbb{P}} I_P(\phi)$ for each ϕ. Note that each I_P is 1-Lipschitz continuous and $\partial I_P(0) = P$.

Take any convergent sequence $(\nabla I_{P_i}(\phi_i))$ where $\phi_i \to 0$, $P_i \in \mathbb{P}$, and $\nabla I_{P_i}(\phi_i)$ exists for each i. Then

$$\nabla I_{P_i}(\phi_i) \in \partial I_{P_i}(\phi_i) \subseteq \partial I_{P_i}(0) = P_i$$

where the set inclusion holds by Lemma A.3. Thus, $\lim_i \nabla I_{P_i}(\phi_i) \in \overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P$. Hence, the desired conclusion follows by applying Lemma A.2 to I. $\square$

Suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. For $\mathbb{P}^*$ as in the sufficiency proof of Theorem 1, we have $\overline{\mathrm{co}} \cup_{P \in \mathbb{P}^*} P \subseteq \partial I(\underline{0})$. Thus, Lemma B.4 immediately implies that $C = \partial I(\underline{0})$ is the unique closed, convex set satisfying (5) for all BEU representations of $\succsim$, with equality for representation $\mathbb{P}^*$. $\square$

B.3 Proof of Corollary 1

Since the proof of Proposition 1 identifies the set of relevant priors as $C = \partial I(\underline{0})$, Corollary 1 is immediate from the following result in GMM:

Lemma B.5 (Theorem 14 in GMM)**.** *Suppose $\succsim$ satisfies Axioms 1–5 and let I and u be as in Lemma B.1. Then the unique closed, convex set D satisfying*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in D$$

is given by $D = \partial I(\underline{0})$.

C Proofs for Section 3

C.1 Proof of Proposition 2

Consider the belief-set collection $\overline{\mathbb{P}}$ defined by

$$\overline{\mathbb{P}} = \{\bar{P}_\phi : \phi \in \mathbb{R}^S\} \text{ with } \bar{P}_\phi = \{\mu \in \Delta(S) : \mu \cdot \phi \geq I(\phi)\}. \tag{18}$$

Arguments similar to those in the proof of Theorem 1 imply that $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. We begin by noting that this representation is $\sqsupseteq$-maximal among all representations of $\succsim$:

Lemma C.1. *Suppose $\succsim$ admits a BEU representation and let $\overline{\mathbb{P}}$ be given by (18). Then $\overline{\mathbb{P}} \sqsupseteq \mathbb{P}$ for all BEU representations $\mathbb{P}$ of $\succsim$.*

Proof. Consider any BEU representation $\mathbb{P}$ of $\succsim$ and any $\bar{P}_\phi \in \overline{\mathbb{P}}$. For I as given by Lemma B.1, there exists $P_\phi \in \mathbb{P}$ such that $\min_{\mu \in P_\phi} \mu \cdot \phi = I(\phi)$. Thus, $P_\phi \subseteq \bar{P}_\phi = \{\mu \in \Delta(S) : \mu \cdot \phi \geq I(\phi)\}$. $\square$

To prove Proposition 2, note first that $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if $u_1 \approx u_2$ and the functionals I_1 and I_2 given by Lemma B.1 satisfy $I_1(\phi) \leq I_2(\phi)$ for all $\phi \in \mathbb{R}^S$.

To prove that (1) implies (2), consider the BEU representations $\overline{\mathbb{P}}_i$ of $\succsim_i$ given by (18). The inequality $I_1 \leq I_2$ implies $\overline{\mathbb{P}}_1 \sqsupseteq \overline{\mathbb{P}}_2$: Indeed, for all ϕ, the fact that $I_1(\phi) \leq I_2(\phi)$ implies $\bar{P}_{\phi,1} \supseteq \bar{P}_{\phi,2}$, whence $\overline{\mathbb{P}}_1 \sqsupseteq \overline{\mathbb{P}}_2$. Consider now any BEU representation $(\mathbb{P}_2, u_2)$ of $\succsim_2$. We have $\overline{\mathbb{P}}_1 \sqsupseteq \overline{\mathbb{P}}_2 \sqsupseteq \mathbb{P}_2$, where the latter inequality comes from the $\sqsupseteq$-maximality of $\overline{\mathbb{P}}_2$. Hence, by transitivity of $\sqsupseteq$, $\overline{\mathbb{P}}_1 \sqsupseteq \mathbb{P}_2$, which proves (2).

To prove that (2) implies (1), consider the BEU representation $(\mathbb{P}_1, u)$ of $\succsim_1$ described in (2) and any representation $(\mathbb{P}_2, u)$ of $\succsim_2$. Fix $\phi \in \mathbb{R}^S$, and let P_1 be any element of $\mathbb{P}_1$ such that $I_1(\phi) = \min_{\mu \in P_1} \mu \cdot \phi$. Since $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$, there exists $P_2 \in \mathbb{P}_2$ with $P_1 \supseteq P_2$, implying $I_2(\phi) \geq \min_{\mu \in P_2} \mu \cdot \phi \geq \min_{\mu \in P_1} \mu \cdot \phi = I_1(\phi)$. Thus, $I_2(\phi) \geq I_1(\phi)$ for all $\phi \in \mathbb{R}^S$, implying that $\succsim_1$ is more ambiguity-averse than $\succsim_2$. $\square$

C.2 Proof of Lemma 1

We combine the proof of Lemma 1 with the proof of Theorem 2 (part 2) below.

C.3 Proof of Theorem 2

C.3.1 Proof of part 1

To prove the "only if" direction, suppose that $\succsim$ satisfies uncertainty aversion. Since it admits the maxmin expected utility representation of Gilboa and Schmeidler (1989), $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$ holds for all ϕ.

We first show that $\cap_{P \in \mathbb{P}} P \supseteq C$. If not, there exists $P \in \mathbb{P}$ such that $P \not\supseteq C$. By the standard property of support functions, this implies the existence of ϕ such that $\min_{\mu \in C} \phi \cdot \mu < \min_{\mu \in P} \phi \cdot \mu$. This leads to $I(\phi) > \min_{\mu \in C} \mu \cdot \phi$, a contradiction.

We now show that $\cap_{P \in \mathbb{P}} P \subseteq C$. If not, there exists $\mu^* \in \cap_{P \in \mathbb{P}} P \setminus C$. Then there exists ϕ such that $\min_{\mu \in C} \phi \cdot \mu < \phi \cdot \mu^*$. But this implies $I(\phi) \leq \phi \cdot \mu^* < \min_{\mu \in C} \mu \cdot \phi$, a contradiction.

To prove the " if" direction, suppose that $\cap_{P \in \mathbb{P}} P = C$. Take any ϕ. It suffices to show that $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$. Note that $I(\phi) \geq \min_{\mu \in C} \mu \cdot \phi$ follows by the construction of $\mathbb{P}^*$ by (3) that we used in the proof of Theorem 1. But the representation based on $\mathbb{P}$ yields the inequality $I(\phi) \leq \min_{\mu \in \cap_{P \in \mathbb{P}} P} \mu \cdot \phi = \min_{\mu \in C} \mu \cdot \phi$, which ensures the desired claim.

C.3.2 Proof of part 2 and Lemma 1

We here prove the equivalence *absolute ambiguity aversion* $\Leftrightarrow$ ∞- *ambiguity aversion* $\Leftrightarrow$ $\cap_{P\in\mathbb{P}}P \neq \emptyset$, which implies both part 2 of Theorem 2 and Lemma 1. We start with the implication *absolute ambiguity aversion* $\Rightarrow$ ∞- *ambiguity aversion*. Suppose that $\succsim$ is more ambiguity-averse than some subjective expected utility preference with belief μ and utility function u (the latter being without loss of generality by Proposition 2). Fix any $k \geq 2$ and any $f_1, \cdots, f_k, p$ such that $f_1 \sim \cdots \sim f_k$ and $\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p$. By comparative ambiguity aversion, for any f_i we have $I(u(f_i)) \leq \mu \cdot u(f_i)$. Since $I(u(f_i)) = I(u(f_1))$ for all i, this implies $kI(u(f_1)) \leq \mu \cdot [u(f_1) + \cdots + u(f_k)] = \mu \cdot [ku(p)] = ku(p)$. As a result, $I(u(f_1)) \leq u(p)$, which implies $f_1 \precsim p$. Thus, $\succsim$ satisfies ∞-ambiguity aversion.

We now turn to the implication ∞- *ambiguity aversion* $\Rightarrow \cap_{P\in\mathbb{P}}P \neq \emptyset$. If $\succsim$ satisfies k-ambiguity aversion for any finite k, by part 3 of the theorem (see the proof below) any BEU representation $(\mathbb{P}, u)$ of $\succsim$ is such that, for any finite k, $\cap_{i=1,\cdots,k}P_i \neq \emptyset$ for any $P_1, \cdots, P_k \in \mathbb{P}$. Since $\Delta(S)$ is compact, any collection of closed subsets of $\Delta(S)$ having the finite intersection property has non-empty intersection. In other words, $\cap_{P\in\mathbb{P}}P \neq \emptyset$.

We conclude the proof with the implication $\cap_{P\in\mathbb{P}}P \neq \emptyset \Rightarrow$ *absolute ambiguity aversion*. Suppose that there exists $\mu^* \in \bigcap_{P\in\mathbb{P}} P$ for some BEU representation $(\mathbb{P}, u)$ of $\succsim$. For any $f \in \mathcal{F}$ and any $P \in \mathbb{P}$, this implies that $\min_{\mu\in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$, and hence $\max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$. As a result,

$$f \succsim p \Longrightarrow \max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot u(f) \geq u(p) \Longrightarrow \mu^* \cdot u(f) \geq u(p) \Longrightarrow f \succsim_{\mu^*} p$$

where $\succsim_{\mu^*}$ is the subjective expected utility preference with belief μ and utility function u. Hence, $\succsim$ is more ambiguity-averse than $\succsim_{\mu^*}$, which proves the result.

C.3.3 Proof of part 3

The proof relies on the following lemma.

Lemma C.2. *Preference $\succsim$ satisfies k-ambiguity aversion if and only if any BEU representation $(\mathbb{P}, u)$ of $\succsim$ is such that*

$$\sum_{i=1}^{k-1} \max_{P_i\in\mathbb{P}} \min_{\mu_i\in P_i} \mu_i \cdot \phi_i \leq \min_{P\in\mathbb{P}} \max_{\mu\in\mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i$$

for all $\phi_1, \cdots, \phi_{k-1} \in \mathbb{R}^S$.

Proof. To prove the "if" part, suppose the inequality in the lemma is satisfied. Consider any $f_1, \cdots, f_k \in \mathcal{F}$ such that $f_1 \sim f_i$ for all i and $\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p$ for some $p \in \Delta(Z)$. We have

$$I(\frac{1}{k}u(f_k)) = I(u(p) - \sum_{i=1}^{k-1} \frac{1}{k} f_i) = u(p) - \min_{P \in \mathbb{P}} \max_{\mu \in P} \sum_{i=1}^{k-1} \frac{1}{k} u(f_i) \cdot \mu$$

$$\leq u(p) - \sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \frac{1}{k} u(f_i) \cdot \mu_i$$

$$= u(p) - \sum_{i=1}^{k-1} I(\frac{1}{k} u(f_i)).$$

Rearranging yields

$$\sum_{i=1}^{k} I(\frac{1}{k} u(f_i)) \leq u(p)$$

which is simply $I(u(f_1)) \leq u(p)$ since $I(u(f_i)) = I(u(f_1))$ for all i. This is turn implies $p \succsim f_1$, and thus $\succsim$ satisfies k-ambiguity aversion.

To prove the "only if" part, suppose that there exists some vectors $\phi_1, \cdots, \phi_{k-1}$ such that the inequality in the statement of Lemma C.2 is violated. By the constant linearity of the max-min and min-max functionals, without loss of generality we can assume that $I(\phi_i) = I(\phi_1)$ for all i, and that each ϕ_i belongs to $[-1, 1]^S$.

Consider now some $c \in \mathbb{R}$ such that $I(c - \phi_1 - \cdots - \phi_{k-1}) = I(\phi_1)$. Such a constant exists by the continuity and monotonicity of I, and satisfies $c \in [-k, k]$. Let $\phi_k \in \mathbb{R}^S$ be defined by $\phi_k = c - \phi_1 - \cdots - \phi_{k-1}$, which implies $\phi_1 + \cdots + \phi_k = c$. Up to rescaling all the ϕ_is and c by a common factor, this vector ϕ_k also belongs to $[-1, 1]^S$. By definition of c, we have $I(\phi_k) = I(\phi_1)$, and

$$I(\phi_k) = I(c - \sum_{i=1}^{k-1} \phi_i) = c - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu \cdot \sum_{i=1}^{k-1} \phi_i > c - \sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i$$

$$= c - \sum_{i=1}^{k-1} I(\phi_i).$$

Rearranging yields $\sum_{i=1}^{k} I(\phi_i) > c$, which implies $I(\phi_1) > \frac{c}{k}$.

To conclude the proof, consider now some best outcome $\bar{z}$ and some worse outcome $\underline{z}$ in Z. Since each ϕ_i belongs to $[-1, 1]^S$, it is possible to find weights (ϵ_i^s) such that the act f_i that maps each state s into the lottery $\epsilon_i^s \delta_{\underline{z}} + (1 - \epsilon_i^s) \delta_{\bar{z}}$ satisfies $u(f_i) = \phi_i$. In addition, the fact that $\sum_{i=1}^{k} u(f_i)$ is a constant vector equal to c shows that $\sum_{i=1}^{k} \frac{1}{k} f_i$ is a constant act that delivers a lottery p supported on $\{\bar{z}, \underline{z}\}$, where $u(p) = \frac{c}{k}$. The collection $(f_1, \cdots, f_k)$ thus satisfies $\frac{1}{k} f_1 + \cdots + \frac{1}{k} f_k = p$, $f_i \sim f_1$ for all i since $I(\phi_i) = I(\phi_1)$, and $f_1 \succ p$ since

$I(\phi_1) > \frac{c}{k} = u(p)$. Hence, $\succsim$ does not satisfy k-ambiguity aversion. $\square$

Let us now prove part 3 of the proposition.

Sufficiency: suppose that $P_1 \cap \cdots \cap P_k \neq \emptyset$ for all $P_1, \cdots, P_k \in \mathbb{P}$. Consider any $P_1, \cdots, P_k$ and some vectors $(\phi_1, \cdots, \phi_{k-1})$. Let $\mu \in P_1 \cap \cdots \cap P_k$. We have

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \sum_{i=1}^{k-1} \mu \cdot \phi_i \leq \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i$$

where the first inequality is due to the fact that $\mu \in P_i$ for all $i \leq k-1$, and the second inequality is due to the fact that $\mu \in P_k$. Since this is true for any $P_1, \cdots, P_k$, this implies

$$\max_{(P_1, \cdots, P_{k-1}) \in \mathbb{P}^{k-1}} \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \min_{P_k \in \mathbb{P}} \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i,$$

i.e.

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \leq \min_{P \in \mathbb{P}} \max_{\mu \in \mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.2 $\succsim$ satisfies k-ambiguity aversion.

Necessity: suppose that there exist $P_1, \cdots, P_k \in \mathbb{P}$ such that $P_1 \cap \cdots \cap P_k = \emptyset$. Consider the sets $A, B \subseteq \mathbb{R}^{S(k-1)}$ defined by

$$A = \{(\mu_1, \cdots, \mu_{k-1}) : \mu_i \in P_i\} \text{ and } B = \{(\mu_k, \cdots, \mu_k) : \mu_k \in P_k\}.$$

The sets A and B are closed and convex. In addition, $A \cap B = \emptyset$ since any $(\mu_k, \cdots, \mu_k) \in A \cap B$ would be such that $\mu_k \in P_1 \cap \cdots \cap P_k$, which is a contradiction. By the separating hyperplane theorem there exists a vector $\phi = (\phi_1, \cdots, \phi_{k-1}) \in \mathbb{R}^{S(k-1)}$, where each $\phi_i \in \mathbb{R}^S$, such that $\min_{a \in A} a \cdot \phi > \max_{b \in B} b \cdot \phi$, which is equivalent to

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i.$$

Hence,

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \geq \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i \geq \min_{P \in \mathbb{P}} \max_{\mu \in \mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.2 $\succsim$ does not satisfy k-ambiguity aversion. $\square$

C.4 Proof of Lemma 2

Note that $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ while $m(E^c) = 1 - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E)$, and thus $AA(E) = \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E) - \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$. This implies that $AA(E) \geq 0$ if and only if all $P, P' \in \mathbb{P}$ satisfy $\max_{\mu \in P} \mu(E) \geq \min_{\mu' \in P'} \mu'(E)$, i.e., if and only if $\{\mu(E) : \mu \in P\} \cap \{\mu'(E) : \mu' \in P'\} \neq \emptyset$. $\square$

C.5 Proof of Proposition 3

For any $\succsim$ with BEU representation $(\mathbb{P}, u)$, note that $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ for all events E. Thus, given $\mathcal{E}$ and $\mathcal{F}$ as in the proposition, it suffices to find $\nu \in \Delta(S)$ and a belief-set collection $\mathbb{P}$ such that $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E) < \nu(E)$ for all $E \in \mathcal{E}$ and $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(F) > \nu(F)$ for all $F \in \mathcal{F}$.

Fix any $\nu \in \Delta^\circ(S)$ and pick $\beta > 0$ with $\beta < \min_{s \in S} \nu(s)$. Define $\mathbb{P}$ by $\mathbb{P} = \{P_F : F \in \mathcal{F}\}$, where for each $F \in \mathcal{F}$,

$$P_F := \{\mu \in \Delta(S) : \mu(F) = \nu(F) + \frac{\beta}{2},\, \mu(E) \in [\nu(E) - \beta, \nu(E) + \beta]\, \forall E \subseteq S\}.$$

Note that each P_F is nonempty: Indeed, pick any $s \in F$ and $s' \in F^c$ (which exist since $F \notin \{S, \emptyset\}$). Then setting $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, and $\mu(s'') = \nu(s'')$ for all $s'' \neq s, s''$ yields $\mu \in P_F$. Since P_F is also closed and convex, $\mathbb{P}$ is a well-defined belief-set collection.

By definition of $\mathbb{P}$, $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(F) \geq \nu(F) + \frac{\beta}{2} > \nu(F)$ for all $F \in \mathcal{F}$. To complete the proof, we show that $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E) \leq \nu(E) - \frac{\beta}{2} < \nu(E)$ for all $E \in \mathcal{E}$. Consider any $E \in \mathcal{E}$, $F \in \mathcal{F}$. Since $E \neq F$ (as $\mathcal{E}$ and $\mathcal{F}$ are disjoint), we either have (a) $F \setminus E \neq \emptyset \neq E \setminus F$; (b) $E \subsetneq F$; or (c) $F \subsetneq E$. In each case, we show that $\min_{\mu \in P_F} \mu(E) \leq \nu(E) - \frac{\beta}{2}$ by constructing a $\mu \in P_F$ such that $\mu(E) = \nu(E) - \frac{\beta}{2}$:

In case (a), pick $s \in F \setminus E$ and $s' \in E \setminus F$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s) - \frac{\beta}{2}$, and $\mu(s'') = \nu(s'')$ for all $s'' \neq s, s'$.

In case (b), pick $s \in F \setminus E$, $s' \in E$, and $s'' \in F^c \subseteq E^c$. Then define μ by $\mu(s) = \nu(s) + \beta$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, $\mu(s'') = \nu(s'') - \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$.

In case (c), pick $s \in F$, $s' \in E \setminus F$, and $s'' \in E^c \subseteq F^c$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \beta$, $\mu(s'') = \nu(s'') + \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$. $\square$

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Supplementary Appendix to "Boolean Expected Utility"

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This supplementary appendix is organized as follows. Section S.1 formalizes the uniqueness properties of BEU representations. Section S.2 focuses on the representation obtained by inverting the order of moves of Optimism and Pessimism and uses this to characterize different degrees of ambiguity seeking. Sections S.3 and S.4 present two generalizations of BEU that correspond to relaxations of certainty independence.

S.1 Uniqueness

For any $\phi \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$, let $H_{\phi,\lambda} := \{\mu \in \Delta(S) : \mu \cdot \phi \geq \lambda\}$ denote the closed half-space in $\Delta(S)$ that is defined by ϕ and λ. For any belief-set collection $\mathbb{P}$, define its ***half-space closure*** by

$$\overline{\mathbb{P}} := \{H \subseteq \Delta(S) : H \text{ is a closed half-space in } \Delta(S) \text{ and } P \subseteq H \text{ for some } P \in \mathbb{P}\}.$$

Proposition S.1.1. *Suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then $(\mathbb{P}', u')$ is a BEU representation of $\succsim$ if and only if $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ and $u \approx u'$.*

Below we fix the unique functional $I : \mathbb{R}^S \to \mathbb{R}$ associated with $\succsim$, as given by Lemma B.1. We begin with the following lemma:

Lemma S.1.1. *Suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then $\overline{\mathbb{P}} = \{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$.*

Proof. First, take any $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$ such that $\lambda \leq I(\phi)$. Since $(\mathbb{P}, u)$ represents $\succsim$, there exists $P \in \mathbb{P}$ such that $\min_{\mu \in P} \mu \cdot \phi = I(\phi)$. Thus, $P \subseteq H_{\phi,I(\phi)} \subseteq H_{\phi,\lambda}$, which implies $H_{\phi,\lambda} \in \overline{\mathbb{P}}$.

Conversely, take any $P \in \overline{\mathbb{P}}$. By definition of $\overline{\mathbb{P}}$, there exist $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$, and $P' \in \mathbb{P}$ such that $P' \subseteq P = H_{\phi,\lambda}$. Since $(\mathbb{P}, u)$ represents $\succsim$, $I(\phi) \geq \min_{\mu \in P'} \mu \cdot \phi \geq \min_{\mu \in H_{\phi,\lambda}} \phi \cdot \mu$. Hence, $\lambda \leq I(\phi)$. □

Proof of Proposition S.1.1. For the "only if" direction, the fact that $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ is immediate from Lemma S.1.1 and uniqueness of I. The proof that $u \approx u'$ is standard.

For the "if" direction, by uniqueness of I, it suffices to show that $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi = I(\phi)$ for all $\phi \in \mathbb{R}^S$. To show this, observe first that by Lemma S.1.1 and since $\overline{\mathbb{P}} = \overline{\mathbb{P}}'$, there exists $P' \in \mathbb{P}'$ such that $P' \subseteq H_{\phi,I(\phi)}$. This ensures $\min_{\mu \in P'} \mu \cdot \phi \geq I(\phi)$. Suppose next that $\min_{\mu \in P''} \mu \cdot \phi - I(\phi) =: \epsilon > 0$ for some $P'' \in \mathbb{P}'$. Then $H_{\phi, I(\phi)+\epsilon} \supseteq P''$, which implies $H_{\phi, I(\phi)+\epsilon} \in \overline{\mathbb{P}'}$. Since $\overline{\mathbb{P}'} = \overline{\mathbb{P}}$, this contradicts Lemma S.1.1. □

S.2 Minmax BEU representation

While BEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for BEU.

Theorem S.2.1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a minmax BEU representation, i.e., there exists a belief-set collection $\mathbb{Q}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$$

represents $\succsim$.

Our construction of the maxmin BEU representation considered in the text uses the belief-set collection $\mathbb{P}^* := \{P^*_\phi : \phi \in \mathbb{R}^S\}$ with $P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$. Analogously, it can be shown that the belief-set collection $\mathbb{Q}^* := \{Q^*_\phi : \phi \in \mathbb{R}^S\}$ with $Q^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$ yields a minmax BEU representation. Paralleling Section 2.3, it is straightforward to show that $C := \partial I(\underline{0})$ again corresponds to the smallest set of priors that is contained in $\overline{\text{co}} \bigcup_{Q \in \mathbb{Q}} Q$ for all minmax BEU representations $\mathbb{Q}$ of $\succsim$, with equality for representation $\mathbb{Q}^*$.

While the different notions of ambiguity aversion are most conveniently characterized using the maxmin BEU representation (cf. Theorem 2), the minmax BEU representation is useful for characterizing their ambiguity-seeking counterparts. Axioms 8 and 9 and Theorem S.2.2 below provide the analogs of Axioms 6 and 7 and Theorem 2, respectively.

Axiom 8 (Uncertainty Seeking)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \precsim f$.

Axiom 9 (k-Ambiguity Seeking)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Rightarrow p \precsim f_1.$$

We say that $\succsim$ is *absolutely ambiguity-seeking* if there exists a nondegenerate subjective expected utility preference that is more ambiguity-averse than $\succsim$. Analogous to Lemma 1, this is characterized by ∞-*ambiguity seeking*, i.e., k-ambiguity seeking for all k.

Theorem S.2.2. *Suppose that $\succsim$ admits a minmax BEU representation $(\mathbb{Q}, u)$.*

1. *$\succsim$ satisfies uncertainty seeking if and only if* $\bigcap_{Q \in \mathbb{Q}} Q = C$.

2. *$\succsim$ is absolutely ambiguity-seeking if and only if* $\bigcap_{Q \in \mathbb{Q}} Q \neq \emptyset$.

3. *$\succsim$ satisfies k-ambiguity seeking if and only if* $\bigcap_{i=1,\cdots,k} Q_i \neq \emptyset$ *for all* $Q_1, \cdots, Q_k \in \mathbb{Q}$.

S.3 Boolean variational representation

The variational model introduced by Maccheroni, Marinacci, and Rustichini (2006) (henceforth, MMR) relies on the following relaxation of certainty independence, which retains the "location invariance" property of preferences but relaxes the "scale invariance" property; we refer to MMR for a discussion.

Axiom 10 (Weak Certainty Independence)**.** For any $f, g \in \mathcal{F}$, $p, q \in \Delta(Z)$, and $\alpha \in (0,1)$,

$$\alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p \Longrightarrow \alpha f + (1-\alpha)q \succsim \alpha g + (1-\alpha)q.$$

We now show that dropping uncertainty aversion from MMR's axioms corresponds to adding a maximization stage into the variational model. A ***cost collection*** is a collection of functions $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ such that each $c \in \mathbb{C}$ is convex and $\mathbb{C}$ is ***grounded*** (i.e., $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$).

Theorem S.3.1. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if $\succsim$ admits a **Boolean variational (BV)** representation, i.e., there exists a cost collection $\mathbb{C}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W_{BV}(f) := \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu) \tag{19}$$

is well-defined and represents $\succsim$.

We note that our characterization of the set of relevant priors under BEU generalizes to the Boolean variational model. Specifically, let $\mathrm{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$ denote the effective domain of any cost function. Then there exists a unique closed, convex set C such that $C \subseteq \overline{\mathrm{co}} \left(\bigcup_{c \in \mathbb{C}} \mathrm{dom}(c) \right)$ for all Boolean variational representations of $\succsim$, with equality for the representation $\mathbb{C}^*$ we construct in the proof of Theorem S.3.1 below. Moreover, it can again be shown that C is the Bewley set of the unambigous preference $\succsim^*$. The argument relies on the observation that $C = \overline{\mathrm{co}} \left(\bigcup_{\phi \in \mathrm{int} U} \partial I(\phi) \right)$, where I is the utility act functional obtained in the proof of Theorem S.3.1. Details are available on request.

S.3.1 Proof of Theorem S.3.1

We first recall the following result from MMR:

Lemma S.3.1 (Lemma 28 in MMR)**.** *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a normalized niveloid $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem S.3.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and Axiom 10. Let I, u, and U be as given by Lemma S.3.1. Since I is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset $\hat{U} \subseteq \text{int}U$ with $U \setminus \hat{U}$ of Lebesgue measure 0 such that I is differentiable on $\hat{U}$. Define $\mu_\psi := \nabla I(\psi)$ and $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$ for each $\psi \in \hat{U}$. By Lemma A.4 and the fact that niveloids are monotonic and constant-additive, $\mu_\psi \in \Delta(S)$ for all $\psi \in \hat{U}$. For each $\psi \in U$, define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let $\mathbb{D} := \{D_\psi : \psi \in U\}$. The following lemma implies that each D_ψ is non-empty; note also that it is closed, convex, and bounded below.

Lemma S.3.2. *For every $\phi, \psi \in U$, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ with equality if $\phi = \psi$.*

Proof. First, consider any $\phi, \psi \in \hat{U}$. Let $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$ be as in Lemma A.6. Note that $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$, so that

$$\inf_{\xi \in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as D_ψ is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi), \tag{20}$$

where (20) holds with equality if $\psi = \phi$ by definition of D_ψ.

Next, consider any $\phi, \psi \in U$. Take sequences $\phi_n \to \phi$, $\psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$ for each n, where we choose $\phi_n = \psi_n$ if $\phi = \psi$. For each n, the previous paragraph yields some $(\mu_n, w_n) \in D_{\psi_n}$ such that $\mu_n \cdot \phi_n + w_n = \min_{(\mu,w)\in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$, with equality if $\phi = \psi$. Since I is a niveloid, it is monotonic. Combined with normalization, this implies that for all n, $I(\phi_n) \in [\min_s \phi_n(s), \max_s \phi_n(s)]$. Hence, $|w_n| \leq \max_s \phi_n(s) - \min_s \phi_n(s)$, which is bounded since $\phi_n \to \phi$. Thus, up to restricting to a suitable subsequence, we can assume that $(\mu_n, w_n) \to (\mu_\infty, w_\infty)$ for some $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$. Then $(\mu_\infty, w_\infty) \in D_\psi$ and $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$ by continuity of I, with equality if $\phi = \psi$. Thus, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w =$

$\inf_{(\mu,w)\in D_\psi} \mu\cdot\phi + w \leq I(\phi)$, with equality if $\phi = \psi$, where the minimum is attained since D_ψ is closed and bounded below. $\square$

Finally, we obtain a Boolean variational representation of $\succsim$ as follows. For each $D \in \mathbb{D}$, define $c_D : \Delta(S) \to \mathbb{R}\cup\{\infty\}$ by $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$ for each $\mu \in \Delta(S)$, where by convention the infimum of the empty set is ∞. Note that c_D is convex for all D by convexity of D. Moreover, for all $\phi \in U$, $\min_{(\mu,w)\in D} \mu\cdot\phi + w = \min_{\mu\in\Delta(S)} \mu\cdot\phi + c_D(\mu)$. Thus, Lemma S.3.2 implies

$$I(\phi) = \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} \mu\cdot\phi + c_D(\mu) \tag{21}$$

for all $\phi \in U$. Since I is normalized, applying (21) to any constant vector $\underline{a} \in U$, yields $I(\underline{a}) = a + \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} c_D(\mu) = a$. Thus, collection $(c_D)_{D\in\mathbb{D}}$ is grounded. Hence, $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$ is a cost collection and $(\mathbb{C}^*, u)$ is a BV representation of $\succsim$ by Lemma S.3.1. $\square$

S.4 Rational Boolean representation

Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) (henceforth, CMMM) maintain uncertainty aversion, but further relax independence to hold only for objective lotteries:

Axiom 11 (Risk Independence)**.** For any $p, q, r \in \Delta(Z)$ and $\alpha \in (0, 1)$,

$$p \succsim q \Longrightarrow \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r.$$

Dropping uncertainty aversion from CMMM's axioms yields the following Boolean generalization of their representation:

Theorem S.4.1. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 11 if and only if $\succsim$ admits a **rational Boolean (RB)**, i.e., there exists a collection $(G_t)_{t\in T}$ of quasiconvex functions $G_t : \mathbb{R}\times\Delta(S) \to \mathbb{R}\cup\{\infty\}$ that are increasing in their first argument and grounded*[25] *and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W_{RB}(f) := \max_{t\in T} \inf_{\mu\in\Delta(S)} G_t(\mathbb{E}_\mu[u(f)], \mu) \tag{22}$$

is well-defined, continuous, and represents $\succsim$.

[25] That is, $\max_{t\in T} \inf_{\mu\in\Delta(S)} G_t(a, \mu) = a$ for all a.

S.4.1 Proof of Theorem S.4.1

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

Lemma S.4.1. *Preference $\succsim$ satisfies Axioms 1–4 and 11 if and only if there exists a non-constant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a monotonic, normalized and continuous functional $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem S.4.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and 11. Let I, u, and U be as given by Lemma S.4.1.

Define $D_\psi := \{(\mu, I(\psi) - \mu \cdot \psi) \in \mathbb{R}^S_+ \times \mathbb{R} : \mu \in \mathbb{R}^S_+\}$ for each $\psi \in U$. Note that D_ψ is non-empty and convex. Let $I_\psi(\phi) := \inf_{(\mu,w)\in D_\psi} \mu \cdot \phi + w$ for each $\phi, \psi \in U$.

Take any $\phi, \psi \in U$. Observe that

$$I_\psi(\phi) = \inf_{\alpha>0, s\in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) \text{ if } \phi \geq \psi \\ -\infty \text{ if } \phi \ngeq \psi \end{cases}$$

Thus, $I(\phi) \geq I_\psi(\phi)$ by monotonicity of I, with equality if $\phi = \psi$. That is, for each $\phi \in U$,

$$I(\phi) = \max_{\psi\in U} I_\psi(\phi). \tag{23}$$

For each $\psi \in U$, define a function $G_\psi : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by

$$G_\psi(t, \mu) = \sup\{I_\psi(\xi) : \xi \in U, \xi \cdot \mu \leq t\}$$

for each (t, μ). The map is quasi-convex (Lemma 31 in CMMM) and increasing in t.

Lemma S.4.2. *$I_\psi(\phi) = \inf_{\mu\in\Delta(S)} G_\psi(\mu \cdot \phi, \mu)$ for each $\phi, \psi \in U$.*

Proof. Observe that RHS $= \inf_{\mu\in\Delta(S)} \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$. To see that LHS $\leq$ RHS, observe that $I_\psi(\phi) \leq \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$ holds for any $\mu \in \Delta(S)$.

To see that LHS $\geq$ RHS, note first that if $\phi \geq \psi$ then LHS $= I(\psi)$ and RHS $\in \{I(\psi), -\infty\}$, so the inequality clearly holds. If $\phi \ngeq \psi$ then $\phi_s < \psi_s$ for some $s \in S$. Thus, by taking $\mu = \delta_s$, any ξ with $\xi \cdot \mu \leq \phi \cdot \mu$ satisfies $\xi_s \leq \phi_s$, which implies $\xi \ngeq \psi$, whence $I_\psi(\xi) = -\infty$. □

Setting $T = U$, Lemma S.4.2 and (23) ensure that W_{RB} given by (22) represents $\succsim$ and is continuous. Finally, to check groundedness, note that since I is normalized, we have $a = I(\underline{a}) = \max_{\psi\in U} \inf_{\mu\in\Delta(S)} G_\psi(a, \mu)$ for any $a \in \mathbb{R}$. □