The Yale Undergraduate Research Journal

Volume 2
Issue 1 *Spring 2021*

Article 31

2021

Incomplete? Or Indefinite? Intuitionism on Gödel's First Incompleteness Theorem

Quinn Crawford
Yale University

Incomplete? Or Indefinite? Intuitionism on Gödel's First Incompleteness Theorem

By Quinn Crawford[1]

[1]*Department of Philosophy, Yale University*

ABSTRACT

This paper analyzes two natural-looking arguments that seek to leverage Gödel's first incompleteness theorem for and against intuitionism, concluding in both cases that the argument is unsound because it equivocates on the meaning of "proof," which differs between formalism and intuitionism. I argue that this difference explains why "proof" has definite extension for the formalist but not for the intuitionist. Sections 1-3 summarize various philosophies of mathematics and Gödel's result. Section 4 argues that, because the Gödel sentence of a formal system is provable to the intuitionist, they are neither aided nor attacked by Gödel's first incompleteness theorem. Section 5 concludes that the intuitionist's notion of proof is indefinitely extensible.

INTRODUCTION

"Naïve set theory" was not always called that — it took Bertrand Russell's observation that the axiom of unrestricted comprehension, with which one can construct a set from any predicate, gives rise to one antinomy after another. The predicate "*x* is an ordinal"? Burali-Forti paradox. "*x* is a cardinal"? Cantor's paradox. "$x \notin x$"? This is the most famous, named after Russell himself: if there were a set *R* containing all sets which are not elements of themselves, we find that *R* is an element of *R* if and only if it is not (Hrbáček & Jech, 1999, p. 2). Gottlob Frege's logicist project, which sought to reduce mathematical truth to logical truth, relied crucially on the axiom of unrestricted comprehension, so the revelation of these paradoxes came as a devastating blow. With mathematical foundations crumbling, how do we proceed?

Two bold, competing stances on the nature of mathematics arose. German mathematician David Hilbert sought to avoid paradox by abandoning semantics, and his **formalist** project attempted to reduce mathematics to a fixed collection of syntactic rules. He strongly believed that *every* true mathematical proposition, and in particular every true proposition about arithmetic, could be proven using only these rules. Dutch mathematician L. E. J. Brouwer sought to avoid paradox by abandoning syntax, insisting that mathematics is about ideas in one's mind. He only allowed constructive proofs, which led his **intuitionist** project to forgo classical logic and reject the law of excluded middle. Formalism seemed like a safer bet, until Gödel's first incompleteness theorem showed that no sufficiently powerful formal system can prove every arithmetical truth — another devastating blow.

Does Gödel's result only affect Hilbert's program, or does it bear on Brouwer's as well? In this paper, I will analyze its effect on intuitionism. To do this, I will first summarize some of the main ideas behind formalism (Section 1), Gödel's first incompleteness theorem (Section 2), and intuitionism (Section 3). In Section 4, I analyze two arguments which, at first glance, tug intuitionism in opposite directions: on the one hand, Gödel's result seems to vindicate the intuitionist's rejection of the law of excluded middle, but on the other hand, it seems to repudiate their connection of mathematical truth with provability. I conclude that these arguments neither support nor threaten Brouwer's program, since the meaning of "proof" is different for intuitionists and formalists. In particular, I argue (in Section 5) that while the formalist's notion of proof has definite extension, the intuitionist's notion of proof does not — rather, it is what Michael Dummett calls an "indefinitely extensible" concept (1978, p. 195). Given any sufficiently powerful formal system of arithmetic, there is a true proposition that cannot be proven except by methods outside of that system, and it is because intuitionists employ these methods that they are left both unsupported and unscathed by Gödel's result.

1. FORMALISM

The formalist maintains that mathematics can be understood through a lens that is purely syntactic, divorced from the semantic meanings we usually attach to mathematical statements. Sentences in the language of arithmetic can be viewed, not as propositions, but as strings of letters on the page, and what mathematics *is*, at bottom, is the process of turning one string of letters into another. **Deduction rules** are the acceptable ways of manipulating the symbols in a string, e.g., pointing out when one can add a dash to the end of a sentence or write a new sentence given two preceding ones. To begin with, we have a recursive set of **axioms**, which are simply the statements we start off with.[1] Together, axioms and deduction rules comprise a **formal system**, and the **theorems** of any particular system are just the sentences that one can produce by applying a finite sequence of deduction rules to its axioms. Once a theorem is proven, it can, in turn, be used as part of the deductions for other theorems, having itself already been established.

[1] We insist that the set of axioms is **recursive**, meaning there is a way to compute whether a sentence *S* is an axiom or not.

Formalists insist that there is no need to view mathematics as anything above and beyond swapping symbols around, erasing them, writing new ones, and so on. This is because these systems work just as well under the assumption that the letters on the page are complete gibberish and do not refer to anything at all. In making this move, formalists avoid hefty questions that arise about the ontological status of mathematical objects. To ask whether a mathematical proposition is true is not to ask about what type of existence, if any, that mathematical objects enjoy, but rather to ask whether, from a particular set of axioms and deduction rules, we are able to derive a sentence which expresses that proposition. For this reason, Hilbert writes, formal systems are valuable for their ability to "determine a position of a given theorem within the system of known truths and their logical connections in such a way that one can clearly see which assumptions are necessary and sufficient to justify the considered truth" (Hilbert, 1902, p. 50. Translation in Murawski, 2002, p. 96).

2. GÖDEL'S FIRST INCOMPLETENESS THEOREM

The two most pressing questions one can ask about a formal system are whether it is **consistent** and whether it is **complete**. If a system is inconsistent, meaning it contains a contradiction, then it is possible to prove any sentence whatsoever, which is disastrous.[2] And an incomplete system is too weak to be welcome, since it leaves out certain statements — there is a sentence *S* such that neither *S* nor ¬*S* is provable. Hilbert's formalist project had as its goal to demonstrate a formal system of arithmetic that was axiomatizable, consistent, and complete. That is, he wanted a recursive set of axioms and deduction rules which could be used to prove, for any statement in the language of arithmetic, either that statement or its negation (but not both). When given the standard interpretation, this system would contain *all* the true arithmetical propositions and *only* the true arithmetical propositions.

"There is no consistent formal system with a recursive set of axioms that can prove all the true propositions about arithmetic."

We now have the conceptual framework in which to appreciate the significance of Gödel's first incompleteness theorem. This result laid bare the hubris of Hilbert's expectation that there would be an axiomatizable, complete, and consistent formal system of arithmetic by showing that none could be all three: if a sufficiently powerful system is axiomatizable and consistent, then it is incomplete. The theorem holds for any consistent formal system that is expressive enough to represent certain predicates about its own sentences, and indeed the most enlightening component of Gödel's proof is his demonstration that "all meta-mathematical statements about the structural properties of expressions in the formal calculus can be accurately *mirrored* within the calculus itself" (Nagel, Newman, & Hofstadter, 2001, p. 80).

Most crucially, if a formal system *F* is consistent and powerful enough to be of real interest to mathematicians, Gödel proved that the property of "being provable within the formal system *F*" is representable *in F itself*. This means that the system is able to talk about its own theorems, even though, when viewed through a different lens, it is just talking about the properties of natural numbers. This collapse of meta-mathematics to mathematics, the ability to reason *about* sentences in the language of arithmetic *in* the language of arithmetic, is at the heart of Gödel's proof.[3] He used this ability to construct a sentence *G*, written in the language of arithmetic, such that neither *G* nor ¬*G* is provable in *F* (the sentence *G* is known as the **Gödel sentence**). This alone directly proves the result: *F* is incomplete in light of its inability to prove *G* and its inability to prove ¬*G*. This, syntactically, is what it means for a system to be incomplete. However, the importance of Gödel's result is magnified in light of the fact that *G* is *true* upon being interpreted meta-mathematically. This has to do with how Gödel constructed *G*, "a sentence purely in the arithmetical vocabulary of number theory that inherits that crucial property of being true if and only if not a theorem of number theory" (Quine, 1976, p. 17). So, because *G* is not a theorem (which is part of how we just showed *F* to be incomplete), it follows immediately that *G* is true. This means there is no consistent formal system with a recursive set of axioms that can prove all the true propositions about arithmetic.

3. INTUITIONISM

We have a clear case for the formalist's dismay about Gödel's first incompleteness theorem. How the intuitionist fares in this regard is also of great importance, and the verdict depends crucially on what it means to them for a mathematical proposition to be *true*. Intuitionists do not believe that mathematical objects enjoy any kind of existence outside the mind, but rather that they "exist only in virtue of our mathematical activity, which consists in mental operations, and have only those properties which they can be recognized by us as having" (Dummett, 2000, p. 5). Thus, a mathematical proposition's truth depends upon our having some idea that proves it, rather than there being a mathematical reality which confers it or a derivation of some string that expresses it within a formal system. Further, their requirement is that this idea be **constructive** in nature. Constructive proofs exhibit the truth of their result directly, rather than by *reductio*-style reasoning.

For the intuitionist, mathematical truths and mathematical proofs are coextensive: a proposition *P* is true if and only if we have a constructive proof which concludes that *P*, and a proposition *P* is

---

[2] It is a quick exercise in classical logic to show that the theoremhood of both *S* and ¬*S* means that every single sentence in the formal language can be proven by the system, which renders the system completely useless.

[3] The way he did this was through a method called **Gödel numbering**, which uniquely assigns a natural number to each sentence in the language of arithmetic. A formal system must be sufficiently powerful to have the expressive ability to reference itself in this way, which is why there are weaker systems (like those which only model addition) that are complete. But because they are so weak, these systems *also* fail to prove all the arithmetical truths, hence the severity of Gödel's result to Hilbert's program.

false if and only if we have a constructive proof which concludes that ¬*P*. What happens when we have neither? A famous example is Goldbach's conjecture, which asserts that every even number greater than 2 can be written as the sum of two primes; at the time of my writing this, there has been no constructive proof either of Goldbach's conjecture or its negation. In the event that someone comes up with a constructive proof of Goldbach's conjecture next Thursday, intuitionists will believe that Goldbach's conjecture is true next Thursday even though it is not true today. More generally, their notion of mathematical truth includes a parameter for tense, and every mathematical proposition is neither true nor false until the time at which it receives the proper kind of proof. That is, the law of excluded middle does not hold.

This is a radical stance on the nature of mathematical truth, one which follows directly from the intuitionist's connection of truth with proof, and its consequences are significant. Mathematicians can no longer bifurcate the world into cases depending on whether a proposition is true or false — a proposition may fail to be true while at the same time failing to be false. Furthermore, "no *a priori* ground exists for supposing we must be capable either of proving the statement or of refuting it," so it is not even inevitable that it will *become* true or false at some point in the future (Dummett, 2000, p. 3).

4. PROOF BEYOND PROOF

How should the intuitionist react to Gödel's first incompleteness theorem? It is not obvious whether they ought to celebrate his result or start engaging in apologetics, so we will consider an argument to each effect, beginning with a celebration. A proposition for which no proof has been given, and no proof of its negation has been given, bears a striking similarity to the sentence *G* at the center of Gödel's proof, which is constructed such that neither *G* nor ¬*G* is a theorem of the formal system *F* and thus shows that *F* is incomplete. It is this similarity which motivates an argument I describe below:

> Formalists insist that mathematical statements in a formal system can be understood merely as strings of symbols with no semantic content, meaning they do not have to be treated as propositions. But the formal systems of interest to mathematicians, like the arithmetic one Hilbert had in mind, are such that, under a given interpretation, the *theorems* express *true* mathematical propositions. Through this lens, the formalist agrees with the intuitionist that truth is coextensive with proof: a mathematical proposition is true if and only if it is provable. The sentence *G* cannot be proven, so we cannot say of *G* that it is true. Likewise, the sentence ¬*G* cannot be proven, so we cannot say of *G* that it is false. So, in the formal system *F*, we cannot assert of every mathematical proposition that it is either true or false (which is what it means, semantically, for *F* to be incomplete). Does this not vindicate the intuitionist's rejection of the law of excluded middle?

This celebratory argument relies on the claim that both formalists and intuitionists view mathematical truth as coextensive with mathematical proof. Strictly speaking, this is correct, but we must be careful not to equivocate on the meaning of "provable." Given their extremely different notions of what constitutes a proof, there will be disagreement about which mathematical propositions are provable, and thus about which mathematical propositions are *true*. In particular, unlike the formalist, the intuitionist is not **system-bound** in their methods of reasoning. The mental constructions that the intuitionist finds acceptable as proofs are not restricted to whatever can be proven through syntactic manipulations within a formal system. Rather, the intuitionist feels at liberty to engage in meta-mathematical reasoning *about* arithmetic that is not restricted to the language of arithmetic itself. The formalist, because of their boundedness to a system's axioms and deduction rules, is not permitted this leeway: every claim they make about arithmetic must be expressed at the level of arithmetic itself, through propositions involving natural numbers. Gödel showed that this is still a very powerful language for one to be able to speak, that it is possible to represent meta-mathematical predicates as propositions which have natural numbers as their object. But because they are not bound to any fixed axiomatization of arithmetic, the intuitionist regards certain mathematical propositions as provable which the formalist does not.

"What falls under the intuitionist's notion of proof, given that it is not captured by a formal system?"

Here is the crucial point: while the formalist cannot assert the truth or falsity of the Gödel sentence *G*, because it is undecidable in their formal system *F*, the intuitionist will *insist* that *G* is true. They claim that they arrive at this conclusion by having the right kind of idea about it, and indeed from the construction of *G* it is clear to anyone who is reasoning *about* the formal system, but who is not limited to the proof methods *of* the formal system, that what *G* says is true. That is, "although the sentence *G* consists only of arithmetic terminology, to establish *G* we must invoke something other than the meanings of arithmetic terminology," which is what the intuitionist, but not the formalist, is able to invoke (Shapiro, 1998, p. 615). Recall that in Gödel's proof, *G* is constructed such that it is true if and only if *G* is not a theorem of the formal system *F*. Because *G* is not a theorem of *F*, it follows immediately that *G* is true. This type of meta-mathematical reasoning, one which inspects what *G* is "saying" above and beyond the arithmetical relationships between natural numbers its symbols express, would constitute the intuitionist's proof of *G*, the one which grounds their claim that *G* expresses a true mathematical proposition.

Thus, the celebratory argument fails: Gödel's first incompleteness theorem does not vindicate the intuitionist's rejection of the law of excluded middle. The intuitionist would not say of *G* that it belongs in the same category as Goldbach's conjecture, a proposition we presently cannot prove or disprove (and may never be able to). Rather, they would surely assert of *G* that it is *true*, employing reasoning that is not system-bound. That is, although *G* fails to be provable in the formal system *F*, it does *not* fail to be a provable mathematical proposition. There is a lingering question, though: what falls under the intuitionist's notion of proof, given that it is not captured by a formal system?

For the moment, we will let this question linger so we can consider an argument seeking to leverage Gödel's result *against* the intuitionists. The dreary argument I present below maintains that incompleteness is just as devastating to intuitionism as it is to formalism:

> Intuitionists view truth as coextensive with proof: a mathematical proposition is true if and only if it is provable. But "as a consequence of Gödel's theorems, truth cannot be equated with provability in any effectively axiomatizable theory," which means it is wrong for the intuitionist to equate truth with provability in this way (Raattkainen, 2005, p. 516). The Gödel sentence *G* is an example of a true but unprovable mathematical proposition, which means, contra the intuitionist, that the extension of truth does not coincide with the extension of proof.

By what we said in response to the first argument, the equivocation here on the meaning of "proof" should be obvious. Call a proposition provable_F if it is provable to a formalist (meaning it is expressed by a string of symbols that can be derived within the formal system *F*) and provable_I if it is provable to an intuitionist (meaning someone has constructed the result in their mind). This dreary argument asserts that because *G* is not provable_F, it is therefore not provable_I. This leap is unfounded, since we have just shown how an intuitionist would reason to prove_I *G*, even though they agree that *G* is not provable_F. Gödel's result shows that not all arithmetical truths are provable_F, but this does not imply that not all arithmetical truths are provable_I. The intuitionist's methods of reasoning are not system-bound, and for this reason they are able to avoid an attack from someone seeking to leverage Gödel's result against them. But, by the same token, they do not get to claim any support from it for their rejection of the law of excluded middle. Here again we can ask, albeit in slightly different language, the same lingering question: what *are* the provable_I propositions?

5. INDEFINITE EXTENSIBILITY

The meaning of "proof_F" clearly differs from the meaning of "proof_I", since the former is about text on a page while the latter is about ideas in one's mind. And we have just shown that the Gödel sentence *G* is provable_I but not provable_F, which means that the extensions of "proof" for the intuitionist and the formalist are also not the same. In fact, I argue there is a more striking difference between them: whereas the formalist's concept of proof has **definite** extension, meaning "it has the same members in all the circumstances in which it exists at all," the intuitionist's concept of proof does not. Even though "its intension is perfectly sharp," there is no fixed set of provable_I arithmetical propositions (Linnebo, 2018, p. 202). "Proof", for the intuitionist, is what Dummett calls an **indefinitely extensible** concept, meaning "for any definite characterization of it, there is a natural extension of this characterization, which yields a more inclusive concept" (Dummett, 1978, p. 195). Any formal system that seeks to include all the provable_I arithmetical propositions will be *missing* a provable_I mathematical proposition, and the reason this is so depends crucially on the meta-mathematical meaning of the Gödel sentence.

It is by virtue of being system-bound that the extension of "proof_F" is definite. Given the axioms and deduction rules of *F*, the theorems of *F* are the same in every possible world — the set of sentences a formal system can prove is modally rigid. But because its methods of proof are fixed in place, they refuse to budge when we might like them to, as in the case of being able to reason meta-mathematically about the Gödel sentence *G*. This is why the full weight of Gödel's first incompleteness theorem comes crashing down on formalists, like a punch they cannot dodge on account of their shoes being nailed to the floor. That their proof methods are fixed means there is a true mathematical proposition the formalist cannot account for: the Gödel sentence, which asserts of itself that it cannot be proven using those methods.

No matter what lengths are taken to avoid it, this is a problem which befalls any sufficiently powerful formal system of arithmetic. One might think that adding the Gödel sentence *G* as an axiom to *F* would make *F* complete, but Gödel showed that formal systems are incomplete not because they *lack* axioms, but rather because they are so expressive that they can represent the predicate "provable_F" about their own sentences. This means that "for any definite characterization of a class of grounds for making an assertion about all natural numbers, there will be a natural extension of it," one which includes its Gödel sentence as a theorem (Dummett, 1978, p. 194). But for each such augmentation, a *new* Gödel sentence strays from the fold. In an extended formal system *F'* which is able to prove *G*, we can form a new Gödel sentence *G'* which asserts its own unprovability in *F'*, making *F'* incomplete. And in an extended formal system *F"* which is able to prove *G'*, we can form a new Gödel sentence *G"* which asserts its own unprovability in *F"*, making *F"* incomplete. And in the extended formal system *F'"*…

In each case, the intuitionist uses meta-mathematical reasoning that is outside the bounds of the formal system at hand in order to prove that the respective Gödel sentence expresses a true proposition. Any formal system of arithmetic has a natural way to be made more inclusive, namely by altering it so that the respective Gödel sentence is a theorem. But this expanded system is also incomplete, because it has *its own* Gödel sentence, and this process can be repeated *ad infinitum*. The fact that, in each case, a fixed axiomatization of acceptable methods of proof can be made more inclusive by meta-mathematically reasoning about the truth of its Gödel sentence means that the extension of "proof_I" is not definite,

> **"Formal systems are bound to a collection of fixed rules in a way that the mind, in its versatile capacity to reason about mathematics, is not."**

but rather is indefinitely extensible. A definite extension would already contain within it *all* of the provable propositions, but in any sufficiently powerful formal system of arithmetic, Gödel showed there is always a true proposition which has yet to be adjoined. The strength of intuitionism and other non-formalist philosophies of mathematics is that they are able to account for the truth of this proposition by reasoning outside the bounds of the formal system.

What intuitionism gets right is that there is something fundamentally informal about mathematical reasoning. The proofs we construct in our minds about natural numbers cannot be fully replicated by any fixed collection of proof methods, since those methods, upon being translated into syntactic rules within a formal system, will fail to prove a true proposition about arithmetic. And no sooner do we augment our characterization of arithmetic to take this proposition into account than a new proposition materializes, asserting of itself that it cannot be proven by our new methods. Intuitionists are unsurprised by Gödel's first incompleteness theorem because they believe the mind's understanding of mathematics has an intricacy that overflows any attempt to axiomatize it – the idea we have of a mathematical truth is "richer and fuller than all the definitions which we can give of it, than all the forms or combinations of signs or of propositions by means of which we can express it" (Boutroux, 1920, p. 203. Translation in Dresden, 1924, p. 36). Formal systems are bound to a collection of fixed rules in a way that the mind, in its versatile capacity to reason about mathematics, is not. Hence, intuitionists are neither aided nor attacked by this ostensibly negative result of Gödel, which applies only to formal systems that are far afield from our intuitive understanding of arithmetic.

ACKNOWLEDGMENTS

Written for Professor Sun-Joo Shin's course PHIL 437: Philosophy of Mathematics.

REFERENCES

Boutroux, P. (1920). *L'idéal scientifique des mathématiciens dans l'antiquité et dans les temps modernes*. Nouveau Monde éditions.

Dresden, A. (1924). Brouwer's contributions to the foundations of mathematics. *Bulletin of the American Mathematical Society*, *30*(1-2).

Dummett, M. (2000). *Elements of intuitionism* (Vol. 39). Oxford University Press.

———. (1978). *Truth and other enigmas*. Harvard University Press.

Hilbert, D. (1902). Über den Satz von der Gleichheit der Basiswinkel im gleich-schenkligen Dreieck. *Proceedings of the London Mathematical Society*, (1).

Hrbáček, K., & Jech, T. (1999). *Introduction to set theory, revised and expanded*. CRC Press.

Linnebo, Ø. (2018). Dummett on indefinite extensibility. *Philosophical Issues*, *28*(1).

Murawski, R. (2002). Truth vs. provability–philosophical and historical remarks. *Logic and Logical Philosophy*, (10).

Nagel, E., Newman, J. R., & Hofstadter, D. R. (2001). *Gödel's proof*. NYU Press.

Quine, W. V. (1976). *The ways of paradox and other essays*. Cambridge, MA: Harvard University Press.

Raattkainen, P. (2005). On the philosophical relevance of Gödel's incompleteness theorems. *Revue internationale de philosophie*, (4).

Shapiro, S. (1998). Induction and indefinite extensibility: The Gödel sentence is true, but did someone change the subject? *Mind*, *107*(427).