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Abstract

A single seller faces a sequence of buyers with unit demand. The buyers are forward-looking and long-lived but vanish (and are replaced) at a constant rate. The arrival time *and* the valuation is private information of each buyer and unobservable to the seller. Any incentive compatible mechanism has to induce truth-telling about the arrival time and the evolution of the valuation.

We derive the optimal stationary mechanism, characterize its qualitative structure, and derive a closed form solution. As the arrival time is private information, the buyer can choose the time at which he reports his arrival. The truth-telling constraint regarding the arrival time can be represented as an optimal stopping problem. The stopping time determines the time at which the buyer decides to participate in the mechanism. The resulting value function of each buyer cannot be too convex and must be continuously differentiable everywhere, reflecting the option value of delaying participation. The optimal mechanism thus induces progressive participation by each buyer: he participates either immediately or at a future random time.

KEYWORDS: Dynamic Mechanism Design, Observable Arrival, Unobservable Arrival, Repeated Sales, Interim Incentive Constraints, Interim Participation Constraints, Stopping Problem, Option Value, Progressive Participation.

JEL CLASSIFICATION: D44, D82, D83.

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1 Introduction

1.1 Motivation

We consider a classic mechanism design problem in a dynamic and stationary environment. The seller wants to repeatedly sell a good (or service) to buyers with randomly evolving valuation. The willingness to pay of each buyer is private information of the buyer and evolves randomly over time. We assume a stationary environment in which each buyer is replaced at random, and with a constant rate, by a new buyer whose initial willingness-to-pay is randomly drawn from a given distribution. The objective of the seller is to find a stationary revenue maximizing policy in this dynamic environment. The choice of policy or mechanism is unrestricted and may consist of leasing contracts, sale contracts, or any other form of dynamic contract.

We depart from the earlier analysis of dynamic mechanisms in our treatment of the participation decision of the buyer. We allow the buyer (he), once he has arrived in the economy, to choose the time at which he enters into a contract with the seller (she). While he can sign a contract with the seller immediately upon arrival, he has the option to postpone the participation decision until a future date. The buyer therefore has the option to wait and sign any contract only after he has received additional information about his willingness to pay. In particular, he can time the acceptance of a contract until he has a sufficiently high willingness to pay. Thus, both the incentive constraints that are in place *after* the buyer has signed the contract and the participation constraints that are in place *before* the buyer has signed the contract are fully responsive to the arrival of new information, and are consequently represented as sequential constraints. In particular, the buyer can enter the contract upon arrival or at any later time. His participation is therefore determined progressively as he receives additional information. For brevity, we sometimes refer to the current setting with interim participation and interim incentive constraints as *progressive mechanism design*.

We can contrast this with the received perspective in *dynamic mechanism design*. There, the seller knows the arrival time of the buyer in the economy and can commit herself to make a single and once-and-for-all offer to the buyer at the moment of arrival. In particular, the seller can commit herself to *never* make another offer to the buyer. These two features: (i) the ability of the seller to time the offer to the arrival time of the buyer and (ii) the ability to refrain from any future offers seem likely to be violated in many economic environments of interest. For example, the consumer clearly has a choice when to sign up for a mobile phone contract, a gym membership, or a service contract for a kitchen appliance. Importantly, as the consumer waits, he may receive more information about his willingness to pay for the product. Thus, relative to the specific assumption in the earlier literature, we allow the arrival time and the identity of the buyer to be private information to the buyer. Consequently, the contract or the menu of contracts cannot be timed to the arrival of the buyer and the contract (or lack of contract) offer cannot be tied to the identity of the buyer. In a stationary environment in which buyers arrive and depart at a balanced rate, we restrict attention to the optimal stationary mechanism.

We view the relaxation of the two above mentioned restrictions as necessary steps to bring the design of dynamic revenue maximizing mechanism closer to many interesting economic applications. To the extent that these restrictions impose additional constraints on the seller, they directly weaken the power of dynamic mechanism design. We therefore investigate the impact of these additional constraints on the ability of the seller to raise revenues from the buyers using dynamic contracts. The additional constraints for the seller are reflected in a larger set of reporting strategies for the buyers. A buyer can misreport both his willingness to pay as well as his arrival time. This creates an option value for the buyer as instead of choosing a contract immediately he can wait and enter into a contract with the seller when it is most favorable for him to do so. Given the menu of contracts offered by the seller, the buyer thus solves an optimal stopping problem to determine when to enter into a contractual relationship with the seller. From the point of view of the buyer, the choice of an optimal contract from the menu therefore has an option element. Subject to the (random) evolution of his type and his willingness to pay, he can choose when to enter into an agreement with the seller. This suggests that the buyer will receive a larger information rent than in the standard dynamic mechanism design framework where the buyer has to sign a contract with the seller immediately.

We develop our analysis in a continuous time setting where the buyer's willingness to pay follows a geometric Brownian motion. The prior distribution of the willingness to pay upon arrival is given exogenously, and paired with the renewal rate in the population, generates an ergodic distribution which forms the stationary environment. The revenue maximizing static mechanism, i.e. the contract which does not condition on a buyer's history, is a leasing contract which offers the good in every period for the posted price that is optimal given the ergodic distribution of the valuations of the buyers.

In the absence of the sequential participation constraint, the revenue maximizing dynamic mechanism would sell the object with probability one and forever at fixed price (see

Bergemann and Strack, 2015). Thus, the object would be sold rather than leased to all buyers who have an initial willingness to pay above a certain threshold. Conversely, all buyers whose initial value is below this threshold would not buy the object, neither at the beginning of time, nor anytime thereafter. In a first pass, we then restrict attention to a sales price policy, which is optimal in the absence of sequential participation constraints, and determine the optimal sales price with the presence of sequential participation constraints. Here, the comparison of thresholds and prices between dynamic and progressive mechanism design are instructive. We find that the threshold for the willingness to pay at which a buyer purchases the object is strictly higher in the progressive model than in the dynamic model without progressive participation constraint. By contrast, the price at which the buyer can acquire the object can be either below or above the price charged in the dynamic setting.

We can gain some initial insight by considering how a buyer would react to the option to buy at a fixed price. In the *dynamic setting*, there would be a threshold type for the buyer who would receive zero expected net surplus at the offered price. In the *progressive setting*, this threshold type could and clearly should delay the purchase until his willingness to pay is sufficiently above the threshold level to guarantee himself a positive net surplus. Thus, at any threshold level, the seller will be able to extract less surplus from the buyer than he could in the presence of a static participation constraint. In response to the weakened ability to extract surplus, the seller has to adjust her policy along the price *and* the quantity margin at the same time. We show that the seller will generally choose to implement a higher threshold for the willingness to pay. Thus, there will be fewer initial sales relative to the static participation constraint. But the seller also adjusts along the dimension of the price and will ask for a price below the price at which the threshold type would have received zero expected net surplus. Interestingly, the price with sequential participation constraints may either be below or above the price charged under the static participation constraint. Most importantly, a gap now arises between the price paid to receive the object and the expected value assigned to the object by the threshold type.

Following the analysis of the optimal price policy under sequential participation constraint, we then show that a single sale price policy is indeed an optimal progressive mechanism in the class of all possible stationary mechanisms. In other words, a single sale price as a specific and simple indirect implementation of a direct mechanism achieves the revenue maximizing optimum. The main challenge for establishing this result is that it is unclear how to handle the progressive participation constraint. As our example with the threshold type illustrates, this constraint will always bind for some type and thus cannot be ignored.

This constraint is non-standard as it states that the value function of the buyer must be the solution to an optimal stopping problem which itself involves the value function. We relax this problem by restricting the buyer to a small set of deviations, namely cut-off strategies which are indexed by the cut-off. This relaxation has the advantage that the buyer's participation strategies can be mapped into $\mathbb{R}$ which allows us to reduce the problem into a static mechanism design problem. This static problem is a variant of the classical setup by Mussa and Rosen (1978) with the non-standard feature that each buyer can (deterministically) increase his type at the cost of multiplicatively decreasing his interim utility. This additional constraint leads to a failure of the first-order approach. We show that the resulting mathematical program can be expressed as a Pontryagin control problem with contact constraints and we develop a verification result for such problems which might be of independent interest. We illustrate the implications that the option to wait has for the effectiveness of dynamic mechanism in a concluding example.

1.2 Related Literature

The analysis of revenue-maximizing mechanism in an environment where the buyer's private information changes over time started with Baron and Besanko (1984) and Besanko (1985). Since these early contributions, the literature has developed considerably in recent years with notable contributions by Courty and Li (2000), Battaglini (2005), Eső and Szentes (2007) and Pavan et al. (2014).[1] These papers derive in increasing generality the dynamic revenue maximizing mechanism. The analysis in these contributions have in common the same set of constraints on the choice of mechanism. The seller has to satisfy all of the sequential incentive constraints, but only a single ex-ante participation constraint. In earlier work, Bergemann and Strack (2015), we considered the same set of constraints in a continuous-time setting where the stochastic process that describes the evolution of the flow utility was governed by a Brownian motion. The continuous-time setting allowed us to obtain additional and explicit results regarding the nature of the optimal allocation policy, which are unavailable in the discrete-time setting. In the present paper, we will use the continuous-time setting again for very similar reasons.

The literature on dynamic mechanism design largely assumes that the arrival time of the buyer is known to the seller and that the seller can make a single, take-it-or-leave-it offer at the moment of the buyer's arrival. In contrast, there is a separate literature that analyzes the

[1] Bergemann and Välimäki (2019) provide a survey into the recent developments of dynamic mechanism design.

optimal sales of a durable good with the recurrent entry of new consumers, and it is directly concerned with the timing of the purchase decision by the buyers. The seminal contribution by Conlisk et al. (1984) considers a durable good model with the entry of a new group of consumers in every period, constant in size and composition. Each buyer has either a low or high value that is persistent. They consider the subgame perfect equilibrium of the game; thus the seller has no commitment. The equilibrium displays a cyclic property. Sobel (1991) considers a durable good model with the entry of new consumers. He extends the equilibrium analysis of Conlisk et al. (1984) to allow for non-stationary equilibria and this enlarges the set of attainable equilibria and payoffs. The model remains restricted to binary and persistent types. The main part of his analysis is concerned with subgame perfect pricing policies by the firm, thus he analyzes the pricing problem for the firm without commitment. In addition, Sobel (1991) describes the optimal sales policy under commitment and establishes that a *stationary price* is the optimal policy (Theorem 4). Board (2008) considers the optimal commitment solution for seller when incoming demand for a durable good varies over time. He characterizes the optimal sequence of prices and allocations in an optimal, possibly time-dependent policy. While he considers a continuum of valuations, he maintains the restriction that the value of each buyer is perfectly persistent and does not change after arrival.[2] Thus, the literature on newly arriving consumer restricts attention to: (i) a sequence of prices rather than general allocation mechanisms, and (ii) perfectly persistent values.

Garrett (2016) offers a notable exception in that he is concerned with unobservable arrival and allows for stochastic values. He considers a stationary environment in continuous time in which each buyer arrives and departs at random times. The private value of each buyers is governed by a Markov process with binary values, low and high. The seller can commit to any deterministic time-dependent sales price policy. The seller maximizes the revenue from a representative buyer. Garrett (2016) provides conditions under which a time-invariant price path is optimal within the class of deterministic price path, and he obtains conditions on the binary values under which a deterministic price cycle prevails in the optimal contract. Garrett (2016) observes that an optimal policy in the class of all dynamic direct mechanisms, one that does not restrict attention to deterministic sale price path (and implied restrictions on reporting types), may lead to very different results and implications.

[2] Besbes and Lobel (2015) consider a related question in a very different environment. They study the revenue-maximizing pricing policy under commitment in a steady state where the consumers have private information across two dimension: the valuation and their willingness to wait. The valuation of the consumer however is constant and the willingness to wait is in terms of a deadline until the value expires. Thus each consumer faces a finite horizon problem without discounting, and the seller maximizes her long-run average revenue.

By contrast, we consider an environment with a continuum of values whose evolution is governed by a geometric Brownian motion. We allow for a general mechanism that can depend in arbitrary ways on the reported values once the buyer has entered the mechanism. We restrict attention to a stationary mechanism. Thus, the seller commits to renew the mechanism in every future period either for newly arriving buyers, or late deciding buyers. In this environment, we establish that a deterministic and time-invariant sale price constitutes a revenue maximizing mechanism in the class of all stationary mechanisms. We discuss the possibility of extending the optimality of our mechanism to allow for general time-dependent mechanism in the penultimate section.

The importance of a privately observed arrival time is also investigated in Deb (2014) and Garrett (2017). In contrast to the present work, these papers do not investigate a stationary environment. Instead, while the mechanism starts at time $t = 0$, the buyer may arrive at a later time. The main concern therefore is how to encourage the early arrivals to contract early. In a setting with either a durable good or a non-durable good, respectively, these authors find that the optimal mechanism treats early arriving participants more favorably than late arriving participants. The late arriving participants face less favorable prices and purchase lower quantities than the early arrivals.

There are related concerns with the emphasis on the ex-ante participation constraints in the literature on dynamic mechanism design that pursue different directions from the one presented here. Lobel and Paes Leme (2019) question the unlimited ability of the seller to commit to make only a single offer to the buyer. They suggest that while the seller may have "positive commitment" power, she may lack in "negative commitment" power. That is, he can commit to any contractual promise, but may not be able to commit never to make any further offer in the future. They show that in a finite horizon model with a sequence of perishable goods, the equilibrium is long-term efficient and that the seller's revenue is a function of the buyer's ex ante utility under a no commitment model. Skreta (2006, 2015) and Deb and Said (2015) also investigate the sequential screening under limited commitment by the seller.

A more radical departure from the ex-ante or interim participation constraint to ex-post participation constraints is suggested in recent work by Krähmer and Strausz (2015) and Bergemann et al. (2017). These papers re-consider the sequential screening model of Courty and Li (2000). In this two-period setting, where information arrives over time and the allocation of a single object can be made in the second period, they impose an ex-post participation rather than an ex-ante participation constraint. In consequence the power of sequential screening is diminished and sometimes the optimal mechanism reduces to the solution of the static mechanism. Ashlagi et al. (2016) investigate the performance guarantees that can be given with ex-post participation constraints in a setting where a monopolist sells k items over k periods.

2 Model

2.1 Payoffs and Allocation

We consider a stationary model with a single seller and a single representative buyer (who we think of as representing a unit continuum of buyers). We alternately refer to the designer as the seller (she) and the agent as the buyer (he).

Time is continuous and indexed by $t \in [0, \infty)$. The buyer departs and gets replaced with a newly arriving buyer at rate $\gamma > 0$. We denote by i the buyer who arrived i-th to the market.[3] We denote the random arrival time of buyer i by $\alpha_i \in \mathbb{R}_+$ and the random departure time by $T_i = \alpha_{i+1} \in \mathbb{R}_+$.

The seller and the buyers discount the future at the same rate $r > 0$. At each point in time t, the buyer demands one unit of the good. The flow valuation of buyer i at time $t \in [\alpha_i, T_i]$ is denoted by $\theta_t^i \in \mathbb{R}_+$, the quantity allocated to buyer i at time t is $x_t^i \in [0, 1]$, and p_t^i is the flow payment from the buyer to the seller. His flow preferences are represented by a (quasi-)linear utility function

$$u_t^i = \theta_t^i x_t^i - p_t^i, \tag{1}$$

We assume that the arrival and departure time of each buyer are independent of his valuation process. The valuation of buyer i, $\theta_{\alpha_i}^i \in \mathbb{R}_+$, at the time of his arrival α_i is distributed according to the cumulative distribution function

$$F : \left[0, \overline{\theta}\right] \to \mathbb{R},$$

with strictly positive, bounded density $f = F' > 0$ on the support. The prior distribution

[3] One can equivalently think of a continuum of buyers that is present at every point in time, where buyers arrive and leave with rate γ. The average behavior of such a continuum of buyers will match the expected behavior of a single representative buyer. The main advantage of the representative buyer model is that it avoids technical issues due to integration over a continuum of independent random variables, which is formally not well defined in standard probability theory, see e.g. Judd (1985).

F is the same for every buyer i and every arrival time α_i.

The valuation of each buyer evolves randomly over time, independent of the valuation of other buyers. We assume that each buyer's valuation $(\theta_t^i)_{t\in[\alpha_i,\infty)}$ follows a geometric Brownian motion, i.e. solves the stochastic differential equation:

$$\mathrm{d}\theta_t^i = \sigma\, \theta_t^i \mathrm{d}W_t\,, \tag{2}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and $\sigma \in \mathbb{R}_+$ is the volatility which measures the speed of information arrival. The geometric Brownian motion forms a martingale and consequently the buyer's best estimate of his valuation at any future point in time is his current valuation, i.e. for all $s \geq t$:

$$\mathbb{E}_t\left[\theta_s^i\right] = \theta_t^i\,.$$

Furthermore, θ_t^i takes only positive values, and so the buyer's valuation for the good is always positive.

Each buyer i seeks to maximize his discounted expected net utility given his valuation $\theta_{\alpha_i}^i$ at his arrival time α_i:

$$\mathbb{E}\left[\int_{\alpha_i}^{T_i} e^{-r(t-\alpha_i)}\left(\theta_t^i x_t^i - p_t^i\right)\mathrm{d}t \mid \theta_{\alpha_i}^i,\, \alpha_i\right].$$

To simplify notation and without loss of generality we assume that the first buyer arrives at time zero $\alpha_0 = 0$.

The seller seeks to maximize the expected discounted net revenue collected from her interaction with the sequence of all buyers:

$$\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{\alpha_i}^{T_i} e^{-r\,t} p_t^i\,\mathrm{d}t\right]\,.$$

This objective captures equivalently the total discounted revenue from a continuum of buyers.

2.2 Stationary Mechanism

A mechanism specifies, after each history, a set of messages for each buyer and the allocation as well as the transfer as a function of the complete history of messages sent by this buyer. Throughout, we impose that the allocation and transfer are independent of a buyers identity i.[4] The allocation process (x_t) specifies whether or not the buyer consumes the good at any point in time. We assume that the allocation of the object is reversible, i.e. the seller can give the buyer an object for some time and then take it away later.

Definition 1 (Mechanism)**.**
A mechanism (x, p) specifies at every point in time $t \in \mathbb{R}_+$, where some buyer i is active $t \in [\alpha_i, T_i)$, the allocation $x_t\left((m_s^i)_{\alpha_i \le s \le t}\right)$ as well as the transfer $p_t\left((m_s^i)_{\alpha_i \le s \le t}\right)$ as a function of the messages $(m_s^i)_{\alpha_i \le s \le t}$ sent by this buyer prior to time t.

A direct mechanism is a mechanism where the buyer reports his arrival and his valuations to the mechanism.

Definition 2 (Direct Mechanism)**.**
A direct mechanism (x, p) specifies at every point in time $t \in \mathbb{R}_+$, where some buyer i is active $t \in [\alpha_i, T_i)$, the allocation $x_t\left(\alpha_i, (\theta_s^i)_{\alpha_i \le s \le t}\right)$ as well as the transfer $p_t\left(\alpha_i, (\theta_s^i)_{\alpha_i \le s \le t}\right)$ as a function of the arrival time α_i and the valuations $(\theta_s^i)_{\alpha_i \le s \le t}$ reported by this buyer prior to time t.

As our environment is stationary, we restrict attention to stationary mechanisms where the allocation and transfers are independent of the arrival time of the buyer. More formally, we require that a buyer who arrives at time α and whose valuations follows the path $(\theta_s)_{s \in [\alpha, T]}$, receives the same allocation as a buyer who arrives at time α' and his valuations follows the same path of valuations shifted by the difference in arrival times, i.e. $\theta'_s = \theta_{s+(\alpha-\alpha')}$ for all $s \in [\alpha', \alpha' + T - \alpha]$. Thus, the seller cannot discriminate the buyer based on his arrival time.

[4]The restriction to mechanisms where each buyer's allocation and transfer is only a function of his own messages is without loss of generality in the following sense: In the continuum interpretation of the model, the seller cannot detect if a buyer misreports his arrival time. Furthermore as the valuations of different buyers are independent the revenue maximization problem is independent between different buyers and so will be the optimal mechanism.

Definition 3 (Stationary Direct Mechanism)**.**
A direct mechanism (x, p) is stationary if for all arrival times α, α' and valuation paths θ:

$$x_t\left(\alpha, (\theta_s)_{\alpha \le s \le t}\right) = x_{t+(\alpha'-\alpha)}\left(\alpha', (\theta_s)_{\alpha \le s \le t}\right),$$
$$p_t\left(\alpha, (\theta_s)_{\alpha \le s \le t}\right) = p_{t+(\alpha'-\alpha)}\left(\alpha', (\theta_s)_{\alpha \le s \le t}\right).$$

2.3 Progressive Mechanism

By the revelation principle we can, without loss of generality, restrict attention to *direct mechanisms* where it is optimal for the buyer to report his arrival time α and his valuation θ_t truthfully at every point in time t. Define the indirect utility $V_\alpha : \mathbb{R}_+ \to \mathbb{R}$ of a buyer who arrives at time α with a value of θ_α and reports his arrival and his valuations $(\theta_t)_t$ truthfully by:

$$\begin{aligned} V_\alpha(\theta) &= \mathbb{E}\left[\int_{\alpha_i}^{T} e^{-r\,(t-\alpha_i)}\left\{\theta_t^i x_t^i - p_t^i\right\} \mathrm{d}t \mid \alpha_i = \alpha, \theta_{\alpha_i}^i = \theta\right] \\ &= \mathbb{E}\left[\int_{\alpha_i}^{\infty} e^{-(r+\gamma)\,(t-\alpha_i)}\left\{\theta_t^i x_t^i - p_t^i\right\} \mathrm{d}t \mid \alpha_i = \alpha, \theta_{\alpha_i}^i = \theta\right]. \end{aligned}$$

The above equality follows immediately from the law of iterated expectations and the fact that the departure time T_i of the buyer is independent of the arrival time α_i and the valuation process θ^i and hence of x_t^i, p_t^i.[5]

It is optimal for the buyer to report truthfully if

$$V_\alpha(\theta_\alpha) \ge \sup_{\hat{\alpha} \ge \alpha_i, (\hat{\theta}_t)} \mathbb{E}\left[\int_{\hat{\alpha}}^{\infty} e^{-(r+\gamma)\,(t-\alpha_i)}\left\{\theta_t^i\, \hat{x}_t^i - \hat{p}_t^i)\right\} \mathrm{d}t \mid \alpha_i = \alpha, \theta_{\alpha_i}^i = \theta\right], \tag{IC}$$

where the allocation $\hat{x}_t^i = x_t(\hat{\alpha}, (\hat{\theta}_s)_{\hat{\alpha} \le s \le t})$ as well as the payment $\hat{p}_t^i = p_t(\hat{\alpha}, (\hat{\theta}_s)_{\hat{\alpha} \le s \le t}$ is a function of the reported arrival time $\hat{\alpha}$ as well as all previously reported valuations $(\hat{\theta}_s)_{\hat{\alpha} \le s \le t}$. We note here that the supremum in (IC) is taken over stopping times $\hat{\alpha}$ as the buyer can condition his reported arrival on his willingness to pay for the good.

We restriction attention to mechanisms where the buyer participates voluntarily, i.e. for all arrival times α and all initial values θ_α, the buyer's expected utility from participating in the mechanism is non-negative:

$$V_\alpha(\theta_\alpha) \ge 0. \tag{PC}$$

[5] See Lemma 4 in the Appendix.

While imposing incentive compatibility constraints (IC) as well as participation constraints (PC) is standard in the literature on (dynamic) mechanism design, we note that the incentive compatibility constrained (IC) imposed here is stronger than the one usually imposed in the literature. As the arrival time α is not observable to the seller, she has to provide incentives for the buyer to report his arrival truthfully. In fact the incentive constraint (IC) we impose implies the participation constraint (PC) as the buyer can always decide to never report his arrival $\hat{\alpha} = \infty$. The seller seeks to maximize her revenue subject to the incentive and participation constraints, and we refer to it as the *progressive mechanism* design problem.

3 Revenue Equivalence

We denote by $\mathcal{M}$ the set of all incentive compatible stationary mechanisms where every buyer participates voluntarily. A first observation that follows from the independence of the values across the buyers is that we can rewrite the objective of the seller only in terms of the revenue collected from the interaction with a *single* buyer.

Lemma 1 (Expected Revenue)**.**
The expected discounted revenue in the optimal mechanism equals

$$\frac{r+\gamma}{r} \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{T_i} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right],$$

where i is an arbitrary buyer.

The proofs for all the results are relegated to the Appendix. In a stationary direct mechanism, the allocation and transfer depend only on the time which elapsed since the buyer arrived. We can therefore, without loss of generality, assume that the buyer arrived at time zero $\alpha = 0$ (or chose $i = 0$), to determine the revenue the seller derives from her interaction with the buyer. To simplify the notation, we will drop the sub-index indicating the buyer's arrival as well as his identity and denote by $V(\theta_0)$, the indirect utility of the buyer when he arrived at time 0 with initial valuation θ_0.

As a first step in the analysis, we establish that the progressive mechanism design problem can be related to a static auxiliary problem. In this static problem the buyer reports only his initial valuation and the seller chooses a discounted expected quantity $q \in \mathbb{R}_+$ to allocate to the buyer. It turns out that in any incentive compatible mechanism, both the value of the buyer as well as the revenue of the seller are only a function of this quantity.

We define the "expected aggregate quantity" $q : \Theta \to \mathbb{R}_+$ which is allocated to a buyer with initial valuation θ_0 by

$$q(\theta_0) \triangleq \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma\, W_t\right) \mathrm{d}t \mid \theta_0\right]. \tag{3}$$

The first term inside the integral is simply the discounted quantity in period t:

$$e^{-(r+\gamma)\,t} x_t.$$

The second term is the derivative of the valuation θ_t in period t with respect to the initial value θ_0:[6]

$$\frac{\mathrm{d}\theta_t}{\mathrm{d}\theta_0} = \exp\left(-\frac{\sigma^2}{2}t + \sigma\, W_t\right),$$

The above derivative represents the influence that the initial value θ_0 has on the future state θ_t. In Bergemann and Strack (2015), we referred to it as stochastic flow, and it is the analogue of the impulse response function in discrete time dynamic mechanism (see Pavan et al. (2014), Definition 3). Thus, the "expected aggregate quantity" $q(\theta_0)$ weighs the discounted quantity with the corresponding stochastic flow, or information rent that emanates from the initial value.

As the quantity x_t is bounded between 0 and 1 and the exponential term is a martingale, it follows that the aggregate quantity is bounded as well, i.e. for all $\theta \in [0, \overline{\theta}]$

$$0 \leq q(\theta) \leq \frac{1}{r+\gamma}. \tag{4}$$

We can complete the description of the static auxiliary problem with the introduction of the virtual value and denote by:

$$J(\theta) \triangleq \theta - \frac{1-F(\theta)}{f(\theta)} \tag{5}$$

the "virtual flow value" of the buyer upon arrival to the mechanism. We denote by

$$\theta^\circ \triangleq \inf\{\theta\colon J(\theta) \geq 0\}, \tag{6}$$

the lowest type with a non-negative virtual value. We assume that the distribution of initial

[6]Here we used that the geometric Brownian motion can be explicitly represented as: $\theta_t = \theta_0 \exp\left(\frac{\sigma^2}{2}t + \sigma W_t\right)$.

valuations is such that $\theta \mapsto \min\{0, f(\theta)J(\theta)\}$ is non-decreasing.[7]

The expected quantity q and the virtual utility J are useful as they allow us to completely summarize the expected discounted revenue of the seller and the value of the buyer:

Proposition 1 (Revenue Equivalence)**.**
In any incentive compatible mechanism, the value of the buyer with initial valuation θ is:

$$V(\theta) = \int_0^\theta q(z)\mathrm{d}z + V(0)\,, \tag{7}$$

and the expected discounted revenue of the seller is:

$$\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} p_t \,\mathrm{d}t\right] = \int_0^{\overline{\theta}} J(\theta)q(\theta)\mathrm{d}F(\theta) - V(0)\,. \tag{8}$$

Proposition 1 allows us to express the objective functions of the buyer and the seller in terms of the discounted quantities q only. This result is an application of a general revenue equivalence result, stated as Theorem 1 in Bergemann and Strack (2015). The particularly transparent reduction here comes from the presence of the geometric Brownian motion and the unit demand. By contrast, the reduction to an auxiliary static program extends to a wide class of stochastic process and allocation problems. The next result establishes that the function q must be increasing in any incentive compatible mechanism.

Proposition 2 (Monotonicity of Discounted Quantity)**.**
In any incentive compatible mechanism the aggregate quantity q increases.

Proposition 1 and 2 follow from the the truth-telling constraint at time zero. Thus, they provide only necessary conditions for incentive compatibility as they omit: (i) the possibility to misreport the arrival, and (ii) the buyer's truth-telling constraints after time zero. In the next section, we establish that these two necessary conditions are sufficient if arrivals are *observable*. Namely, for every monotone expected aggregate quantity q there exists a mechanism that implements it (the mechanism where x_t is constant). But significantly, the monotonicity of q will not be sufficient with *unobservable* arrival.

[7]This is a weak technical assumption which is satisfied for most standard distributions like the uniform distribution, the exponential distribution, or the log-normal distribution. For example for the uniform distribution $U([0,\overline{\theta}])$ we have that $f(\theta)J(\theta) = \frac{2\theta-\overline{\theta}}{\overline{\theta}}$ which is increasing in θ. For the exponential distribution with mean $\mu > 0$ we have that $\min\{0, f(\theta)J(\theta)\} = \min\left\{0, \left(\frac{\theta}{\mu} - 1\right)\exp\left(-\frac{1}{\mu}\theta\right)\right\}$ which is also increasing in θ.

4 Sales Contract

We will derive the revenue maximizing mechanism for the seller when she does not observe the arrival time of the buyer in Section 5. As a point of reference, it will be instructive for us to first understand what the seller would do if the (individual) arrival time of each buyer would be observable by the seller. With observable arrival, the optimal solution can be implemented by sales contract. We first review these results in Section 4.1 and then investigate in Section 4.2 how a sales contract performs with unobservable arrival .

4.1 Optimal Contract with Observable Arrival

With observable arrival time by the buyer, we are in the canonical dynamic mechanism design environment. In earlier work, Bergemann and Strack (2015), we derived the revenue maximizing mechanism for a general environment of allocation problems and stochastic processes in continuous time. The current problem of interest, unit demand with values governed by a geometric Brownian motion was a canonical problem of this general environment. We establish that an implementation of the optimal mechanism is to offer the product for sale at an optimally determined price P, see Proposition 8 of Bergemann and Strack (2015).

We described the revenue of the seller in Proposition 1. By the revenue equivalence formula (8), the revenue of the seller is determined by the flow virtual value when the uncertainty is given by the geometric Brownian motion. Thus, the optimal mechanism awards the object to the buyer if his virtual value is positive upon arrival, formally if and only if

$$J(\theta_0) \geq 0.$$

Hence, it is optimal to maximize $q(\theta_0)$ if $J(\theta_0) \geq 0$ and minimize it otherwise. The optimal allocation then awards the object to the buyer at all times $s \geq 0$ if and only if his initial valuation θ_0 at arrival time $t = 0$ is sufficiently high:

$$x_s = \begin{cases} 1, & \text{if } \theta_0 \geq \theta^\circ; \\ 0, & \text{otherwise;} \end{cases}$$

where the critical value threshold θ° is determined by

$$J(\theta^\circ) = 0.$$

Thus, the buyer receives the object *forever* whenever his initial valuation θ_0 is above the threshold value θ°. With observable arrivals this allocation can be implemented in a sales contract where the seller charges a sales price of $\theta^\circ/(r+\gamma)$, which entitles the buyer to ownership and continued consumption at all future times. An revenue-equivalent implementation would be to sell the good at time $t = 0$ and then charge the buyer a constant flow price of

$$p^\circ = \theta^\circ,$$

in all future periods, independent of his future value θ_s, for all $s \geq 0$. Thus, the indirect utility of the buyer when his arrival is observable equals

$$V(\theta_0) = \max\left\{0, \frac{\theta_0 - \theta^\circ}{r+\gamma}\right\}.$$

4.2 Sales Contract with Unobservable Arrival

We now abandon the restrictive informational assumption of observable arrival and let the arrival time be private information to each buyer. Let us first consider what would happen if the seller were to maintain the above sales policy and offer the object for sale at the flow price p, which could be the optimal observable price p°, as a stationary contract, at time $t = 0$ and all future times. As we show in this section, any newly arriving buyers with value close to p would conclude that rather than buy immediately, he should wait until he learns more about his value, and purchase the object if and only if he learned that he has a sufficiently high valuation for the object. Thus, the sale would occur (i) later and (ii) to fewer buyers.

When deciding on the optimal purchase time in a sales contract the buyer faces an optimal stopping problem. Purchasing the good later forces the buyer to delay his consumption, but allows him to learn more about his value for the good and to potentially avoid an ex-post sub-optimal purchasing decision. Recall that we denote by T the random time at which the buyer leaves the market. If the buyer acquires the good at time t with valuation θ_t his expected continuation utility is given by

$$\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}(\theta_s - p)\,\mathrm{d}s\right] = (\theta_t - p)\,\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}\mathrm{d}s\right] = \frac{\theta_t - p}{r+\gamma}.$$

The first equality in the above equation follows from the fact that θ is a martingale (independent of T) and thus the buyer's value at time t is his best estimate of his value at later points in time. The second equality follows as the time T at which the buyer leaves the market and thus stops consuming the good is (from a time t perspective) exponentially distributed with mean $t+1/\gamma$. The time τ at which the buyer optimally purchases the good thus solves the stopping problem:

$$\sup_{\tau} \frac{1}{r+\gamma} \mathbb{E}\left[e^{-r\,\tau}\, \mathbf{1}_{\{\tau<T\}}\left(\theta_\tau - p\right)\right] .$$

As the buyer leaves the market with rate γ this problem is equivalent to the problem where the discount rate is given by $(r+\gamma)$, i.e. the buyer solves the stopping problem

$$\sup_{\tau} \frac{1}{r+\gamma} \mathbb{E}\left[e^{-(r+\gamma)\,\tau}\ \left(\theta_\tau - p\right)\right] . \tag{9}$$

The stopping problem given in (9) is the classical irreversible investment problem analyzed in Dixit and Pindyck (1994, Chapter 5, p.135 ff.). For a given sales price p, it leads to a characterization of the buyer's behavior in terms of a threshold $w(p)$ that his valuation θ_τ needs to reach at the stopping time τ.[8]

To simplify notation, we define a constant β that summarizes the discount rate r, the renewal rate γ and the variance σ^2 in a manner relevant for the stopping problem :

$$\beta \triangleq \frac{1}{2} + \frac{1}{2}\sqrt{1+8\frac{r+\gamma}{\sigma^2}} > 1\,. \tag{10}$$

Proposition 3 (Sales Contract)**.**
In a sales contract with flow price p, the buyer acquires the object once his valuation θ reaches a time independent threshold $w(p)$ given by

$$w(p) \triangleq \frac{\beta}{\beta-1}p\,. \tag{11}$$

The buyer's value in this sales contract is given by

$$V(\theta) = \begin{cases} \frac{1}{1+\gamma}\left(\frac{\theta}{w(p)}\right)^{\beta}(w(p)-p)\,, & \text{if } \theta \le w(p);\\ \frac{1}{1+\gamma}\,(w(p)-p)\,, & \text{if } \theta \ge w(p). \end{cases}$$

[8] Dixit and Pindyck (1994) consider an investment problem with a real asset. There, the geometric Brownian motion may have a positive drift, $\alpha > 0$. The positive quadratic root in their equation (16) becomes (10) after setting the growth rate α, the drift of the geometric Brownian, to zero, or $\alpha = 0$. Their discount rate ρ becomes in our setting the sum of discount rate and renewal rate, thus $\rho = r+\gamma$, and the difference between discount rate and growth, $\delta = \rho - \alpha$, is then simply the discount rate, or $\delta = \rho$.

We assumed that the valuation evolves according to a geometric Brownian motion for two reasons that have now become apparent: First, it is invaluable for tractability. It allows us to calculate the option value the buyer receives from delaying his decision to enter a contract in closed form. It is exactly for this reason that most of the literature on irreversible investment and the option value has focused on the geometric Brownian motion (c.f. Dixit and Pindyck, 1994). As a second benefit, this enables us to relate and interpret some of our results through the lens of this classical literature. In particular, the buyer's stopping problem when he is offered a fixed price sales contract becomes a special case of the framework analyzed in Dixit and Pindyck (1994).

We denote by $\tau_{w(p)}$ the (random) time at which the buyer purchases the good:

$$\tau_{w(p)} \triangleq \inf\{t \colon \theta_t \geq w(p)\}.$$

As $w(p) > p$, the buyer only purchases the good once his valuation is sufficiently above the price p charged for the object. Thus, a buyer who starts with an initial value of θ_0 below the threshold $w(p)$ expects to wait some random time until he hits any given threshold $w(p)$. With the geometric Brownian motion, we can compute the expected discounted time for a buyer with initial value θ_0 to hit any arbitrary valuation threshold x.

Lemma 2 (Expected Discounted Time)**.**
The expected discounted time $\tau_x = \inf\{t \colon \theta_t \geq x\}$ until a buyer's valuation exceed a threshold x conditional on the initial valuation θ_0 is given by

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} \mid \theta_0\right] = \min\left\{\left(\frac{\theta_0}{x}\right)^{\beta}, 1\right\}. \tag{12}$$

Thus if the initial value θ_0 exceeds the threshold x, then the expected discounted time is simply 1, in other words there is no waiting at all. By contrast, if the initial value θ_0 is below the threshold x, then the expected discounted time is smaller when the gap between the initial value θ_0 and target threshold x is larger. The magnitude of the discounting is again determined entirely by the constant β which summarizes the primitives of the dynamic environment, namely r, γ and σ^2, as defined earlier in (10).

Intuitively, the buyer has an option value of waiting and learning more about his valuation of the good and only purchases once the forgone utility of not purchasing the good is sufficiently high. This is in sharp contrast to the classical dynamic mechanism design approach where the arrival time of the buyer is observable. When the arrival time is observable the seller can commit herself to not sell to the buyer in the future if the buyer does not purchase the good immediately and thus the buyer can not delay his purchasing decision. The buyer thus always buys the good immediately if his valuation exceeds the price p. The information rent that the buyer gains from his ability to delay his purchasing decision is called the "option value" and equals

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_{w(p)}}(w(p)-p)\right] - \max\left\{(w(p)-p), 0\right\} . \tag{13}$$

From a dynamic mechanism design perspective the option value given in (13) corresponds to an additional information rent the buyer receives due to his ability to delay entering a contractual relation with the seller. As the option value is always positive, the buyer is, for any fixed mechanism, unambiguously better off if he can delay his purchasing decision.

In contrast the effect of the buyer's ability to delay the purchase on the seller's revenue is ambiguous in a sales contract. When the buyer delays his purchase the revenue of the seller decreases. But to the extent, that some types of the buyer who would not have bought the object upon arrival will do now later on, and after a sufficiently large positive shock on their valuation, there are now additional revenues accruing to the seller.

Using the characterization of the purchase behavior of the buyer in Proposition 3 and standard stochastic calculus arguments, we can completely describe the seller's average revenue for a given sales contract.

Proposition 4 (Revenue of Sales Contract)**.**
The flow revenue per time in a sales contract with flow price p is given by

$$R_{sales}(p) = \frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\,\frac{\theta}{p}\right)^{\beta}, 1\right\} f(\theta)\,\mathrm{d}\theta\,. \tag{14}$$

Equation (14) reduces the problem of finding an optimal sales contract to a simple single dimensional maximization problem over the price. It is worth noting that the revenue up to a linear scaling depends on r, γ, σ only through β which implies that the optimal sales price is only a function of β and the distribution of initial valuations F.

The expression inside the integral of (14) represent the expected quantity to be sold to a buyer with initial value θ. In contrast to a standard revenue function under unit demand, the realized quantities are not merely 0 or 1. Rather, the sellers offers positive quantity to all buyers, namely

$$\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^{\beta},1\right\}. \tag{15}$$

This expression reflects the expected discounted time the object is consumed for those buyers who have an initial value below the optimal purchase threshold $w(p) = \frac{\beta}{\beta-1}p$ derived in Proposition 3. The complete expression (15) then follow from Lemma 2 as the expected discounted probability of a sale to a buyer with initial value θ. Thus, an increase in the sales price p uniform lowers the probability of a sale for every value θ. The problem for the seller with unobservable arrivals is therefore how to respond to slower and more selective sales.

4.3 Failure of Incentive Compatibility with Unobservable Arrival

We can now summarize some of our findings regarding the performance of a sales contract in an environment with unobservable arrival. Suppose the seller would like to implement the sales contract that was revenue maximizing under observable arrival in the new environment. This sales contract sets the flow price p° equal to the threshold value θ°. In this contract, any buyer with an initial value θ_0 below θ° never gets the object and makes no payment and thus his utility from the contract equals zero. We can now ask whether this contract remains incentive compatible in the environment with unobservable arrival times. As shown in Proposition 3 the buyer reacts to this contract by reporting his arrival only once his value exceeds

$$w(p^{\circ}) = w(\theta^{\circ}) = \frac{\beta}{\beta-1}\theta^{\circ} > \theta^{\circ}.$$

Hence all buyers with low valuations would deviate by not reporting their arrival immediately, and the optimal contract with observable arrivals can not be implemented with unobservable arrivals.

We can illustrate the payoff consequences by comparing the value functions of the buyer across the two informational environments. The blue line depicts the value function for the buyer in the setting with observable arrival time. The value is zero for all values below the threshold θ° and then a linear function of the initial value. Notably, the value function has a kink at the threshold level θ°. The red curve depicts the value function when the sales contract is offered at the above terms as a stationary contract. Now, the value function is smooth everywhere, and coincides with blue curve whenever the initial value weakly exceeds the critical type $w(p^{\circ})$ Importantly, for all values θ_0 below $w(p^{\circ})$, the red curve is above the blue curve, which depicts the option value as expressed by (13). Notably, the value is strictly

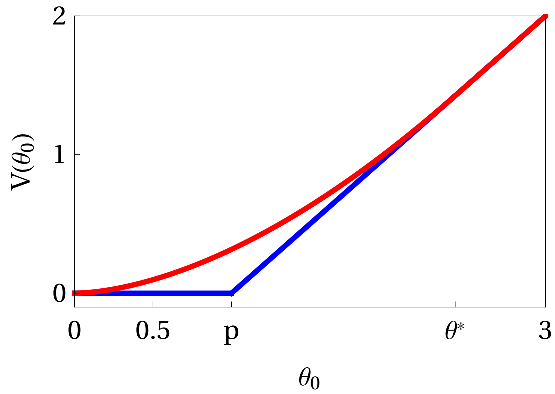


Figure 1: This figure displays the value of the buyer as a function of her initial valuation in a sales contract with flow price $p = 1$ when she has to participate immediately (blue) and when she can delay her arrival (red) when $\beta = 1.7$.

positive for all initial values which expresses the fact that the option value guarantees every value θ_0 an information rent, quite distinct from the environment with observable arrival.

Perhaps surprisingly then, using a sequence of relaxation arguments we prove in Section 5, that the optimal mechanism (in the space of all incentive compatible mechanisms when the buyers arrival to the mechanism is unobservable) remains a sales contract. Thus, (14) can be used to identify the optimal mechanism. But importantly, as the current analysis suggests, there is going to bea large gap between the optimal flow price p and the optimal threshold $w(p)$ with $p < w(p)$.

5 The Optimal Mechanism

The discussion in the previous section illustrates that the first order approach will in general fail once the buyer can misreport his arrival time. To solve this problem we will employ the following strategy: First, we will identify particularly tractable necessary conditions for the truthful reporting of arrivals, by considering a specific class of deviations in the arrival time dimension. We then find the optimal mechanisms for the relaxed problem where we impose only these necessary conditions using a novel result on optimization theory we develop. Finally, we will verify that in this mechanism it is indeed optimal to report the arrival time truthfully.

5.1 Truthful Reporting of Arrivals

In the first step we find a necessary condition such that the buyer wants to report his arrival immediately. Observe that if it were optimal for the buyer to reveal his presence to the mechanism immediately, then the value from revealing his presence at any stopping time $\hat{\alpha}$ must be smaller than revealing his presence at time zero. As the buyer can condition the time at which he reports his arrival to the mechanism on his past valuations, the following constraint must hold for all stopping times $\hat{\alpha}$ which may depend on the buyers valuation path $(\theta_t)_t$:[9]

$$V(\theta_0) \geq \sup_{\hat{\alpha}} \mathbb{E}\left[e^{-(r+\gamma)\,\hat{\alpha}} V(\theta_{\hat{\alpha}}) \mid \theta_0\right] . \tag{IC-A}$$

We first show that the buyer's value function V in any incentive compatible mechanism must be continuously differentiable and convex.

Proposition 5 (Convexity of Value Function)**.**
The value function in any incentive compatible mechanism is continuously differentiable and convex.

The discussion in Subsection 4.3 illustrated that the indirect utility need not to be continuously differentiable in the optimal mechanism if the buyers arrival time is observable. Intuitively, the constraint that the buyer must find it optimally to report his arrival immediately, (IC-A) implies that there cannot be kinks in the value function as this would imply a first order gain for the buyer from the information he would get by waiting to report his arrival. As the cost of waiting due to discounting are second order this implies that a mechanism with a kinked indirect utility can not be incentive compatible.

In the next step, we will relax the problem by restricting the buyer to a small class of deviations in reporting his arrival. The class of deviations we are going to consider is to have the buyer report his arrival the first time his valuation crosses a time independent cut-off $x > \theta_0$:

$$\tau_x = \inf\{t \geq 0 \colon \theta_t \geq x\} .$$

Note, that the optimal deviation of the buyer will not (necessarily) be of this form for every direct mechanism. By restricting to deviations of this form we hope that in the optimal mechanism the optimal deviation will be of this form and the restriction is non-binding.

[9] This is a version of the revelation principle as the seller can replicate every outcome where the buyer does not report his arrival immediately in a contract where the buyer reveals his arrival immediately, but never gets the object before he would have revealed his arrival in the original contract.

5.2 Information Rents Associated with Unobservable Arrival

We established in Lemma 2 that the payoff from deviating to τ_x when reporting the arrival time, while maintaining to report values truthfully, is given by:

$$\mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} V(v_{\tau_x}) \mid \theta_0\right] = V(x)\left(\frac{\theta_0}{x}\right)^{\beta}$$

where $\beta > 1$ was defined in (10). The term

$$\left(\frac{\theta_0}{x}\right)^{\beta}$$

captures the discount factor caused by the time the buyer has to wait to reach a value of x before participating in the mechanism. When the buyer then participates in the mechanism he receives the indirect utility $V(x)$ of a buyer whose initial value equals x. Now, in any mechanism where (IC-A) is satisfied the buyer does not want to deviate to the strategy τ_x we must have

$$V(\theta_0) \geq V(x)\left(\frac{\theta_0}{x}\right)^{\beta} \Leftrightarrow V(x)x^{-\beta} \leq V(\theta_0)\theta_0^{-\beta}. \tag{16}$$

As (16) holds for all θ_0 and $x > \theta_0$, we have that the buyer does not want to deviate to any reporting strategy $(\tau_x)_{x>\theta_0}$ if and only if $V(x)x^{-\beta}$ is decreasing. Taking derivatives yields that this is the case whenever[10]

$$V'(x) \leq \beta \frac{V(x)}{x}\,. \tag{17}$$

By the earlier revenue equivalence result, see Proposition 1, the derivative of the value function $V(\theta)$ is equal to the aggregate quantity $q(\theta)$. We therefore have the following proposition that derives a necessary condition on the aggregate quantity q for it to be optimal for the buyer to report his arrival truthfully.

Proposition 6 (Upper Bound on Discounted Quantities)**.**
The aggregate quantity is bounded from above by

$$q(\theta_0) = V'(\theta_0) \leq \beta \frac{V(\theta_0)}{\theta_0} \tag{18}$$

in any mechanism where it is optimal to report arrivals truthfully, i.e. that satisfies (IC-A)*.*

Intuitively, (18) bounds the discounted quantity a buyer of initial type θ_0 can receive.

[10] $0 \geq V'(x)x^{-\beta} - \beta x^{-\beta-1}V(x) \Rightarrow V'(x) \leq \beta \frac{V(x)}{x}$.

Note, that (18) is always satisfied if the value function of all initial values θ_0 of the buyer from participating in the mechanism is sufficiently high. Intuitively, due to discounting the buyer does not want to delay reporting his arrival when the value from participating is high. As we can always increase the value to all types of the buyer, by possibly offering a subsidy to the lowest type, we can reformulate (18) as a lower bound on the value $V(0)$ of the lowest type $\theta_0 = 0$.

Proposition 7 (Lower Bound on Information Rent)**.**
In any mechanism which satisfies (IC-A) *we have that*

$$V(0) \geq \sup_{\theta \in \Theta} \left(\frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z) \mathrm{d}z \right).$$

The above result establishes a lower bound on the cost of providing the buyer with incentives to report his arrival time truthfully. This lower bound depends only on the allocation q. Intuitively, the seller may need to pay subsidies independent of the buyer's type to provide incentives for the buyer to report his arrival time truthfully if the quantity q is too convex and the option value of waiting is thus too high.[11] The subsidy would correspond to a payment made to the buyer upon arrival and independent of his reported value θ_0. Such a scheme makes delaying the arrival costly to the buyer due to discounting and it is potentially very costly as it requires the seller to pay the buyer just for "showing up". We will show that in the optimal mechanism this issue will not be relevant as the optimal mechanism does not reward the buyer merely for arriving.[12]

As a consequence of Proposition 7 we get an upper bound on the revenue in any incentive compatible mechanism.

Corollary 1 (Revenue Bound)**.**
An upper bound on the revenue in any incentive compatible mechanism is given by

$$\int_0^{\overline{\theta}} q(z) J(z) \mathrm{d}F(z) - \max_{\theta \in [0, \overline{\theta}]} \left(\frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z) \mathrm{d}z \right). \tag{19}$$

[11] An immediate corollary from this formula is that it is infinitely costly to implement a policy which leads to a value function V that admits a convex kink and thus has an infinite derivative $V' = q$ at some point as argued in Proposition 5.

[12] Such subsidy schemes are also discussed in Gershkov et al. (2015, 2018) in a context where the buyer's values do not evolve over time. In contrast to our result Gershkov et al. (2015) show that to implement the efficient allocation in this context such subsidies are sometimes necessary in order to to incentivize the buyers to report their arrival time truthfully.

The upper bound on revenue in (19) is obtained by considering only a small class of deviations. In particular, the buyer is only allowed to misreport his arrival via simple threshold strategies, where he enters the mechanism once his valuation is sufficiently high. Economically,

$$V(0) = \max_{\theta\in[0,\bar{\theta}]} \left(\frac{\theta q(\theta)}{\beta} - \int_0^\theta q(z)\mathrm{d}z \right)$$

is a lower bound on the information rent the buyer must receive to ensure that he reports his arrival truthfully in a mechanism which implements the allocation q. As discussed before, this information rent is payed to the buyer in the form of a transfer that is independent of his consumption and thus even those types receive who never consume the object. We note that due to the maximum this information rent can not be rewritten as an expectation and thus is fundamentally different from the classical information rent term. As a consequence, pointwise maximization can not be used to find the optimal contract even in the relaxed problem. We next develop the mathematical tools to deal with this type of non-standard maximization problem.

5.3 The Optimal Mechanism

We now characterize the optimal mechanism. To do so we proceed by first finding the allocation q that maximizes the upper bound on revenue (19). Second, we are going to construct an incentive compatible mechanism that implements this allocation. As (19) is an upper bound on the revenue, in any incentive compatible mechanism, we then found a revenue maximizing mechanism.

A mathematical challenge is that, due to the information rent from arrivals, the relaxed problem (19) is non-local and non-linear in the quantity q. A change of the quantity for one type can affect the surplus extracted from all higher and lower types. Consider the relaxed problem of finding the revenue maximizing mechanism such that the buyer never wants to misreport his arrival using a cut-off stopping time. By Proposition 6, the indirect utility V

of the buyer in this mechanism solves the optimization problem:

$$\max_{V} \int_{\underline{\theta}}^{\overline{\theta}} V'(z) J(z)\, f(z)\, \mathrm{d}z - V(0), \tag{20}$$

subject to

$$V'(\theta) \in \left[0, \frac{1}{r+\gamma}\right] \text{ for all } \theta, \tag{21}$$

$$V \text{ is convex,} \tag{22}$$

$$V'(\theta) \le \beta \frac{V(\theta)}{\theta} \text{ for all } \theta\,. \tag{23}$$

We will further relax the problem by initially ignoring the convexity constraint (22) and later verifying that the relaxed solution indeed satisfies the convexity condition. By the revenue equivalence result, Proposition 1, we can restate the allocation problem in terms the indirect utility of the buyer. The novel and important restriction is given by the inequality (23) that states that the information rent of the buyer cannot grow too fast. The inequality thus present an upper bound on the allocated quantity $q(\theta) = V'(\theta)$.

We could approach the above problem as an optimal control problem where $V(\theta)$ is the state variable and $V'(\theta)$ is the control variable. The presence of the derivative constraint (23) which combines, in an inequality, the state and the control variable renders this problem intractable. In particular, to the best of our knowledge the current problem is not directly covered by any standard result in optimization theory.[13] In particular, while a non-standard version of the Pontryagin maximum principle with state dependent control constraints could in principle be used to deal with the derivative constraint (23),[14] this approach would lead to a description of the optimal policy in terms of a multi-dimensional ordinary differential equation (ODE). There seems to be no obvious way to infer the optimal policy from the resulting ODE, and we could make this approach work only in special cases.

To avoid these issues, we adopt a proof technique that has proved useful in stochastic optimal control as established by Peng (1992), see also Karoui et al. (1997) for a wide range of applications of this technique. A comparison principle asserts a specific property of a differential inequality if an auxiliary inequality has a certain property. An important comparison result is Gronwall's inequality that allows us to bound a function that is know to satisfy a certain differential inequality by the solution of the corresponding differential

[13]This constraint is fundamentally different from the Border constraint appearing in multi-buyer mechanism design problems which is a (weak) majorization constraint.

[14]See for example Evans (1983) for a detailed introduction into the Pontryagin maximum principle.

equation. Following standard arguments in the literature on comparison principles, we use Gronwall's inequality to establish the following lemma.

Lemma 3 (Comparison Principle)**.**
Let $g, h : [0, \overline{\theta}] \to \mathbb{R}$ *be absolutely continuous and satisfy* $g'(\theta) \le \Phi(g(\theta), \theta)$ *and* $h'(\theta) \ge \Phi(h(\theta), \theta)$ *where* $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}$ *is uniformly Lipschitz continuous in the first variable. If* $g(0) \le h(0)$ *we have that* $g(\theta) \le h(\theta)$ *for all* $\theta \in [0, \overline{\theta}]$.

We can then use the comparison principle to apply it the differential inequality constraint (23) and give a characterization of the optimal solution.

Proposition 8 (An Optimization Problem with State Dependent Control Constraints)**.**
Let $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}_+$ *be increasing and uniformly Lipschitz continuous in the first variable as well as continuous in the second on every interval* $[a, \overline{\theta}]$ *for* $a > 0$.[15] *Let* $\mathcal{J} : [0, \overline{\theta}] \to \mathbb{R}$ *be continuous, satisfy* $\mathcal{J}(0) = -1$ *and* $z \mapsto \min\{\mathcal{J}(z), 0\}$ *be non-decreasing. Consider the maximization problem:*

$$\max_{w} \int_0^{\overline{\theta}} \mathcal{J}(\theta) w'(\theta) \mathrm{d}\theta - w(0)\,. \tag{24}$$

over all differentiable functions $w : [0, \overline{\theta}] \to \mathbb{R}$ *that satisfy* $w'(\theta) \le \Phi(w(\theta), \theta)$*. There exists an optimal policy* w *to this problem such that for all* $\theta \in (0, \overline{\theta}]$

$$w'(\theta) = \Phi(w(\theta), \theta)\,.$$

To apply Proposition 8 to the optimization problem given by (20), (21) and (23) we define

$$\mathcal{J}(\theta) \triangleq f(\theta) J(\theta),$$

and

$$\Psi(v, \theta) \triangleq \min\left\{\beta \frac{v}{\theta}, \frac{1}{r + \gamma}\right\}.$$

An immediate observation is that $\mathcal{J}(0) = -1$. Applying Proposition 8 to the optimization problem (20)-(23) by ex-post verifying that the solution is non-negative and convex, and hence feasible, yields the following characterization of the relaxed optimal mechanism.

[15] This means that for every $a > 0$, there exists a constant $L_a < \infty$ such that $|\Phi(v, \theta) - \Phi(w, \theta)| \le L_a \cdot |v - w|$ for all $\theta \in [a, \overline{\theta}]$.

Theorem 1 (Optimal Control)**.**
There exists a $\theta^\star \in [0, \overline{\theta}]$ *such that a solution to the control problem* (20)-(23) *is given*

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta^\star}\right)^\beta \frac{\theta^\star/\beta}{r+\gamma}, & \textit{for } \theta \leq \theta^\star, \\ \frac{\theta^\star/\beta}{r+\gamma} + \frac{\theta-\theta^\star}{r+\gamma}, & \textit{for } \theta^\star \leq \theta, \end{cases}. \tag{25}$$

We arrived at the optimization problem (20)-(23) by relaxing the original mechanism design problem in two ways. First, we allowed the buyer to misreport his arrival only using cut-off stopping times. Second, we ignored the monotonicity constraint associated with truthful reporting of the initial value.

Proposition 9 (Posted Price Implementation)**.**
The indirect utility given in (25) *is implemented by a sales contract with a flow price of*

$$p^\star = \frac{\beta - 1}{\beta}\theta^\star.$$

By Proposition 9, the allocation which maximizes the expected revenue under those relaxed incentive constraints can be implemented using a sales contract. As the revenue with relaxed incentive constraints is an upper bound on the revenue in the original problem and this upper bound is achieved by some sales contract it follows that a sales contract is a revenue maximizing mechanism.

Theorem 2 (Sales Contracts are Revenue Maximizing)**.**
The flow price $p^\star$ *and the associated sales contract is a revenue maximizing mechanism with unobservable arrivals.*

We observe that the optimal allocation gives the object to the buyer forever. Hence, any irreversibility constraint on the allocation is non-binding and thus the problem of irreversibly selling the buyer an object yields the same solution.[16] Thus, our optimal mechanism is also revenue maximizing in a problem where the buyer consumes the object once and immediately, the buyer is privately informed about his arrival, and the buyer's valuation evolves over time.

[16] For the case of observable arrivals this problem was analyzed in Board (2007) and Kruse and Strack (2015).

5.4 An Example: The Uniform Prior

We illustrate the results now for the case of the uniform prior, and assume that $\theta_0 \sim \mathcal{U}[0,1]$ throughout this section.With the uniform prior we can then directly compute from the revenue formula (14) the value threshold $\theta^\star$ and the associated flow price $p^\star$ in the optimal progressive participation mechanism:

$$\theta^\star = \frac{1}{2}\frac{1+\beta}{\beta},$$
$$p^\star = \frac{1}{2}\frac{\beta^2-1}{\beta^2}.$$

In the dynamic mechanism, the value threshold and the associated price are determined exclusively by the virtual value at $t = 0$, and thus under the uniform distribution, the corresponding threshold and flow price are given by

$$\theta^\circ = p^\circ = \frac{1}{2}.$$

Thus the price in the progressive mechanism is below the dynamic mechanism whereas the threshold of the progressive mechanism is above the dynamic mechanism:

$$p^\star < p^\circ = \theta^\circ < \theta^\star. \tag{26}$$

In Figure 2 we display the behavior of the thresholds and the prices as a function of $\beta \in (1,\infty)$. As β increases, the discounting rate and the renewal rate are increasing, and the buyer becomes less forward-looking. As β decreases towards one, the gap between the value threshold $\theta^\star$ and the price p^* increases. As the option value becomes more significant, the buyer chooses to wait until his value has reached a higher threshold, thus he will wait longer to enter into a relationship with the seller. Faced with a more hesitant buyer, the seller decreases the flow price as β decreases. Yet, the decrease in the flow price only partially offsets the option value, and the buyer still waits longer to enter into the relationship with the seller. In contrast, the threshold value, and the price, in the dynamic mechanism, θ° and p°, respectively remain invariant with respect to the patience of the buyer β. An important aspect of the progressive mechanism is that the buyer enters the relationship gradually rather than once and for all, as in the dynamic mechanism. In Figure 3a we plot the probability that an initial type drawn from the uniform distribution consumesthe object as a function of the time since his arrival. In the dynamic mechanism, this probability is constant over

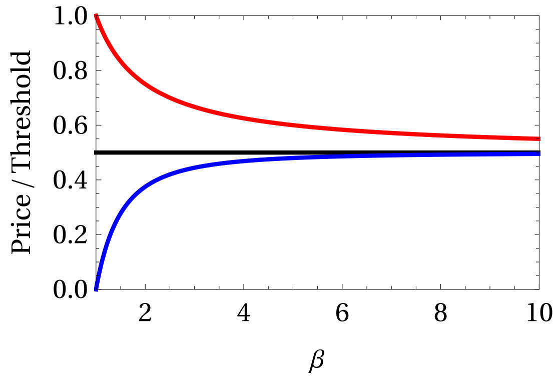
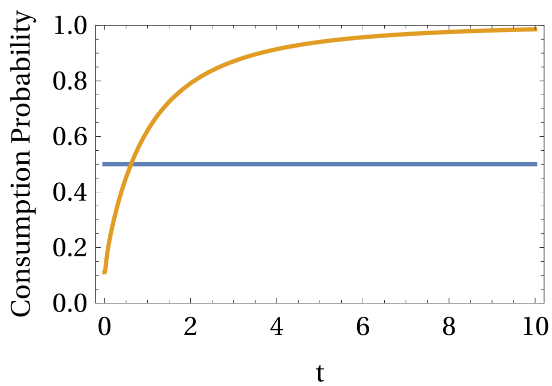
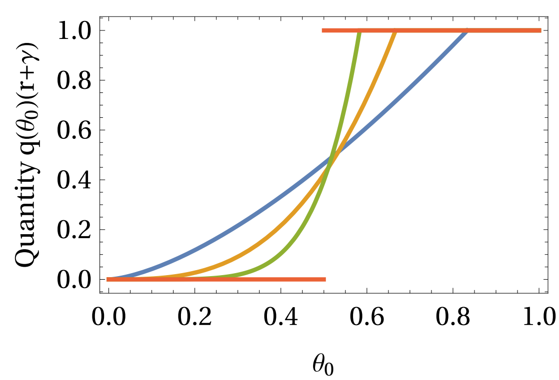


Figure 2: Progressive threshold (red), Dynamic threshold and price (black), and Progressive price (blue)



(a) Consumption Probability over Time, progressive (orange), dynamic (blue).



(b) Quantities assigned in dynamic and progressive mechanism. $\beta = 1.5$ in blue, $\beta = 3$ in yellow, $\beta = 6$ in green, observable arrival in red.

Figure 3

time. As all values θ_0 above $\theta^\circ = 1/2$ buy the object, and all those with initial values $\theta_0 < \theta^\circ = 1/2$ never buy the object, the probability of consumption does not change over time, and is always equal to $1/2$. By contrast, in the progressive mechanism, the probability of participation is progressing over time, and thus the probability of consumption is increasing over time. The geometric Brownian motion displays sufficient variance, so that eventually every buyer purchases the product.

We now zoom in on the purchase behavior of the initial types θ_0. Figure 3b, quantity, . illustrates the discounted expected consumption quantity $q(\theta_0)$ as a function of the initial valuation θ_0 for various values of β. We find again that in the dynamic mechanism there is a sharp distinction in the consumption quantities between the initial values below and above the threshold of $\theta^\circ = 1/2$. By contrast in the progressive mechanism, the consumption quantity is continuous and monotone increasing in the initial value θ_0. As the buyer becomes more patient, and hence as β decreases, the slope of the consumption quantity flattens outs and the threshold $\theta^\star$ upon which consumption occurs immediately is increasing.

The differing thresholds and allocation probabilities give us some indication regarding the contrasts in welfare properties between progressive and dynamic mechanism. As the price in the progressive mechanism is uniformly lower, this allows us to immediately conclude that the consumer surplus is larger in the progressive mechanism than in the corresponding dynamic mechanism. Conversely, as the seller could have offered the progressive mechanism in the dynamic setting, but did not, it follows that the revenue of the seller is uniformly lower in the progressive mechanism. Thus, the option of the buyer to postpone his allocation is indeed valuable and increases the consumer surplus significantly. This leaves open the question as to how the social surplus is impacted by these different participation constraints. With the uniform prior, we can further compute that the social welfare is uniformly larger in the progressive than in the dynamic mechanism.

Significantly, the social welfare comparison does not extend to all prior distributions. In particular, if there is only a small amount of private information, so that the static virtual utility is non-negative for all initial values, then the dynamic mechanism will not distort the allocation, and thus support the first best social welfare. For example, in the class of uniform distribution on the interval $[a, 1]$, the static virtual utility:

$$\theta - \frac{1 - F(\theta)}{f(\theta)}$$

is positive for all $\theta \in [a, 1]$ if the lower bound a in the support of the distribution is sufficiently large, or $a > 1/2$. In these circumstances, the seller in the dynamic environment will cease to discriminate against any initial value, and rather sell the object forever to all initial types $\theta \in [a, 1]$. By contrast, in the progressive mechanism, the option value remains an attractive opportunity for all buyers, and thus the seller will never sell to all buyers irrespective of their initial value $\theta \in [a, 1]$. In consequence, the revenue maximizing progressive mechanism leads to some initial inefficiency, and thus will not attain the first best.

6 Conclusion

We considered a dynamic mechanism problem where each buyer is described by two dimensions of private information, his willingness to pay (which may change over time) *and* his arrival time. We considered a stationary environment – in which the buyers arrive and depart at random – and a stationary contract. In this arguably more realistic setting for revenue management, the seller has to guarantee both interim incentive as well as interim participation constraints. As the buyer has the valuable option of delaying his participation, the mechanism has to offer incentives to enter into the relationship.

One challenge in our environment is that the first-order approach and other standard methods fail as global incentive constraints bind in the optimal contract. We were able to solve this multi-dimensional incentive problem by rephrasing the participation decision of the buyer as a stopping problem, and then solve a new optimal control problem. More precisely, we decomposed the progressive mechanism problem into an intertemporal participation (entry) problem and an intertemporal incentive problem. Given the separability between these two problems, our approach can be possibly extended to allocation problems beyond the unit demand problem considered here. There are (at least) three natural directions to extend the analysis. First, the stochastic evolution of the value was governed by the geometric Brownian motion, and clearly other stochastic process could be considered. Second, the allocation problem could be extended to nonlinear allocation problems rather than the unit demand problem considered here. Third, a natural next step is to extend the techniques developed here to multi-buyer environments, say competing bidders for a scarce resource. The final generalization will pose new challenges as we will have to investigate whether the solution of the individual stopping problem can be decentralized or distributed in a consistent manner across the buyers. This is a problem similar to the reduced form auction as posed by Border (1991) but now in dynamic rather than static allocation problem.

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A Appendix

Lemma 4. *We have that*

$$V_\alpha(\theta) = \mathbb{E}\left[\int_{\alpha_i}^{\infty} e^{-(r+\gamma)\,(t-\alpha_i)} \left\{\theta_t^i x_t^i - p_t^i\right\} \mathrm{d}t \mid \alpha_i = \alpha, \theta_{\alpha_i} = \theta\right].$$

Proof. By the law of iterated expectations and the fact that the departure time of the buyer T_i is independent of the arrival time α_i and the valuation process θ^i and hence of x_t^i, p_t^i and $u_t^i = \theta_t^i x_t^i - p_t^i$

$$\begin{aligned}
\mathbb{E}\left[\int_{\alpha_i}^{T} e^{-r\,(t-\alpha_i)} u_t^i \mathrm{d}t\right] &= \mathbb{E}\left[\int_{\alpha_i}^{\infty} \mathbf{1}_{\{T\geq t\}} e^{-r\,(t-\alpha_i)} u_t^i \mathrm{d}t\right] = \mathbb{E}\left[\int_{\alpha_i}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{T_i\geq t\}}\right] e^{-r\,(t-\alpha_i)} u_t^i \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_{\alpha_i}^{\infty} \mathbb{P}\left[T \geq t\right] e^{-r\,(t-\alpha_i)} u_t^i \mathrm{d}t\right] = \mathbb{E}\left[\int_{\alpha_i}^{\infty} e^{-\gamma\,(t-\alpha_i)} e^{-r\,(t-\alpha_i)} u_t^i \mathrm{d}t\right].
\end{aligned}$$

□

Proof of Lemma 1. As each buyer's allocation is only a function of his own reports and the willingness to pay is independent between different buyers the law of iterated expectations implies that the revenue can be rewritten as

$$\mathbb{E}\left[\sum_{i=0}^{\infty} \int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,t} p_t^i \mathrm{d}t\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i} \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right]\right].$$

As buyer are ex-ante identical they are necessarily treated the same in the optimal mechanism which yields that the revenue equals

$$\max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\sum_{i=0}^{\infty} \int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,t} p_t^i \mathrm{d}t\right] = \max_{(x,p)\in\mathcal{M}} \mathbb{E}\left[\int_{\alpha_i}^{\alpha_{i+1}} e^{-r\,(t-\alpha_i)} p_t^i \mathrm{d}t\right] \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right].$$

Note, that $\alpha_{i+1} - \alpha_i = \tau_i - \alpha_i$ are independently and identically exponentially distributed with rate γ it follows from this that

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_i}\right] &= \mathbb{E}\left[\sum_{i=0}^{\infty} e^{-r\alpha_0} \prod_{j=0}^{i-1} e^{-r(\alpha_{j+1}-\alpha_j)}\right] = \mathbb{E}\left[e^{-r\alpha_0}\right] \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} \mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right] \\
&= \sum_{i=0}^{\infty} \mathbb{E}\left[e^{-r(\alpha_{j+1}-\alpha_j)}\right]^i = \sum_{i=0}^{\infty} \left(\frac{\gamma}{r+\gamma}\right)^i = \frac{r+\gamma}{r}.
\end{aligned}$$

This yields the result. $\square$

Proof of Proposition 1. As $(\theta_t)_{t\geq 0}$ is a geometric Brownian motion it can be explicitly represented as:

$$\theta_t = \theta_0 \exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right) . \tag{27}$$

The first part of the Proposition is a special case of Proposition 1 in Bergemann and Strack (2015) and follows by applying the envelope theorem. As it is optimal for the buyer to report his initial value θ_0 truthfully we have that the derivative of the buyer's indirect utility can be calculated by treating (x, p) as independent of the buyers report

$$\begin{aligned} V'(\theta) &= \frac{\partial}{\partial \theta_0} \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t} \{x_t\, \theta_t - p_t\}\, \mathrm{d}t \mid \theta_0\right] = \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t} x_t \left(\frac{\partial}{\partial \theta_0}\, \theta_t\right) \mathrm{d}t \mid \theta_0\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t} x_t \left(\frac{\partial}{\partial \theta_0} \left\{\theta_0 \cdot \exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right)\right\}\right) \mathrm{d}t \mid \theta_0\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t} x_t \exp\left(-\frac{\sigma^2}{2} t + \sigma W_t\right) \mathrm{d}t \mid \theta_0\right] = q(\theta_0) . \end{aligned}$$

As shown in Bergemann and Strack (2015, Theorem 1, Proposition 8, and Equation 27) the virtual value is given by $\theta_t \left(1 - \frac{1-F(\theta_0)}{f(\theta_0)}\right)$ and the expected revenue of the seller equals

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t} p_t \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t}\, x_t\, \theta_t \left(1 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t}\, x_t\, \frac{\theta_t}{\theta_0} \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathrm{d}t\right] - V(0) \\ &= \int \mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\, t}\, x_t\, \frac{\theta_t}{\theta_0} \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \mathrm{d}t \mid \theta_0\right] f(\theta_0) \mathrm{d}\theta_0 - V(0) \\ &= \left(\theta_0 - \frac{1-F(\theta_0)}{f(\theta_0)}\right) \int \mathbb{E}\left[\int_0^\infty e^{-r\, t}\, x_t\, \frac{\theta_t}{\theta_0} \mathrm{d}t \mid \theta_0\right] f(\theta_0) \mathrm{d}\theta_0 - V(0) . \end{aligned}$$

Plugging in the explicit representation of θ_t given by (27) yields that the expected revenue satisfies

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} p_t \mathrm{d}t\right] \\
&= \int_0^{\overline{\theta}} J(\theta_0) \underbrace{\mathbb{E}\left[\int_0^\infty e^{-(r+\gamma)\,t} x_t \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) \mathrm{d}t \mid \theta_0\right]}_{q(\theta_0)} f(\theta_0)\mathrm{d}\theta_0 - V(0)\,. \qquad \square
\end{aligned}$$

Proof of Proposition 2. The result follows as q plays the same role as the quantity in a static allocation problem. $\square$

Proof of Proposition 3. The result follows from Dixit and Pindyck (1994), Section 5.2. $\square$

Proof of Lemma 2. For $\theta_0 \geq x$, the buyer stops immediately and thus the statement is true. For $\theta_0 < x$ we have that

$$\begin{aligned}
\mathbb{E}\left[e^{-(r+\gamma)\,\tau_x} \mid \theta_0\right] &= \mathbb{E}\left[e^{-(r+\gamma)\,\tau_x}\left(\frac{\theta_{\tau_x}}{\theta_{\tau_x}}\right)^\beta \mid \theta_0\right] = \mathbb{E}\left[e^{-(r+\gamma)\,\tau_x}\left(\frac{\theta_0 e^{-\frac{\sigma^2}{2}\tau_x + \sigma W_{\tau_x}}}{x}\right)^\beta \mid \theta_0\right] \\
&= \mathbb{E}\left[e^{-\left[(r+\gamma) - \frac{\sigma^2}{2}\beta\right]\tau_x + \beta\sigma W_{\tau_x}}\left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right] \\
&= \mathbb{E}\left[e^{-\left[(r+\gamma) + \frac{\sigma^2}{2}\beta - \frac{\sigma^2\beta^2}{2}\right]\tau_x} e^{-\frac{\sigma^2\beta^2}{2}\tau_x + \beta\,\sigma\,W_{\tau_x}}\left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right].
\end{aligned}$$

As $(r+\gamma) + \frac{\sigma^2}{2}\beta - \frac{\sigma^2\beta^2}{2} = 0$ and $t \mapsto e^{-\frac{\sigma^2\beta^2}{2}t + \beta\,\sigma\,W_t}$ is a uniformly integrable martingale it follows from Doob's optional sampling theorem that

$$\mathbb{E}\left[e^{-r\,\tau_x} \mid \theta_0\right] = \mathbb{E}\left[\left(\frac{\theta_0}{x}\right)^\beta \mid \theta_0\right].$$

$\square$

Proof of Proposition 4. By Proposition 3 the buyer acquires the object once his valuation exceeds $\theta^\star = \frac{\beta}{\beta-1}p$. By Lemma 1 the expected revenue the seller generates from a single buyer with initial valuation θ_0 is given by

$$\begin{aligned}\frac{r+\gamma}{r}\mathbb{E}\left[\int_{\tau_{\theta^\star}}^{\infty} e^{-(r+\gamma)\,t} p \,\mathrm{d}t \mid \theta_0\right] &= \frac{1}{r}\mathbb{E}\left[e^{-(r+\gamma)\,\tau_{\theta^\star}}\, p \mid \theta_0\right] = \frac{p}{r}\,\mathbb{E}\left[e^{-(r+\gamma)\;\tau_{\theta^\star}} \mid \theta_0\right] \\ &= \frac{p}{r}\,\min\left\{\left(\frac{\theta_0}{\theta^\star}\right)^{\beta}, 1\right\} = \frac{p}{r}\,\min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta_0}{p}\right)^{\beta}, 1\right\}.\end{aligned}$$

Consequently, the expected discounted revenue from buyer with random initial valuation distributed according to F is given by

$$\frac{p}{r}\int_0^\infty \min\left\{\left(\frac{\beta-1}{\beta}\frac{\theta}{p}\right)^{\beta}, 1\right\} f(\theta)\,\mathrm{d}\theta\,. \qquad \square$$

Proof of Proposition 5. It follows from the envelope theorem that the value function is continuous and convex in any mechanism where truthfully reporting the initial valuation is incentive compatible. Furthermore, the envelope theorem implies that V is absolutely continuous, thus any non-differentiability must take the form of a convex kink. As it is never optimal to stop in a convex kink it follows that V is differentiable. $\square$

Proof of Proposition 7. By Proposition 1 and 6 we have that IC-A implies that for all θ

$$\begin{aligned}\frac{\theta\, q(\theta)}{\beta} &\leq V(\theta) = V(0) + \int_0^\theta q(z)\mathrm{d}z \\ \Leftrightarrow \frac{\theta\, q(\theta)}{\beta} - \int_0^\theta q(z)\mathrm{d}z &\leq V(0)\,.\end{aligned}$$

Taking the supremum over θ yields the results. $\square$

Proof of Lemma 3. Define $\Delta \equiv g - h$. Suppose, that there exists a point θ' such that $\Delta(\theta') > 0$. As $\Delta(0) \leq 0$ and by the absolute continuity of Δ there exists a point θ'' such that $\Delta(\theta'') = 0$ and as $\Delta' \geq 0$ we have that $\Delta(\theta) \geq 0$ for all $\theta \in [\theta'', \theta']$. This implies that there exists a constant $L > 0$ such that for all $\theta \in [\theta'', \theta']$

$$\begin{aligned}\Delta'(\theta) = g'(\theta) - h'(\theta) &\leq \Phi(g(\theta),\theta) - \Phi(h(\theta),\theta) \leq |\Phi(g(\theta),\theta) - \Phi(h(\theta),\theta)| \\ &\leq L\,|g(\theta) - h(\theta)| = L\,|\Delta(\theta)| = L\,\Delta(\theta)\,.\end{aligned}$$

By Gronwall's inequality we thus have that $\Delta(\theta') \leq \Delta(\theta'')e^{L(\theta'-\theta'')} = 0$ which contradicts the assumption that $\Delta(\theta') > 0$. $\square$

Lemma 5 (Generalized Comparison Principle)**.**
Let $g, h : [0, \overline{\theta}] \to \mathbb{R}$ be absolutely continuous and satisfy $g'(\theta) \leq \Phi(g(\theta), \theta)$ and $h'(\theta) \geq \Phi(h(\theta), \theta)$ where $\Phi : \mathbb{R} \times [0, \overline{\theta}] \to \mathbb{R}$ is uniformly Lipschitz continuous in the first variable. If $g(\hat{\theta}) = h(\hat{\theta})$ we have that $g(\theta) \leq h(\theta)$ for all $\theta \in [\hat{\theta}, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for all $\theta \in [0, \hat{\theta}]$.

Proof. The first part of the result follows by considering the functions $\tilde{g}(s) = g\left(\hat{\theta} + s\right), h(s) = \bar{y}\left(\hat{\theta} + s\right)$ and applying Lemma 3. The second part follows by considering the functions $\tilde{g}(s) = -g\left(\hat{\theta} - s\right), \tilde{h}(s) = -h\left(\hat{\theta} - s\right)$ for $s \in [0, \hat{\theta}]$ and applying Lemma 3 which implies that for all $s \in [0, \hat{\theta}]$

$$\tilde{g}(s) \leq \tilde{h}(s) \Leftrightarrow -g\left(\hat{\theta} - s\right) \leq -h\left(\hat{\theta} - s\right) \Leftrightarrow g\left(\hat{\theta} - s\right) \geq h\left(\hat{\theta} - s\right). \qquad \square$$

Lemma 6.
Suppose that $\mathcal{J} : [0, \overline{\theta}]$ is a non-decreasing function with $\mathcal{J}(\overline{\theta}) \leq 0$, and $g, h : [0, \overline{\theta}] \to \mathbb{R}$ are absolutely continuous with $g \geqq h$ then

$$\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \mathrm{d}t + \mathcal{J}(0) g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \mathrm{d}\theta + \mathcal{J}(0) h(0).$$

Proof. The result follows from partial integration, the assumption that $\mathcal{J}(\overline{\theta}) \geq 0$

$$\begin{aligned}
\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \mathrm{d}t + \mathcal{J}(0) g(0) &= [\mathcal{J}(\theta) g(\theta)]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} g(\theta) \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) g(0) \\
&= \mathcal{J}(\overline{\theta}) g(\overline{\theta}) - \mathcal{J}(0) g(0) - \int_0^{\overline{\theta}} g(\theta) \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) g(0) \\
&\leq \mathcal{J}(\overline{\theta}) h(\overline{\theta}) - \mathcal{J}(0) h(0) - \int_0^{\overline{\theta}} h(\theta) \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) h(0) \\
&= [\mathcal{J}(\theta) h(\theta)]_{\theta=0}^{\theta=\overline{\theta}} - \int_0^{\overline{\theta}} h(\theta) \mathrm{d}\mathcal{J}(\theta) + \mathcal{J}(0) h(0) \\
&= \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \mathrm{d}\theta + \mathcal{J}(0) h(0). \qquad \square
\end{aligned}$$

Proof of Proposition 8. Let g be an arbitrary feasible policy inthe optimization problem (24). Define $\theta^\star = \inf\{\theta \colon \mathcal{J}(\theta) \geq 0\}$. As $\mathcal{J}$ is continuous $\mathcal{J}(\theta^\star) = 0$. Let $h : [0, \overline{\theta}] \to \mathbb{R}$ be the solution to

$$
\begin{aligned}
h'(\theta) &= \Phi(h(\theta), \theta), \\
h(\theta^\star) &= g(\theta^\star) .
\end{aligned}
$$

The proof proceeds in two step: first we establish that h leads to a higher value of the integral (24) above $\theta^\star$ and in the second step we establish the analogous result below $\theta^\star$.

Step 1: As $g'(\theta) \leq \Psi(g(\theta), \theta)$ it follows from Lemma 5 that $g(\theta) \leq h(\theta)$ for $\theta \in [\theta^\star, \overline{\theta}]$ and $g(\theta) \geq h(\theta)$ for $\theta \in [a, \theta^\star]$ for every $a > 0$. As g and h are continuous it follows that $g(0) \geq h(0)$. The monotonicity of Φ in the first variable implies that for $\theta \geq \theta^\star$

$$
g'(\theta) \leq \Phi(g(\theta), \theta) \leq \Phi(h(\theta), \theta) = h'(\theta) .
$$

As $\mathcal{J}(\theta^\star) = 0$ and $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is non-decreasing we have that $\mathcal{J}(\theta) \geq 0$ for $\theta \geq \theta^\star$ we have that

$$
\int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \mathrm{d}\theta \leq \int_{\theta^\star}^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \mathrm{d}\theta . \tag{28}
$$

Step 2: Note, that by Lemma 5 $g(\theta) \geq h(\theta)$ for $\theta \leq \theta^\star$. Furthermore, by definition of $\theta^\star$ we have that $\mathcal{J}(\theta) = \min\{\mathcal{J}(\theta), 0\}$ for $\theta \leq \theta^\star$. As $\theta \mapsto \min\{\mathcal{J}(\theta), 0\}$ is non-decreasing $\mathcal{J}(\theta)$ is non-decreasing for $\theta \leq \theta^\star$. Lemma 6 implies that

$$
\int_0^{\theta^\star} \mathcal{J}(\theta) g'(\theta) \mathrm{d}\theta + \mathcal{J}(0) g(0) \leq \int_0^{\theta^\star} \mathcal{J}(\theta) h'(\theta) \mathrm{d}\theta + \mathcal{J}(0) h(0) . \tag{29}
$$

Combining the inequalities (28) and (29) with the assumption that $\mathcal{J}(0) = -1$ yields that

$$
\int_0^{\overline{\theta}} \mathcal{J}(\theta) g'(\theta) \mathrm{d}t - g(0) \leq \int_0^{\overline{\theta}} \mathcal{J}(\theta) h'(\theta) \mathrm{d}\theta - h(0) .
$$

As Φ is continuous in both variables it follows that h is continuously differentiable and thus feasible and an optimal policy. $\square$

Proof of Proposition 1. Define $\mathcal{J}(\theta) = J(\theta) f(\theta)$ and recall that $\theta^\circ = \min\{\theta \colon \mathcal{J}(\theta) = 0\}$. We first note, that $\mathcal{J}(\theta)$ is negative for $\theta < \theta^\circ$ and $\mathcal{J}(0) = -1$. Consider the problem of solving

$$\max_{V} \int_0^{\bar\theta} V'(z)\mathcal{J}(z)\,\mathrm{d}z - V(0) \,.$$
$$\text{subject to } V'(\theta) \le \Psi(V(\theta),\theta) \text{ for all } \theta \in \left[\underline{\theta}_k, \overline{\theta}\right],$$

where $\Psi(v,\theta) = \min\left\{\beta\frac{v}{\theta}, \frac{1}{r+\gamma}\right\}$. By Proposition 3 we have that there exists an optimal policy that solves

$$V'(\theta) = \Psi(v,\theta) \tag{30}$$

We have that all solutions to the ODE (30) are of the form

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta'}\right)^{\beta} V(\theta^\star) & \text{for } \theta \le \theta^\star \\ V(\theta^\star) + \frac{\theta-\theta^\star}{r+\gamma} & \text{for } \theta \le \theta^\star \end{cases}$$

where $\frac{1}{r+\gamma} = V'(\theta^\star) = \frac{\beta}{\theta'}V(\theta^\star)$. Thus, plugging in $V(\theta^\star)$ yields that

$$V(\theta) = \begin{cases} \left(\frac{\theta}{\theta^\star}\right)^{\beta} \frac{\theta^\star/\beta}{r+\gamma} & \text{for } \theta \le \theta^\star \\ \frac{\theta^\star/\beta}{r+\gamma} + \frac{\theta-\theta^\star}{r+\gamma} & \text{for } \theta \le \theta^\star \end{cases}.$$

We note that $V \geqq 0$ and V' is increasing. It is thus feasible in the control problem (20)-(23) and we hence have found an optimal policy.

Consider the sales contract where the object is sold at a flow price of $p = \frac{\beta-1}{\beta}\theta^\star$. Proposition 3 yields that the buyer's value is given by

$$V(\theta) = \begin{cases} \frac{1}{r+\gamma}\left(\frac{\theta}{\theta^\star}\right)^{\beta} \frac{1}{\beta}\theta^\star & \text{for } \theta \le \theta^\star \\ \frac{1}{r+\gamma}\left(\theta - \frac{\beta-1}{\beta}\theta^\star\right) & \text{for } \theta \ge \theta^\star \end{cases},$$

and thus satisfies (25) which establishes the result. □