Essays on Financial Intermediation and Collateral Requirements

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Abstract

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Chuan Du

2021

This collection of essays examines financial intermediation and collateral requirements in economies where limited enforcement of repayment necessitates loans to be backed by eligible collateral. Collateralized lending was a key feature of the 2008 financial crisis, with significant policy interventions in major economies directed towards restoring leverage and financial intermediation. Similarly, given the unprecedented shock of the COVID-19 pandemic, firms that face binding collateral constraints and find it difficult to secure loans on affordable terms may struggle to weather the crisis. Governments and central banks worldwide have instigated large scale policy responses to ease credit conditions. Theoretical models of financial frictions that account for endogenous variations in leverage across the business cycle are thus important for guiding policy design and for analyzing the propagation and amplification of shocks through the financial sector.

In Chapter 1, I examine how central banks should intervene to improve credit conditions during a downturn. Specifically, I analyze the setting of collateral requirements in central bank lending facilities. Traditionally, central bank lending during times of crisis followed Bagehot’s rule. That is to lend freely to solvent institutions, against collateral that is good in normal times, and at “high interest rates”. The rule is designed so that the central bank can improve credit conditions without taking on any credit risk. Lately, central banks began to deviate from this approach, conducting more direct lending to firms, against a broader class of collateral, and reducing the haircuts imposed on the collateral posted. Which is the more appropriate response? Should central banks take on greater credit risk in order to provide a larger stimulus?
To answer the question, I develop a model of central bank intervention in collateralized credit markets. I find that when the downturn is severe it is optimal for the central bank to take on greater credit risk. The analysis suggests that credit facilities set up by the Federal Reserve in response to COVID-19, such as the Main Street Lending Program, can achieve greater participation and effectiveness by easing their terms of lending.

In Chapter 2, I present joint work with Agostino Capponi and Stefano Giglio on the determinants of collateral requirements in central clearinghouses for credit default swaps (CDSs). The empirical results in Capponi et al. (2020) demonstrate that extreme tail risk measures have higher explanatory power for observed collateral requirements than the standard Value-at-Risk rule. To provide a theoretical foundation for these findings, we develop a model of endogenous collateral requirements in the CDS market, where counterparties trade state-contingent promises backed by cash as collateral. Trading occurs due to differences in market participants’ beliefs about the uncertain states of the world.

We show that it is the nature – rather than the degree – of these belief differences that determines collateral requirements in equilibrium. For instance, the equilibrium level of collateral increases both when the optimist becomes more pessimistic (which reduces the extent of the disagreement), and when the pessimist becomes more concerned about tail events (which increases the extent of the disagreement). We can thus point to the clearinghouse’s concerns about extreme tail events as an explanation for the highly conservative levels of collateral observed in practice.

In Chapter 3, I propose a generalization of the Binomial No-Default Theorem of Fostel and Geanakoplos (2015). The Binomial No-Default Theorem states that “in binomial economies with financial assets serving as collateral, any equilibrium is equivalent in real allocations and prices to another equilibrium in which there is no default”. I extend this theorem to economies with more than two states of
nature when debt can be ordered by seniority. For instance, with three states of nature, borrowers can issue both senior secured debt and junior unsecured debt. The senior secured debt is explicitly backed by the risky financial asset held by the firm, whereas the junior unsecured debt is implicitly backed by the residual value of the firm after the senior creditors satisfy their claims. The interest rates and credit risks associated with each creditor tier are endogenously determined in equilibrium. The Multinomial Max-Min theorem I prove states that any equilibrium is equivalent to another equilibrium where the senior tranche never defaults, and the junior tranche only defaults in the worst state of the world. The expanded theorem allows for the application of the endogenous leverage framework to a richer set of models with more than two states and where some loans are not contractually secured.

Finally, in a joint paper with David Miles presented in Chapter 4, I examine the interaction between the real interest rate and capital requirements in the risk-taking behavior of banks. In this model, banks can undertake costly screening to discover private information about the probability of success on their potential lending projects. Once this probability is known, each bank sets a cut-off threshold for the likelihood of success on a project that determines whether or not to lend. When banks are highly leveraged, a combination of asymmetric information and limited liability means that banks screen too little (insufficient “participation”) and accept projects that are too risky (insufficient “prudence”) relative to the first-best benchmark.

We show that the real interest rate and capital requirements function as imperfect substitutes. Qualitatively, raising either the real interest rate or the capital requirement on banks can increase “prudence” at the cost of decreased “participation”. But the two policy instruments work through very different mechanisms. An increase in capital requirements forces banks to hold “more skin in the game” and is targeted at the subset of poorly capitalized banks in the population. In contrast, an increase in the real interest rate increases the opportunity cost of lending for all banks, and
is thus a much blunter instrument. The interaction between these two policy levers means that the optimal capital requirement on banks rises as the interest rate falls, suggesting tougher macro-prudential capital standards in a “lower-for-longer” interest rate environment.
Essays on Financial Intermediation and Collateral Requirements

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by

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\(^1\)This is joint work with Agostino Capponi and Stefano Giglio.
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\(^2\)Joint work with David Miles.
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1 Collateral Requirements in Central Bank Lending

1.1 Introduction

Central bank lending during times of crisis traditionally followed Bagehot’s rule: lend freely to solvent institutions, against collateral that is good in normal times, and at high interest rates. The rule is designed so that the central bank can improve credit conditions without taking on any credit risk while also limiting moral hazard. Lately, central banks have begun to deviate from this approach in their response to COVID-19, conducting more direct lending to firms, against a broader class of collateral, and reducing the haircuts imposed on the collateral posted. Which is the more appropriate response? Should central banks take on greater credit risk in order to provide a larger stimulus?

I develop a model of central bank intervention in collateralized credit markets that combines the Credit Cycles in Kiyotaki and Moore (1997) and the Leverage Cycle in Geanakoplos (1997). I find that when the downturn is severe, it is optimal for the central bank to take on greater credit risk. Specifically, the central bank should intervene by lending at more favorable interest rates compared to the private market, while simultaneously lowering collateral requirements. The potential losses for the central bank on the loans extended falls on taxpayers.

In the model, firms borrow in order to purchase capital for production. There are two sources of financial frictions. First, firms can only borrow using simple debt contracts that are non-state-contingent. Second, each debt contract must be backed by one unit of capital as collateral in order to enforce repayment. Crucially, firms can choose one or more contracts from an entire spectrum of such simple debt contracts that differ only in the size of the promised repayment. When the promise exceeds the value of the collateral at the point of delivery, firms default. Since all debt contracts are backed by one unit of capital as collateral, a debt contract with a higher promised
repayment implies greater credit risk for the lender. The price of each debt contract, the firms’ choice of contracts, and thus the credit risk faced by the lenders are fully endogenized in competitive equilibrium. This is in contrast to many papers in the literature where the amount the firm can borrow against each unit of collateral is exogenously given and the lenders face fixed value-at-risk when lending.

The borrowing constraints faced by firms amplify negative aggregate productivity shocks. During a downturn, firms experience an endogenous reduction in their liquid wealth and their ability to borrow. As a consequence, firms hold too little capital in the downturn relative to the socially efficient level.

A central bank can intervene in this case by lending to firms against collateral at more favorable terms relative to the market. Central bank loans are funded through the issuance of a public liability to households that carries the safe rate of interest, and crucially, without the need to post collateral. The central bank is able to borrow at the risk-free rate without posting collateral because it is backed by the ability of the government to tax. As such, any ex post losses incurred on central bank loans will need to be recouped through recourse to the Treasury.

Since the amount firms can borrow against each unit of collateral is fully endogenized in the model, we can assess two different types of central bank interventions. The central bank can either reduce the risk-free interest rate on low loan-to-value debt contracts, or subsidize riskier loans with high loan-to-value. If the central bank is unwilling to bear any credit risk, then the size of the stimulus may be limited. Instead, optimal intervention during severe downturns requires the central bank to take on greater risk in order to provide a larger stimulus. By offering contracts that private lenders are able, but unwilling, to make during a downturn, the central bank can achieve significant gains in productive efficiency through its intervention.
The paper suggests that credit facilities set up by the Federal Reserve in response to COVID-19, such as the Main Street Lending Program, can achieve greater participation and effectiveness by easing their terms of lending.

**Background: Central banks response to COVID-19**

Given the enormous effect of the COVID-19 pandemic on the real economy, central banks across the world are intervening aggressively in credit markets to cushion the impact. In addition to pushing the benchmark policy rate to historic lows and conducting very large asset purchases, the Federal Reserve launched a number of new credit facilities in March 2020. For instance, the Primary and Secondary Market Corporate Credit Facilities are joint programs set up by the Federal Reserve and the US Treasury, whereby a Special Purpose Vehicle (SPV) is established to purchase qualifying bonds from eligible issuers either directly, or through the secondary market. The Treasury made the initial $10bn equity investment and the Federal Reserve committed to lend to the SPV on a recourse basis.\(^3\) The stated goal of these facilities is to "support credit to employers" through either bond issuance, or by providing liquidity to the market for outstanding corporate bonds.\(^4\) Similarly, the Term Asset-Backed Securities Loan Facility extends loans secured by eligible asset-backed securities; and the Main Street Lending Program purchases participations in loans originated by eligible lenders. In the UK, the Bank of England and HM Treasury announced a similar suite of measures, including the COVID-19 Corporate Financing Facility and the Term Funding Scheme with additional incentives for SMEs.

This paper joins a growing literature that examines the design and efficacy of these dramatic interventions.


\(^4\)More details can be found at: www.federalreserve.gov
Key Themes and Related Literature

The analytical framework in this paper draws from two seminal models that study how financial frictions amplify shocks to the real economy: *Credit Cycles* by Kiyotaki and Moore (1997); and *The Leverage Cycle* by Geanakoplos (1997, 2003, 2010). In Credit Cycles, borrowers secure loans subject to a borrowing constraint that is tied to their net worth. During a downturn, the fall in borrower’s net worth reduces their ability to borrow, leading to a fall in their capital holdings which reduces future revenue and net worth, thus completing a dynamic feedback loop to even lower asset prices and net worth today. Crucially however, in the Credit Cycles model the collateral constraint faced by borrowers is exogenously given: the loan to value (LTV) against each unit of collateral is fixed, and the interest rate always equals the lender’s rate of time preference (i.e. the risk-free interest rate). Geanakoplos (1997) provides a natural framework for endogenizing the collateral constraint. In the Leverage Cycle, loan to value - and equivalently leverage - is too high in normal times, and crashes when bad news arrive. The bad news about the value of borrower’s collateral also reduce their liquid wealth and further restricts their purchasing power today.

In this paper I combine the key features of both the *Credit Cycle* and the *Leverage Cycle* frameworks. Firms and households are differentially productive with a durable capital good, and firms face an *endogenous* borrowing constraint when they try to secure loans against capital as collateral. A key theme here is that firms have the option to choose from an entire spectrum of collateralized debt contracts that differ in LTV and the contractual/promised interest rate. For given collateral, a high LTV debt contract trades off a larger loan today at the cost of a higher promised rate of interest. Plotting the promised interest rate against the LTV on each debt contract that arises in the competitive collateral equilibrium generates the *credit surface* (Figure 1.1). The credit surface summarizes the prevailing credit condition at each point in time; and shifts in the credit surface reflect changing circumstances. During a downturn,
credit conditions deteriorate and the corresponding credit surface shifts inwards and upwards. So for a firm that was originally highly leveraged, it must now either accept a much larger haircut on the same collateral in order to maintain the same interest rate; or retain a similar (albeit slightly reduced) LTV at the cost of a much higher interest rate. Figure 1.2 plots the option-adjusted spread on US corporate debt against credit rating, and highlights how credit conditions - especially for riskier loans - could deteriorate during recessions in this fashion, in spite of central bank interventions.

The analytical framework in this paper accounts for the richer picture of credit market conditions that we observe in practice. Firms optimize simultaneously over their desired capital holding and the type of debt contract they wish to issue. Choosing a high LTV contract means a lower haircut and a larger sized loan today, in return for a higher promised interest rate to compensate lenders for the increased credit risk. Importantly, firms’ choices over leveraged debt contracts determine what the optimal central bank intervention should look like during a downturn (Figure 1.3). Interventions targeted at the low-LTV end of the credit surface can reduce the risk-free interest rate faced by firms without exposing the central bank to significant credit risk. In contrast, subsidizing riskier high-LTV loans can provide a larger stimulus, but at the cost of potential losses for the central bank that will ultimately fall on taxpayers. I show that when the downturn is severe firms demand larger and riskier loans against their dwindling pool of collateral. In such cases, it is optimal for the central bank to take on more credit risk.

Figure 1.4 shows that during the current COVID-19 recession high-yield corporate debt issuance in the US, as a proportion of investment grade issuance, first collapsed at the onset of the crisis before recovering quickly after the Federal Reserve intervened aggressively in the credit markets. Gilchrist et al. (2020) find that the Federal Reserve’s Secondary Market Corporate Credit Facility (SMCCF) has been effective
The Credit Surface plots the promised interest rate on a collateralized debt contract against its loan to value (LTV) - both of which are determined endogenously in equilibrium. The haircut imposed on the collateral posted is defined as one minus the loan to value. The credit surface is composed of three parts. The horizontal segment to the left represents the risk-free spectrum of the market where the haircut is so high on the collateral posted that the loan is effectively risk-free. The increasing function in the middle of the credit surface captures the fact that as the haircut falls (and the LTV rises), the lender starts to take on more credit risk and must be compensated through a higher promised interest rate. Lastly, the credit surface becomes vertical once the maximum loan-to-value is reached. During normal times, interest rates are low across the credit surface. In a downturn, credit conditions deteriorate and the corresponding credit surface shifts inwards and upwards. So for a firm that was originally highly leveraged, it must now either accept a much larger haircut on the same collateral in order to maintain the same interest rate; or retain a similar (albeit slightly reduced) LTV at the cost of a much higher interest rate.
Figure 1.2 plots the option-adjusted spread on US corporate debt against credit rating, for the three most recent recessions in the US. It compares the average spread during each recession against the average in the preceding years. The figure shows that, in spite of central bank interventions, credit conditions deteriorate during recessions. The interest rate spreads on riskier loans are especially elevated.

in reducing credit spreads on corporate bonds. The findings in my paper support calls for the Federal Reserve to further ease lending terms on existing facilities, such as the Main Street Lending Program (English and Liang (2020), Anderson (2020)), in order to increase participation and efficacy.

Hanson et al. (2020) is a recent and closely related paper which models the business credit programs conducted by the Federal Reserve during the current recession. The authors also conclude that “in contrast to the classic lender-of-last-resort thinking that underpinned much of the response to the 2007–2009 global financial crisis, an effective policy response to the pandemic will require the government to accept the prospect of significant losses on credit extended to private sector firms”. Similarly, Koulischer and Struyven (2014) argued for looser central bank collateral requirements during credit crunches to reduce risk spreads and increase output. In both of these papers, the collateral constraints faced by borrowers are exogenously fixed. By endogenizing
In the model, a central bank can intervene in the collateralized debt market during a downturn by either reducing the interest rate on low loan-to-value (LTV) debt contracts (blue dashed line); or by reducing the interest rate on high LTV loans (red dotted line). Interventions aimed at the high LTV end of the market entail greater credit risk for the central bank.

Source: SIFMA (Securities Industry and Financial Markets Association) and own computations. Figure 1.4 shows that during the current COVID-19 recession high-yield corporate debt issuance in the US, as a proportion of investment grade issuance, first collapsed at the onset of the crisis before recovering quickly after the Federal Reserve intervened aggressively in the credit markets.
the size of the loan that can be secured against each unit of collateral, I provide a novel and richer framework to assess the design of central bank credit facilities during a crisis.

1.2 The Model

Agents, goods and uncertainty

Consider a discrete time general equilibrium model with two types of representative agents: households and firms, both risk-neutral and price taking. There are two goods: a numeraire consumption good which depreciates fully between periods (a “fruit”) and a durable capital good (a “tree”). Let $x_t$ and $K_t$ denote the firms’ holding of the consumption good and the capital good in period $t$ respectively. The corresponding terms for the households are denoted by $\bar{x}_t$ and $\bar{K}_t$.$^5$ The total supply of capital in the economy is exogenously fixed at $\bar{K} = 1$.

Both households and firms can use the capital good as an input to produce the consumption good. The production technology is different between households and firms, but in both cases production occurs with one period delay. Specifically, households have a concave production technology $\bar{y}_{t+1} = G \left( \bar{K}_t \right)$, with $G \in C^2$, $G' (\cdot) > 0$ and $G'' (\cdot) < 0$, where $\bar{y}_{t+1}$ is the units of the consumption good produced. Firms have a linear production technology, but one that is subject to uncertainty: $y_{t+1} (s_{t+1}) = a_{t+1} (s_{t+1}) \bar{K}_t (s^t)$, where the productivity coefficient $a \in \{a_U, a_D\}$ can take either a high value $a_U$ or a low value $a_D$ depending on the state of nature in period $t+1$. The productivity of firms is the main source of uncertainty in the model. Figure 1.5 summarizes the timing and the structure of this uncertainty.

The model starts at period $t = 0$ with a certain state $S_0 = \{0\}$, before any production has occurred. From period $t = 1$ onward, there are at most two possible

$^5$Throughout the paper, I will use $\sim$ to differentiate variables associated with households and those associated with firms.
Figure 1.5: Timing and Uncertainty

states $s_t \in S_t = \{U, D\}$: an Up-state with $a_t(U) = a_U$ and a Down-state with $a_t(D) = a_D$. The path the economy can take is summarized by its history $s^t \in S^t$.

In period $t = 1$, there are two possible histories $S^1 = \{U, D\}$, that are reached with probability $p$ and $(1 - p)$ respectively. If the Up-state is reached at $t = 1$ (i.e. $s^1 = U$), then the economy will stay in state $s_t = U$, $\forall t > 1$. If instead the Down-state is reached at $t = 1$ (i.e. $s^1 = D$), then the state of the economy can either switch back to $U$ with probability $p$ in period $t = 2$ and stay there forever, or remain at $D$ for all $t \geq 2$ with probability $(1 - p)$. Therefore by $t = 2$ all uncertainty has resolved and there are three possible histories: $S^2 = \{UU, DU, DD\}$. These three histories captures the three possible paths for this economy. On the first path $0 \rightarrow U \rightarrow UU \rightarrow \ldots$, the firm is found to be highly productive. On the second path $0 \rightarrow D \rightarrow DU \rightarrow \ldots$, the firm suffers a temporary negative productivity shock in period 1, but recovers from

---

Technical side note: while the firm’s productivity coefficient is state dependent and path/history independent: $a_t(s^t) = a_t(s_t)$ $\forall t, s^t$ (i.e. realized productivity today is independent of whether there was a Down-state previously); the price of capital is history dependent. Temporary shocks will have a persistent effect on real outcomes.
period 2 onward. On the third path \(0 \rightarrow D \rightarrow DD \rightarrow \ldots\), the negative productivity shock is permanent.

**Markets**

Firms and households trade in three markets in each period \(t\) and history \(s^t\). The first is a spot market for the numeraire consumption good, with price normalized to 1. The second is a spot market for the capital good, with price \(q_t(s^t)\). Third, and most importantly, there is a credit market for one-period loans with the capital good \(K\) serving as collateral. The key assumption here is that in this economy the only way to enforce repayment of loans is by requiring collateral.

Specifically, a loan contract at time \(t\) is composed of a *promise* to repay \(j\) units of the consumption good in all states of the world in period \(t + 1\), backed by 1 unit of the capital good as *collateral*. Given the limited enforcement of repayment, whenever the promised amount exceeds the price of the collateral posted, the borrower will simply default on the loan and hand over the collateral posted. The actual delivery on contract \(j\) at time \(t + 1\) is therefore given by:

\[
\delta_{j,t+1}(s^{t+1}) = \min \{j, q_{t+1}(s^{t+1})\}
\]  

At each period \(t\) and each history \(s^t\) an entire spectrum of such debt contracts are available, indexed by \(j\) the size of the promise (and all backed by 1 unit of the capital good as collateral). The price of each loan contract \(j\), denoted by \(\pi_{j,t}(s^t)\), is determined endogenously in equilibrium. Equivalently, one can interpret \(\pi_{j,t}(s^t)\) as the size of the collateralized loan that promises to repay \(j\) next period.
A key feature of this set-up for the credit market is that for each loan $j$, we can also compute the promised interest rate:

$$1 + r_{j,t} (s^t) := \frac{j}{\pi_{j,t} (s^t)}$$

(1.2)

and the loan to value (or equivalently 1 minus the haircut imposed on the collateral):

$$LTV_{j,t} (s^t) := \frac{\pi_{j,t} (s^t)}{q_t (s^t)} =: 1 - Haircut_{j,t} (s^t)$$

(1.3)

Plotting the loan to value on the $x$-axis against the promised interest rate on the $y$ axis for each contract $j$ generates the credit surface in Figure 1.6. The credit surface is composed of three parts.\(^7\) The horizontal segment to the left represents the risk-free spectrum of the market where the haircut is so high on the collateral for given promise $j \leq \bar{j} := \min_{t+1} (q_{t+1} (s^{t+1}))$ that the loan is effectively risk-free. I refer to $\bar{j}$ as the max-min leverage contract, because it is the maximum promise against one unit of collateral that still minimizes credit risk to the lender. The increasing function in the middle of the credit surface captures the fact that as the haircut falls (and the LTV rises), the lender starts to take on more credit risk and must be compensated through a higher promised interest rate. Lastly, the credit surface becomes vertical once the maximum loan-to-value is reached. This cap on loan-to-value arises endogenously because the highest credible promise $\bar{j} := \max_{t+1} (q_{t+1} (s^{t+1}))$ is given by the maximum possible valuation of the collateral next period. I refer to $\bar{j}$ as the maximum leverage contract. From the definition of $j$ and $\bar{j}$ it is evident that how much a borrower can promise to repay on their collateralized debt contract depends on the future price of capital. So any changes in the expected

\(^7\)This concept of the “Credit Surface” was introduced in Geanakoplos and Zame (2014) and Fostel and Geanakoplos (2015). Geanakoplos (2016) discusses the implications of the credit surface for monetary policy.
price of capital tomorrow can influence credit conditions today. Firms’ choice of contracts along the credit surface also determines the credit risk faced by lenders in equilibrium. The possibility, and the occurrence, of defaults play an essential role in the model.

Without loss of generality, I show later that the optimal contract for firms lies on the increasing segment of the credit surface: \( j^* \in [\bar{j}, \bar{j}] \). Intuitively, when collateral is scarce, the max-min leverage contract \( \bar{j} \) is preferable to contracts \( j < \bar{j} \) because it allows the firm to borrow more at the same interest rate against the same unit of collateral. On the other side of the credit surface, any contract \( j > \bar{j} \) will have the exact same expected delivery as contract \( \bar{j} \) and will thus be priced the same: \( \pi_{j>\bar{j}}(s^t) = \pi_{\bar{j}}(s^t) \). So using the maximum leverage contract \( \bar{j} \) allows the firm to borrow the same amount as contracts \( j > \bar{j} \) but with a lower promised interest rate.

Finally, let \( \varphi_{j,t}(s^t) > 0 \) denote that the firm is selling the contract \( j \) (i.e. borrowing an amount equal to \( |\varphi_j|\pi_j \)), and \( \varphi_{j,t}(s^t) < 0 \) indicate the firm is buying the contract \( j \) (i.e. lending \( |\varphi_j|\pi_j \)). The corresponding notation for households is \( \tilde{\varphi}_{j,t}(s^t) \).

**Agent Optimization**

By assumption both the representative firm and household are price taking and risk neutral \( (u(x) = x) \). The firm’s optimization problem is given by:

\[
\max_{\{x_t(s^t), K_t(s^t), \{\varphi_{j,t}(s^t)\}\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \left[ u \left( x_t(s^t) \right) \Pr \left( s^t|s^{t-1} \right) \right] 
\]

subject to:
The credit surface plots the loan to value on the $x$-axis against the promised interest rate on the $y$ axis for each collateralized contract $j$. Contract $j^* := \min_{s^t+1} (q_{t+1} (s_t^{t+1}))$ promises to repay the lender an amount equal to the minimum possible value of the collateral next period. Therefore $j^*$ is the maximum promise that minimizes credit risk to the lender - a max-min leverage contract. Contract $\bar{j} := \max_{s^t+1} (q_{t+1} (s_t^{t+1}))$ promises to repay the lender the maximum possible value of the collateral next period. $\bar{j}$ is the maximum credible promise the borrower can make, and is therefore the maximum leverage contract. Due to the scarcity of collateral, firms will optimal choose a contract in the interval $[j^*, \bar{j}]$ in equilibrium.
1. Flow of funds constraint:

\[
\begin{align*}
\text{consumption} & \quad x_t(s^t) + \text{K expenditure} \quad K_t(s^t) - \sum_{j \in J} \varphi_{j,t}(s^t) \pi_{j,t}(s^t) \\
\text{loans} & \quad \text{endowment} + \text{K income} - \sum_{j \in J} \varphi_{j,t-1}(s^{t-1}) \delta_{j,t}(s^t) \\
= & \quad e_t(s^t) + (a(s_t) + q_t(s^t)) K_{t-1}(s^{t-1}) - \sum_{j \in J} \varphi_{j,t-1}(s^{t-1}) \delta_{j,t}(s^t) \\
=: & \quad w_t \quad \text{[liquid wealth at start of period]} \\
\end{align*}
\]

(1.5)

2. Collateral constraint:

\[
\begin{align*}
\sum_j \max(\varphi_{j,t}(s^t), 0) \leq K_t(s^t) \\
\end{align*}
\]

(1.6)

Relative to the standard general equilibrium model, the \textit{collateral constraint} is the key addition here. The collateral constraint states that the total number of loans the firm takes out across all contracts \(j\), \(\sum_j \max(\varphi_{j,t}(s^t), 0)\), must be weakly less than the units of capital it holds. This is due to the requirement that each loan must be backed by one unit of capital. The collateral constraint is asymmetric: only borrowers need to post collateral, and lenders do not; hence the \(\max\) operator in the expression.

The representative household solves a similar optimization problem, replacing only the uncertain CRS production function for firms with households’ certain but concave production function \(G(\cdot)\). For brevity, I omit the \((s^t)\) notation where appropriate from this point onward:

\[
\max \left\{ x_t, K_t, (\bar{\varphi}_{j,t}) \right\} \quad E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(\bar{x}_t) \right].
\]

(1.7)
subject to:

\[ \tilde{x}_t + q_t \tilde{K}_t - \sum_{j \in J} \tilde{\varphi}_{j,t} \pi_{j,t} = \tilde{e}_t + G\left(\tilde{K}_{t-1}\right) + q_t \tilde{K}_{t-1} - \sum_{j \in J} \tilde{\varphi}_{j,t-1} \min \{j, q_t\} =: \tilde{w}_t \]

(1.8)

\[ \sum_j \max (\tilde{\varphi}_{j,t}, 0) \leq \tilde{K}_t \]

(1.9)

Collateral Equilibrium

The solution concept of interest is a collateral equilibrium. Formally, a **Collateral Equilibrium** is a vector consisting of the price of capital, contract prices, consumption, capital holdings and contract trades \((q(s^t), \pi(s^t), (x(s^t), K(s^t), \varphi(s^t)), (\tilde{x}(s^t), \tilde{K}(s^t), \tilde{\varphi}(s^t))\)

\(\in (R_+ \times R_+^J) \times (R_+ \times R_+ \times R^J) \times (R_+ \times R_+ \times R^J) \) s.t. at all period \(t\) and all history \(s^t\):

1. All agents optimize (equations 1.4 to 1.9); and

2. All markets clear:

(a) Consumption good market:

\[ (x_t(s^t) - e_t(s^t)) + (\tilde{x}_t(s^t) - \tilde{e}_t(s^t)) = G\left(\tilde{K}_{t-1}(s^{t-1})\right) + a(s_t) K_{t-1}(s^{t-1}) \]

(1.10)

(b) Capital good market:

\[ \tilde{K}_t(s^t) + K_t(s^t) = \bar{K} \equiv 1 \]

(1.11)

(c) Collateralized debt market:

\[ \tilde{\varphi}_{j,t}(s^t) + \varphi_{j,t}(s^t) = 0 \quad \forall j \]

(1.12)
First-best Benchmark

Before characterizing the collateral equilibrium of the model, it is useful to establish the first-best outcome as a frame of comparison. In the first-best, I assume there is a central planner who assigns capital between firms and households in each period and each history in order to maximize the sum of their discounted utility (suppressing the \((s^t)\) notation again for brevity):

\[
\max\left\{K_t, \tilde{K}_t\right\}_{t \geq -1, s^t \in S^t} \quad E_0 \left[ \sum_{t=0}^{\infty} \beta^t (\ddot{x}_t + x_t) \right]
\]

s.t. \( \ddot{x}_t + x_t = \ddot{e}_t + e_t + G\left( \ddot{K}_{t-1} \right) + a_t K_{t-1} \quad \forall t \geq 0 \)

\( K_t + \tilde{K}_t = \bar{K} = 1 \quad \forall t \geq -1 \)

The first-best level of capital holdings therefore equalizes the marginal productivity of capital between households and firms:

\[
G' \left( \dddot{K}^{fb}_t \right) = E_t [a_{t+1}] \quad \text{(1.13)}
\]

\[
\dddot{K}^{fb}_t = \dddot{K} - \dddot{K}^{fb}_t \quad \text{(1.14)}
\]

The first-best benchmark also coincides with the decentralized solution when we remove the two main sources of productive inefficiencies in the collateral equilibrium: (1) the endogenous collateral constraint which restricts agents’ ability to borrow; and (2) limitations in the firm’s liquid wealth (which arise both exogenously at the start, and then endogenously in certain histories, e.g. during the downturn at \(s^1 = D\)). Specifically, if we assume (1) firms and households are sufficiently well-endowed in every history to ensure they can always consume in addition to any desired capital purchases and lending; and (2) there exists a perfect enforcement mechanism for debt repayments, so promises are always honored and there is no need for posting any
collateral; then the equilibrium price of capital \( q_t \) will adjust to equate the marginal return of capital between firms and households: 
\[
\beta E_t \left[ G' \left( \tilde{K}_t \right) + q_{t+1} \right] = q_t \left( s^t \right) = \beta E_t \left[ a_{t+1} + q_{t+1} \right],
\]
and it is easy to verify that \( \tilde{K}_t = \tilde{K}^f \) \( t \geq 0 \).

### 1.3 Characterizing the Collateral Equilibrium

The first-order conditions for both the firm and household’s optimization problems are reported in Appendix 1.6. I impose assumptions on preferences, production functions and endowments as follows. These assumptions are fairly weak (with risk neutrality arguably being the strongest). The assumptions provide analytical tractability and help restrict attention to equilibria of interest.

**Assumptions and household’s behavior in equilibrium**

**Assumption** \( A1 \) [Risk neutrality]: \( u'(x) = u'(\bar{x}) = 1 \).

**Assumption** \( A2 \) [Common discounting]: \( 1 > \tilde{\beta} = \beta > 0 \).

**Assumption** \( A3 \) [Production functions]:

1. \( 1 > p > \frac{a_D}{a_U} \)
2. \( G \in C^2 \), with \( G'(0) = a_U > a_D = G' \left( \bar{K} \right) \) and \( G''(\cdot) < 0 \).

Assumption \( A3.1 \) imposes an upper bound on the probability of the downstate \((1-p)\). Assumption \( A3.2 \) states that the households are as productive as the firm in the Up-state if they hold no capital; but if they hold the entire stock then their marginal productivity becomes as low as the firm in the Down-state.

**Assumption** \( A4 \) [Endowments]:

1. \( \tilde{K}_{-1} = \bar{K} \) and \( K_{-1} = 0 \).
2. \( e_0 = w_0 \in (0, \beta^2 a_D K^f_0] \) and \( e_t \left( s^t \right) = 0 \) \( \forall s^t \in S^t \) and \( \forall t \geq 1 \).
3. \( \tilde{e}_t(s^t) > \frac{\beta}{1-\beta} a_U, \forall t \geq 0 \) and \( \forall s^t \in S^t. \)

The first part of assumption A4 states that households start period 0 with the entire stock of capital. The second part restricts the firms’ endowment of consumption good in period 0 (their starting liquid wealth) and stipulates further that firms receive no more exogenous endowments from period 1 onward. These two parts combined imply that the firms will borrow from households in order to invest in the capital good. The third part of assumption A4 ensures that households are sufficiently well endowed to lend and consume in all periods and histories.

Lastly, I impose a no-bubble condition to rule out an ever increasing price for capital in equilibrium, whereby any arbitrary price for capital today can be justified by an sufficiently high expected price tomorrow:

\[
\lim_{t \to \infty} q_t(s^t) < \infty \quad \forall s^t \in S^t \tag{1.15}
\]

Under assumptions A1-4 and the no-bubble condition, we can characterize the behavior of the representative household as follows:

**Lemma 1. [Household behavior in equilibrium]**

1. The household always consumes: \( \tilde{x}_t(s^t) > 0 \quad \forall t \geq 0 \) and \( \forall s^t \in S^t. \)

2. The household never borrows: \( \tilde{\varphi}_{j,t}(s^t) < 0 \quad \forall t \geq 0 \) and \( \forall s^t \in S^t. \)

3. The household is always indifferent between consuming and purchasing another marginal unit of capital, so the equilibrium price of capital is given by:

\[
q_t = E_t \left[ \frac{\tilde{\beta} u'(\tilde{x}_{t+1})}{u'(\tilde{x}_t)} \left( G' \left( \tilde{K}_t \right) + q_{t+1} \right) \right] \tag{1.16}
\]

\[
= E_t \left[ \beta \left( G' \left( \tilde{K}_t \right) + q_{t+1} \right) \right]
\]
4. The household is always indifferent between consuming and lending, so the equilibrium price of contract $j$ (i.e. the size of the loan granted for a promised repayment of $j$ next period, collateralized by 1 unit of capital) is given by:

$$
\pi_{j,t} = E_t \left[ \tilde{\beta} u' \left( \tilde{x}_{t+1} \right) \frac{\delta_{j,t+1}}{u' \left( \tilde{x}_t \right)} \right] = E_t [\beta \min \{j, q_{t+1}\}] \quad \forall j
$$

(1.17)

**Proof.** Very briefly, the household always consumes and never borrows because by assumption they are given very large exogenous endowments in every period and every history. The price of capital and the price of contract $j$ are both expressed in the form of standard asset pricing equations (stochastically discounted cash flows, from the household’s perspective), which can be derived directly from household’s first order conditions (Appendix 1.6). More details can be found in Appendix 1.6. 

Collaterals Value and the Optimal Choice of Leverage for firms

For households, who never need to borrow, the capital good is simply a means to transfer wealth into the next period through production or resale. For firms however, the capital good serves as both an investment opportunity and the only means to secure loans (courtesy of the collateral constraint - equation 1.6). Consequently, from the firm’s perspective, the price of capital reflects both its stochastically discounted cash flow and its value as collateral.

When the firm purchases capital on leverage using contract $j$, the standard asset pricing equation yields:

$$
q_t - \pi_{j,t} = E_t \left[ \frac{\beta \gamma_{t+1}}{\gamma_t} \left( a_{t+1} + q_{t+1} - \delta_{j,t+1} \right) \right]
$$
where the left-hand-side of the expression is the down-payment required, and the right-hand-side gives the stochastically discounted cash flow from the transaction, composed of dividends plus the price next period and minus actual delivery on debt next period. \( \gamma_t \) denotes the marginal utility of income of firms in period \( t \) and history \( s^t \). Re-arranging this equation illustrates how the price of capital can be decomposed into its fundamental value and its collateral value to firms.

**Lemma 2. [Collateral Value]** When firms’ capital holding is strictly positive \( K_t > 0 \) and the collateral constraint is binding, the price of capital can be expressed as:

\[
q_t = E_t \left[ \frac{\beta \gamma_{t+1}}{\gamma_t} (a_{t+1} + q_{t+1}) \right] + \left\{ \pi_{j,t} - E_t \left[ \frac{\beta \gamma_{t+1}}{\gamma_t} \delta_{j,t+1} \right] \right\}
\]

where \( \gamma_t \) denotes the marginal utility of income of firms in period \( t \) and history \( s^t \); and \( \tilde{\gamma}_t = \tilde{\gamma}_{t+1} = 1 \) for households. Note that since the firm does not necessarily consume in every history, in general \( \gamma_t \neq u'(x_t) = 1 \).

Let \( \lambda_t \) denote the Lagrangian multiplier for the firm’s collateral constraint at period \( t \) and history \( s^t \), and \( \gamma_t \) the multiplier for its budget constraint, then the equilibrium collateral value can be expressed as:

\[
\text{Collateral Value} := \max_j \left\{ E_t \left[ \left( \frac{\beta \tilde{\gamma}_{t+1}}{\tilde{\gamma}_t} - \frac{\beta \gamma_{t+1}}{\gamma_t} \right) \delta_{j,t+1} \right], 0 \right\} = \frac{\lambda_t}{\tilde{\gamma}_t} \tag{1.19}
\]

In other words, collateral value is the dollar value the firm attaches to a marginal relaxation of its collateral constraint.

**Proof.** Both equations can be derived directly from firm’s first order conditions (Appendix 1.6).
Equation 1.18 highlights the dual role capital plays for firms, and equation 1.19 shows that the collateral value is positive whenever the firm’s collateral constraint is binding ($\lambda_t > 0$). Furthermore, when the collateral constraint is binding, the firm will choose the debt contract $j$ that maximizes collateral value. Thus even though an entire spectrum of contracts $j \in R_+$ is priced, potentially only a single contract (if any) will be actively traded in equilibrium. Lemma 3 below examines how the firm chooses the optimal contract $j^*$.

**Lemma 3. [Optimal leverage]** Suppose in equilibrium the collateral constraint is binding ($\lambda_t > 0$), then:

1. Without loss of generality, the firm will only consider contracts:

   $$ j \in [\tilde{j}, \bar{j}] $$

   where $\tilde{j} := \min_{s_{t+1}} (q_t (s_{t+1}))$ is the max-min leverage contract, and $\bar{j} := \max_{s_{t+1}} (q_t (s_{t+1}))$ is the maximum leverage contract.

2. The firm’s optimal choice of debt contract is given by:

   $$ j^* = \begin{cases} 
   \tilde{j} & \text{if } \gamma_t > \gamma_{t+1}^U \\
   \bar{j} & \text{if } \gamma_t < \gamma_{t+1}^U \\
   [\tilde{j}, \bar{j}] & \text{if } \gamma_t = \gamma_{t+1}^U
   \end{cases} $$

   where $\gamma_{t+1}^U$ is the firm’s marginal utility of income in period $t+1$ if the Up-state is realized (i.e. $s_{t+1} = U$).

Lemma 3 states that in general the firm will choose between either the left-hand or the right-hand side kink of the credit surface (see Figure 1.7). In the knife edge case where $\gamma_t = \gamma_{t+1}^U$, the firm is indifferent between all debt contracts $j \in [\tilde{j}, \bar{j}]$. A formal proof can be found in Appendix 1.6.
Due to linear preference, when firms borrow against capital as collateral, they do so with either the max-min leverage contract $\bar{j}$ or the maximum leverage contract $\tilde{j}$. Firms prefer the max-min leverage contract when their marginal utility of income today $\gamma_t$ is lower than their marginal utility of income tomorrow in the Up-state $\gamma_{t+1}^U$. Intuitively, this is because the max-min leverage contract entails a larger downpayment today, but allows the firm to transfer more resources into the Up-state tomorrow. In contrast, in any ensuing Down-state, firms default - regardless of whether they borrowed using the max-min leverage contract or the maximum leverage contract today.
To see why the choice between the max-min leverage contract $\bar{j}$ and the maximum leverage contract $j$ depends only on the marginal utility of income in the Up-state tomorrow $\gamma_{t+1}^U$, and not the Down-state $\gamma_{t+1}^D$, note that with buying $K$ using contract $\bar{j} = q_{t+1}^D$ the firm’s cash flow in period $t+1$ is given by:

$$
\begin{pmatrix}
a_U + q_{t+1}^U - q_{t+1}^D \\
a_D
\end{pmatrix}.
$$

In comparison, if the firm bought $K$ using contract $j = q_{t+1}^U$ instead, its cash flow would be given by $\begin{pmatrix} a_U \\ a_D \end{pmatrix}$. The net difference between the two is $\begin{pmatrix} q_{t+1}^U - q_{t+1}^D \\ 0 \end{pmatrix}$, an Up-Arrow security that delivers only in the Up-state. Therefore the firm would choose the max-min contract $\bar{j}$ when the marginal utility of income in the Up-state tomorrow is sufficiently high.

**Illustrative Example**

Having described households’ behavior and the choice of firms with regard to collateralized debt contracts, we can now proceed to characterize the collateral equilibrium along the different possible paths of the economy. The main complication that arises when solving the model is that prices and choices today depend on the expectation of prices tomorrow. The model is thus solved through backward induction, from deterministic steady states that can be eventually reached after the uncertainty has been fully resolved\(^8\), back to the uncertain histories in periods $t = 0, 1$. The path $(0 \rightarrow D \rightarrow DU \rightarrow DUU \ldots)$, whereby the firm’s productivity suffers a temporary negative shock in history $D$ before recovering permanently to $a_U$ in period $t \geq 2$, is the most interesting trajectory for our purpose. I illustrate the key features of the collateral equilibrium with a simple numerical example below, before extending the key results in the form of general propositions.

Consider an economy characterized by the set of parameters shown in Table 1.1. Restricting attention to the path where firms suffer a negative, but temporary,\(^8\)A full characterization of the deterministic steady states can be found in Appendix 1.6.
Table 1.1: Parameters for Illustrative Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>0.75</td>
<td>$\beta$</td>
<td>0.8</td>
</tr>
<tr>
<td>$a_U$</td>
<td>1.5</td>
<td>$\beta$</td>
<td>0.8</td>
</tr>
<tr>
<td>$a_D$</td>
<td>0.75</td>
<td>$e_0$</td>
<td>0.75</td>
</tr>
<tr>
<td>$G(\bar{K})$</td>
<td>$a_U\bar{K} - \frac{(a_U - a_D)\bar{K}}{2}\bar{K}^2$</td>
<td>$\bar{e}_{t \geq 0}$</td>
<td>6</td>
</tr>
</tbody>
</table>

Note: While assumption A4.2: $e_0 < \beta^2 a_D K_F^b$, which restricts firms’ starting endowment in period 0, is useful in the analytical proofs of my Propositions, the assumption is much stricter than necessary. In this illustrative example, I relax this assumption significantly and demonstrate that the key implications of the model are robust.

productivity shock ($0 \rightarrow D \rightarrow DU \rightarrow DUU \ldots$), Figure 1.8 plots the dynamics of the firms’ capital holding in the collateral equilibrium against the first-best benchmark. In the initial period 0, firms start with no capital and very limited endowment of the numeraire consumption good, so they borrow using the maximum leverage contract to purchase as much capital as they can. Endogenously, the model generates two further instances of productive inefficiency along this temporary downturn path. First, firms default in history $D$, their ability to borrow and leverage collapse and firms’ holding of capital falls even further. Second, even when the uncertainty has been fully resolved at history $DU$ and beyond, the recovery to the first-best level is gradual. I address the causes of each of these two distortions in turn.

The Downturn ($s^1 = D$)

During the downturn (history $D$), firms experience low productivity. The price of capital falls because a larger proportion of capital is transferred to households, whose production function exhibits diminishing returns. Firms would like to hold more capital in order to produce next period during the potential recovery phase, but cannot due to a tightening borrowing constraint arising from the reduction in their liquid wealth and a fall in the borrowing capacity of capital as collateral. Crucially, even though at both history 0 and $D$ there is a common probability $p$ that aggregate productivity will be high ($a_U$) next period and $(1 - p)$ that it will be low, the situation
Figure 1.8: Proportion of capital held by firms

The representative firm’s capital holding collapses during the downturn (history $D$) before gradually recovering in subsequent periods towards the first-best benchmark. Even though by history $DU$ all aggregate productivity uncertainty has been resolved, a full recovery is not achieved immediately.
Credit conditions deteriorate during the downturn (orange dashed line) relative to normal times (blue solid line). Firms that wish to borrow using the maximum leverage contract face both a higher interest rate and a higher haircut for each unit of capital posted as collateral.

for firms at history $D$ is much worse. This is because in history $DD$ the negative aggregate productivity shock would be permanent. The price of capital at history $DD$, $q_{DD}$, is significantly lower than that in history $D$. Consequently, each unit of capital is much more valuable as collateral at history 0 than at history $D$. Thus credit conditions deteriorate during the downturn relative to normal times (Figure 1.9). Firms that wish to borrow using the maximum leverage contract face both a higher interest rate and a higher haircut for each unit of capital posted as collateral. The collateral constraint faced by firms and this endogenous fall in the borrowing capacity of their collateral amplify the adverse aggregate productivity shock to the real economy, and feed back into even lower capital holding by firms.

I generalize these findings in the two propositions below. In the statement of these propositions (and corresponding proofs) I adopt a simplified set of notation,
combining the time subscript $t$ and the dependence of variables on histories $(s^t)$ into a single subscript whenever the context is clear. For instance, let $K_D := K_{t=1}(s^1 = D)$.

**Proposition 1. [Productive inefficiency and leverage during the downturn]**

Under assumptions A1 − 4 and given the no-bubble condition (equation 1.15), at history $D$:

1. $K_D < K_D^{fb}$: firms hold too little capital relative to the first best; and
2. $\sum_{j \in J} \max \{\varphi_{D,j}, 0\} = K_D$: the collateral constraint is binding (i.e. firms purchase capital using leveraged debt contracts).

**Proof.** Intuitively, at history $D$ credit conditions are so tight and firms’ liquid wealth are so low that firms are unable to purchase the first-best level of capital even if they borrow using the maximum leverage contract. Credit conditions tighten at history $D$ because the negative productivity shock significantly reduces the expected price of capital in the next period. Specifically, if the shock proves to be permanent, the price of capital in history $DD$ would be significantly lower. The arrival of the bad news at $D$ therefore endogenously reduces the borrowing capacity of firms against each unit of collateral. Moreover, firms have limited means to transfer their limited liquid wealth from the initial history 0 to history $D$. If firms used leverage during history 0 to purchase capital, they would default at history $D$, hand over the collateral to lenders and retain only the consumption goods produced. The resulting amount of liquid wealth is insufficient to purchase the first best level of capital at $D$ given the prevailing credit conditions. On the other hand, firms may try to transfer more resources into history $D$ by purchasing capital without leverage at history 0 (and thus avoiding default at $D$).\(^9\) But for firms to have enough liquid wealth at history $D$

\(^9\)Another way for firms to transfer resources into history $D$ is by lending to households at history 0. However, since households start with the entire stock of capital and their productivity is concave, it is easy to show that firms strictly prefer purchasing capital without leverage to lending at history 0.
to purchase $K_{D}^{fb}$ with this strategy, the price of capital in the downturn $q_{D}$ must be sufficiently high relative to its initial price $q_{0}$. In such situations, the household would also like to purchase more capital in period 0, which would push $q_{0}$ beyond the level required for the firms’ strategy to succeed. For the second part of the proposition, because firms hold too little capital relative to the first-best benchmark at history $D$, they are in expectations more productive than households and would like to utilize leveraged debt contracts to increase their holding of capital. A formal proof can be found in Appendix 1.6.

Proposition 1 states that firms would use leveraged debt contracts during the downturn, but is silent on the optimal choice of contracts $j_{D}^{*}$. From corollary 3 we know that the choice is essentially between the maximum leverage contract $\tilde{j}_{D} = q_{DU}$ and the max-min (risk-free) leverage contract $\hat{j}_{D} = q_{DD}$. The following proposition shows that firms will use maximum leverage when the downturn is “severe”, and max-min leverage when it is more “moderate”.

**Proposition 2.** *Optimal leverage during the downturn* Under assumptions $A1-A4$ and given the no-bubble condition (equation 1.15), at history $D \exists \hat{K}_{D} \in (0, K_{D}^{fb}]$ s.t.

$$j_{D}^{*} = \begin{cases} 
\tilde{j}_{D} := q_{DU} & \text{if } K_{D} < \hat{K}_{D} \\
\hat{j}_{D} := q_{DD} & \text{if } K_{D} > \hat{K}_{D} \\
\{\tilde{j}_{D}, \hat{j}_{D}\} & \text{if } K_{D} = \hat{K}_{D}
\end{cases} \quad (1.20)$$

*Proof.* From Lemma 3, we have previously established that firms prefer the maximum leverage contract over the max-min leverage contract when their marginal utility of income today is sufficiently high relative to their marginal utility of income tomorrow in the Up-State. When firms start history $D$ with very limited liquid wealth, households will hold the majority of capital in equilibrium and firms will hold very
little (high $\bar{K}_D$ and low $K_D$). This implies that the equilibrium price of capital will be very low from the perspective of the firm, so firms’ marginal utility of income at history $D$ is very high and they would like to borrow the maximum amount possible to take advantage of these fire sale prices. In contrast, if firms have access to more resources at history $D$, the price of capital would be higher and firms may find it optimal to use the max-min leverage contract to transfer more resources into the Up-state at history $DU$ where the uncertainty surrounding their productivity has been resolved favorably. Given the linearity in preferences and the continuity of households’ production function $G(\cdot)$, there exists a threshold value of $K_D$ whereby the firms are indifferent between the max-min leverage contract and the maximum leverage contract. A formal proof can be found in Appendix 1.6.

In summary, Proposition 2 shows that when the production distortions in the downturn is severe (i.e. very low $K_D$) firms would like to use maximum leverage contracts to maximize their purchasing power. But when the production distortion is more moderate ($K_D$ closer to $K_D^{lb}$), then the max-min leverage contract is optimal instead.

**The Recovery** ($s^2 = DU$)

In the illustrative example (Figure 1.8) above, in the recovery phase firms honor the promise made in history $D$ and repay the loan ($\bar{j}_D := q_{DU}$) by either selling their capital holding or equivalently handing over the collateral posted. This leaves firms with just the output from production $w_{DU} = a_U K_D$ with which to rebuild their stock of capital. Unfortunately, this amount of liquid wealth is insufficient to purchase the first-best level of capital, even if they use the maximum leverage contract to minimize the size of the down-payment required. Thus full recovery is not achieved at history $DU$ even though the uncertainty surrounding aggregate productivity has been fully resolved. Lemma 4 generalizes this result.
Lemma 4. *[Gradual Recovery]*:

1. If firms used maximum leverage at \( s^1 = D \), then \( K_{DU} < K_{DU}^{fb} \) if:

\[
K_D < \beta K_{DU}^{fb} \equiv \beta \left[ 1 - G^{-1} \left( a_U \right) \right] \quad (1.21)
\]

2. If firms used max-min leverage at \( s^1 = D \), then \( K_{DU} < K_{DU}^{fb} \) if:

\[
K_D < \beta K_{DU}^{fb} \left[ \frac{1 - \beta}{1 - \beta a_D} \right] \quad (1.22)
\]

**Proof.** If firms used maximum leverage contracts at \( s^1 = D \), then their liquid wealth at history \( DU \) is given by \( w_{DU,(j_D^* = j_D)} = a_U K_D \). In order to purchase the first-best level of capital \( K_{DU}^{fb} \), the minimum down-payment required in equilibrium is given by: \( q_{DU} - \pi_{j_D} = \beta G' \left( \hat{K}_{DU}^{fb} \right) = \beta a_U \) per unit of capital. So \( K_{DU} \) is less than \( K_{DU}^{fb} \) whenever \( K_D < \beta K_{DU}^{fb} \). If instead firms used maximum leverage contracts at \( s^1 = D \), then their liquid wealth at history \( DU \) is given by \( w_{DU,(j_D^* = j_D)} = (a_U + q_{DU} - q_{DD}) K_D \), where \( (q_{DU} - q_{DD}) \) is bounded above by \( \left( \frac{\beta}{1 - \beta} a_U - \frac{\beta}{1 - \beta} a_D \right) \). Comparing \( w_{DU} \) with the minimum down-payment required again yields inequality 1.22. \( \square \)

Recovery from a downturn will typically be gradual because firms held too little capital in history \( D \) and will need more time to build up sufficient liquid wealth to purchase the efficient level of capital at equilibrium prices. Recovery from a moderate downturn will be faster than that from a severe downturns for two reasons. First, trivially, firms retain a higher stock of capital in a moderate downturn (by definition) and this increases their production in history \( DU \). Second, more subtly, firms optimally choose a lower level of leverage during moderate downturns (the max-min leverage contract \( j_D^* \) instead of the maximum leverage contract \( \tilde{j}_D \) - see Proposition 2). This more conservative choice of leverage helps firms transfer a higher
level of liquid wealth into the recovery phase, reducing the time required to achieve the efficient level of production.

### 1.4 Central Bank Intervention

In the previous section, we saw how the negative aggregate productivity shock to the real economy is amplified by the borrowing constraints faced by firms. A combination of reduced liquid wealth and decline in the borrowing capacity of collateral leads to firms holding too little capital during the downturn relative to the first-best. A central bank can intervene in this case by lending to firms against collateral at more favorable terms relative to the market \( \pi_{j,s}^{CB} := (1 + \chi_{j,s}) \pi_{j,s} \). Central bank lending is funded through the issuance of a public liability \( m \) to households that carries the risk-free rate of interest, and crucially, without the need to post collateral. The central bank is able to borrow at the risk-free rate without posting collateral because it is backed by the ability of the government to tax. As such, any ex post losses incurred on central bank loans will need to be recouped through recourse to the Treasury. By circumventing the collateral constraints faced by firms, a central bank can fully alleviate the aggregate productive inefficiencies in this model if it is willing to take on the necessary level of credit risks.

Formally, the central bank aims to achieve the first-best efficient level of aggregate production, by choosing subsidy \( \{\chi_{j,t}\} \) on loan \( j \) at history \( s^t \) and the amount of public liability issued \( m_t \) at history \( s^t \) (omitting the \( (s^t) \) notation henceforth for brevity):

\[
\max_{\{\chi_{j,t}\}, \{m_t\}} \sum_{\tau=0}^{\infty} \beta^{\tau+1} E_t \left[ G \left( \bar{K}_{t+\tau} \right) + a_{t+\tau+1} K_{t+\tau} \right]
\]  

(1.23)
subject to households’ optimization (equations 1.4 - 1.6), firms’ optimization (equations 1.7 - 1.9) and the public sector budget constraint \( \forall \tau \geq 0 \):

\[
\begin{align*}
\frac{1}{\beta} m_{t+\tau-1} & - \sum_j \varphi_{j,t+\tau} \pi_{j,t+\tau} \\
\text{repayment on outstanding public liability} & - \sum_j \varphi_{j,t+\tau} \pi_{j,t+\tau} \\
\leq m_{t+\tau} & - \sum_j \varphi_{j,t+\tau} \delta_{j,t+\tau} + T_{t+\tau} \\
\text{issuance of new public liability} & + \text{transfer from/to Treasury} \\
\text{delivery on outstanding loans}
\end{align*}
\] (1.24)

where \( \varphi_{j}^{CB} \) denotes the number of collateralized debt contract \( j \) held by the central bank, with \( \varphi_{j}^{CB} < 0 \) indicating the central bank is selling contract \( j \) (i.e. lending).

The size of the loan offered on contract \( j \): \( \pi_{j}^{CB} := (1 + \chi_{j}) \pi_{j} \) depends on the size of the subsidy offered \( (\chi_{j}) \) relative to prevailing market rates. Lastly, \( T \) is the lump sum tax required to balance the budget in every history (representing recourse to the Treasury when positive and transfer of surplus when negative). The new market clearing condition for collateralized debt contracts is given by:

\[
\varphi_{j,t+\tau} + \varphi_{j,t+\tau} + \tilde{\varphi}_{j,t+\tau} = 0 \quad \forall \tau \geq 0, \forall j \in J
\] (1.25)

A key point here is that the central bank can intervene both across time \( t \) and across the credit surface. By choosing which segment of the credit surface, or which specific contract \( j \), it is willing to subsidize \( \{ \chi_{j,t} \} \), the central bank is simultaneously setting both the interest rate and the haircut it imposes on the collateralized loan to firms:

\[
(1 + r_{j,t}) := \frac{j}{\pi_{j,t}^{CB}}
\] (1.26)

\[
haircut_{j,t} := 1 - LTV_{j,t} = 1 - \frac{\pi_{j,t}^{CB}}{q_{t}}
\] (1.27)
The central bank can also choose the timing of the intervention. Since period 0 production is heavily dependent on starting endowments (which are exogenous parameters in the model), we will instead focus attention on the optimal policy intervention during the downturn, when credit conditions tighten and firms’ holding of capital falls endogenously. In the following sections, we will first examine the optimal policy intervention during the downturn (history D) when the central bank’s actions are unanticipated, before turning to the period 0 impact of such policies if they were anticipated instead.

Unanticipated intervention at history D

From Proposition 2 we see that firms will choose either the max-min leverage contract \( j_D \) or the maximum leverage contract \( \bar{j}_D \) during the downturn depending on the severity of the recession. Correspondingly, central bank responses can be grouped into two broad categories: (1) intervene at the risk-free, low LTV end of the credit surface \( j_D \); or (2) intervene at the high LTV end \( \bar{j}_D \).

Proposition 3 states that when the downturn is severe and firms wish to borrow using the maximally leveraged loans, then the central bank can bring about the first-best level of capital holding by subsidizing contract \( \bar{j}_D \). In contrast, intervening at the risk-free end of the credit surface may not be sufficient to achieve the first-best outcome, when the central bank abides by an effective-lower-bound on interest rates\(^{10} \) and never lends more than the promised repayment amount (\( \pi^{CB}_j \leq j, \forall j \)).

Proposition 3. [Unanticipated Intervention] Under assumptions A1 – 4 and given the no-bubble condition (equation 1.15), at history D if \( j^*_D = \bar{j}_D \), then:

1. There exists \( \chi^{*}_{JD} > 0 \) s.t. \( \pi^{CB}_{jD} \leq \bar{j}_D \) and \( K^{CB}_D = K^{fb}_D \); and

\(^{10}\)The effective-lower-bound here is not a constraint in the strict sense, because the central bank can always choose to lend at an negative interest rate that is below its cost of funding. But doing so guarantees a loss of public funds on every loan made.
2. There may not exist $\chi^*_{LD} > 0$ s.t. $\pi^*_{LD} \leq \bar{f}_D$ and $K_{CB}^D = K_{fb}^D$.

Proof. Intuitively, when firms find it optimal to use the maximum leverage contract $j^*_D = \bar{j}_D$ during the downturn, they will purchase as much capital as they can with their available liquid wealth $w_D$. The central bank can increase the amount of capital the firms can buy for given $w_D$ by subsidizing loans to the firms and thus effectively reducing the down-payment required on each unit of capital. The intervention will raise the equilibrium price of capital during the downturn ($q_{CB}^D > q_D^D$) as well as during any subsequent recovery ($q_{CU}^D > q_{DU}$), which will affect credit conditions in the private market during the downturn (see Figure 1.10, dashed line). The central bank can offset the impact of these price increases on the down-payment required for firms by simultaneously reducing the haircuts and the interest rates on high LTV loans, pushing the credit surface downwards and outwards (Figure 1.10, dotted line). In fact, when the central bank is willing to intervene at the high LTV segment of the market, it is always possible to reduce the down-payment faced by firms such that they can purchase the first-best level of capital for given liquid wealth $w_D$. In contrast, when the central bank intervenes only at the risk-free segment of the market, the maximum loan amount it can offer while ensuring full repayment is given by the price of capital in history $DD$, where the firms are permanently unproductive. In general, the price of capital at $DD$, $q_{DD}$, is so low that the loan amount - even with the central bank subsidy - is insufficient to allow firms to purchase the first-best level of capital. The full proof can be found in Appendix 1.6.

Figure 1.11 plots firms’ capital holding given optimal central bank intervention, along the path where the productivity shock is temporary, using the same illustrative parameters as previously. It shows that when the central bank is willing to take on credit risks, it becomes possible for firms to hold the socially efficient level of capital.
The solid orange line plots the credit surface during the downturn in the absence of any central bank intervention. Given this deterioration in credit conditions, the central bank intervenes at the riskier segment of the credit market. This intervention raises the price of capital immediately ($q_{CB} > q_D$) as well as during any subsequent recovery ($q_{CB}^{DU} > q_{DU}$). The net effect of the price increase is that for given contract $j < j_D$ on the private market, the interest rate remains unchanged but the loan to value (LTV) falls (green dashed line). But the central bank intervention simultaneously lowers the interest rate and the haircut on high LTV loans, so the actual credit surface faced by firms is given by the green dotted line. As a consequence of the intervention, firms can borrow at a higher LTV and at a lower interest rate.
Figure 1.11: Unanticipated Intervention: Firms’ Capital Holding

The unanticipated central bank intervention during the downturn ensures that firms can purchase the first-best benchmark level of capital at history $D$. This increase in capital holding improves firms’ financial position at history $DU$ as well. Consequently, the recovery at history $DU$ (green dotted line) is more robust than that under the no-intervention case (orange dashed line).

Moreover, the recovery is more robust in the following period because firms hold more capital during the downturn.

**Sustained support at history $DU$ and its announcement effect at history $D$**

From Figure 1.11 we see that even though the recovery at history $DU$ is stronger with central bank intervention at $D$ than without, firms’ capital holding does not immediately reach the first-best benchmark once the uncertainty surrounding aggregate productivity has been resolved.\(^{11}\) This is because even if the firms held the first-best level of capital during the downturn ($K_{CB}^{D} = K_{fb}^{D}$), their available liquid

\(^{11}\)In the illustrative example, full recovery occurs at period $t = 3$ and history $DUU$, one period after the resolution of uncertainty at period $t = 2$. A more protracted recovery period is possible with alternative parameterizations, with full recovery not reached until period $t = 4$ or beyond.
wealth during the recovery may not be sufficient, in general, to ramp up production fully to the new, higher, efficient level \((K^{fb}_{DU} > K^{fb}_D)\).

An immediate implication of the above observation is that it is optimal for the central bank to continue its credit support during the recovery phase: \(\chi^*_jDU=q_{DUU} > 0\).

**Lemma 5.** [Sustained credit support] If \(K_{DU} < K^{fb}_{DU}\), then \(\exists \chi^*_jDU=q_{DUU} > 0\) s.t. \(\pi_{jDU}^{CB} \leq q_{DUU}\) and \(K_{DU}^{CB} = K^{fb}_{DU}\).

**Proof.** The proof of Lemma 5 is analogous to that for the first part of Proposition 3. \(\square\)

A second, more subtle, implication is that whether or not the private sector anticipates continued credit support at history \(DU\) can have important fiscal implications for the public sector. Specifically, suppose the central bank credibly announces its commitment to sustained credit support if and only if the resolution of uncertainty next period is favorable (i.e. at history \(DU\) but not at \(DD\)). The anticipation of credit support during the recovery raises the expected price of capital at \(DU\), which in turn increases the borrowing capacity of each unit of capital at \(D\). Credit conditions in the private market ease and capital prices rise immediately at \(D\). The net result is a reduction in the rate at which central bank loans must be subsidized (\(\chi\) falls), but an overall increase in the total loan amount \(((1 + \chi)\pi\) increases). With or without the announcement, the central bank still ensures firms hold the first-best level of capital at \(D\); but with the announcement the larger total loan size means that the return on public funds in the following period becomes a mean-preserving spread.

Formally, when the central bank makes the announcement, let \(q_{DU}^{An}\) denote the price of capital at history \(D\); \(q_{DU}^{An}\) the price at history \(DU\); \(\pi_{jD=q_{DU}^{An}}\) the price of the maximum leverage contract on the private credit market at \(D\); and \(\chi_{jD=q_{DU}^{An}}\) the level of central bank subsidy required to achieve the first-best benchmark at \(D\). Furthermore, let central bank lending at \(D\) be financed through the issuance of the risk-free
public liability: 

$$m_D^{An} = \left(1 + \chi_{j_D=q_{DU}}^{An}\right) \pi_{j_D=q_{DU}}.$$ 

And let any shortfalls/windfalls in the following period ($t = 2$) be met through taxation/transfers: 

$$E_D \left[ T_2 | \chi_{j_D}^{An} \right] = \left[ \frac{1}{q} \left(1 + \chi_{j_D=q_{DU}}^{An}\right) \pi_{j_D=q_{DU}} \right] - E \left[ q_2^{An} \right],$$ 

where $E \left[ q_2^{An} \right]$ is the expected price of capital next period and is equal to the expected delivery on the maximum leverage contract. I prove the following proposition:

**Proposition 4. [The Announcement Effect]** If firms used leverage at history 0: 

$$j_0^* \in \left[ j_0, \bar{j}_0 \right],$$ 

and prefer the maximum leverage contract during the downturn $j_D^* = \bar{j}_D$, then, when the central bank announces conditional credit support at history $D$, such that $\chi_{j_{DU}=q_{DU}}^{An} > 0$ and $\chi_{j_{DD}}^{An} = 0$:

1. The optimal rate of subsidy falls, 

$$\frac{\chi_{j_D}^{An}}{\chi_{j_D}^{An}} = \frac{\pi_{j_D=q_{DU}}^{CB}}{\pi_{j_D=q_{DU}}^{An}} < 1,$$

but the optimal total loan size increases, 

$$\left(1 + \chi_{j_D}^{An}\right) \pi_{j_D=q_{DU}} > \left(1 + \chi_{j_D}^{An}\right) \pi_{j_D=q_{DU}}^{CB}.$$ 

2. The return on public funds becomes a mean-preserving spread relative to the no announcement case: 

$$E \left[ T_2 | \chi_{j_D}^{An} \right] = E \left[ T_2 | \chi_{j_D}^{An}\right],$$ 

and 

$$T_{DU} | \chi_{j_D}^{An} < T_{DU} | \chi_{j_D}^{An} \text{ (larger windfall in the recovery phase)}$$ 

and 

$$T_{DD} | \chi_{j_D}^{An} > T_{DD} | \chi_{j_D}^{An} \text{ (larger shortfall in the permanent downturn).}$$

**Proof.** The intuition of the proof is as follows. A credible announcement of continued credit support during any subsequent recovery raises the price of capital during the down-state. Conventionally, one would expect this price increase to improve the balance sheet position of firms and reduce the amount of central bank lending required. However, if firms used leverage at history 0: 

$$j_0^* \in \left[ j_0, \bar{j}_0 \right],$$ 

they would default in history $D$ and hand over their entire stock of capital to lenders. As a result, firms must rebuild their stock of capital from scratch at history $D$. The increase in capital prices therefore increases the total amount the central bank must lend to firms, even as private market credit conditions improve and the required rate of subsidy falls. A larger initial loan implies a larger windfall in the recovery phase but also a larger shortfall in any permanent downturns. Lastly, since with or without
the announcement the central bank will ensure the firms hold the first-best level of capital at history $D$, the expected cost to taxpayers is unchanged. A formal proof can be found in Appendix 1.6.

Proposition 4 shows that even though a credible announcement of sustained credit support in the future can have significant immediate effects on asset prices and credit conditions, the expected return/loss on public funds may stay the same with or without the announcement. This is a somewhat surprising result. In traditional Diamond-Dybvig style models with multiple equilibria, a commitment to “do whatever it takes” can shift the economy to a more virtuous equilibrium and reduce the initial stimulus required. In my model, having abstracted away from the asymmetry of information that is central to models of financial crises and panics, we see that a credible announcement of future support provides no additional gains when responding to big shocks to the real economy. In fact, if the public sector is instead risk-averse with taxpayer funds, it might prefer to “surprise” the market with additional stimulus instead of announcing them in advance. The model therefore suggests that the optimal central bank response might look very different when the shocks are predominantly hitting the real economy (such as during the COVID-19 crisis) as opposed to the financial sector.

Anticipated Intervention

What if the intervention at history $D$ and beyond are anticipated by market participants in advance? Conventional wisdom suggests that if agents expect the central bank to provide credit support during a downturn, they may take on excessive leverage during normal times, potentially leading to greater productive distortions and exacerbating the severity of the downturn. There exists however an alternative narrative. Anticipating low asset prices and central bank credit support during the
downturn, firms may be incentivized to reduce leverage in normal times in order to exploit more favorable investment opportunities during the downturn.

In the following Proposition, I show that if firms are using maximum leverage contracts \( j_0^* = j_0^U = q_U \) at history 0, then the anticipation of central bank intervention at history \( D \) will lead to either: (1) firms continuing to use maximum leverage contracts at period 0 and hold the same level of capital at history 0; or (2) a jump to an equilibrium whereby firms are purchasing capital without leverage at history 0.

**Proposition 5.** *Waiting for the downturn* If \( j_0^* = j_0^U = q_U \) in the absence of central bank intervention, and it becomes common knowledge that \( \chi_{j_D}^* > 0 \), then either:

1. \( j_0^* = j_0^{CB} := q_U^{CB} \) and \( K_0 = K_0^{CB} \); or
2. \( \varphi_{0,j}^{CB} = 0, \forall j \).

where \( q_U^{CB}, K_0^{CB} \) and \( \varphi_{0,j}^{CB} \) denote respectively the equilibrium price for capital in history \( U \), the firms’ optimal capital holding and contract sales at history 0, when central bank intervention is fully anticipated.

**Proof.** For the first part of the proposition, I show that if firms continue to find it optimal to use the maximum leverage contract at \( t = 0 \), then their capital holding in equilibrium remains unchanged. This is because when firms use maximum leverage contracts, the required down-payment is given by the opportunity cost of production faced by households (see Corollary 1 and equation 1.41). Therefore \( K_0 = \frac{e_0}{q_0 - \pi_0} = \frac{e_0}{\beta G(K - K_0)} \) and \( K_0^{CB} = \frac{e_0}{q_0^{CB} - \pi_0^{CB}} = \frac{e_0}{\beta G(K - K_0^{CB})} \). And given \( G'(\cdot) < 0 \), we have \( K_0 = K_0^{CB} \). The capital holding of firms remains unchanged despite the change in equilibrium prices and credit conditions.

For the second part of the proposition, recall that the payoff to the firm from purchasing capital at history 0 without leverage is given by \( \left( \begin{array}{c} a_U + q_U \\ a_D + q_D \end{array} \right) \) in the two
possible states. Correspondingly the payoff to the firm from purchasing capital on
leverage is \( \begin{pmatrix} a_U + q_U - q_D \\ a_D \end{pmatrix} \) with the max-min leverage contract, and \( \begin{pmatrix} a_U \\ a_D \end{pmatrix} \) with the maximum leverage contract. We see that the difference in payoffs between the
max-min leverage contract and the maximum leverage contract is simply an Up-Arrow
security \( \begin{pmatrix} q_U - q_D \\ 0 \end{pmatrix} \), which is reflected in a higher down-payment required with the
max-min leverage contract. Consequently, firms will only switch from the maximum
leverage contract to the max-min leverage contract when their marginal utility of
income in the up-state is sufficiently higher than their marginal utility of income in
the current period\(^{12} \). The central bank intervention at history \( D \) is aimed at reducing
the down-payment required when purchasing capital on leverage during the downturn.
This drives up the firms’ marginal utility of income at \( D \): \( \gamma^{CB}_0 > \gamma_D \), but leaves their
marginal utility of income at \( U \) unchanged. Thus the firm will either continue with
the maximum leverage contract at history 0, or switch to purchasing capital without
leverage. Purchasing capital without leverage generates the highest payoff in the
down-state for each unit of capital, and may become the optimal choice for firms
when their marginal utility of income at \( D \) becomes sufficiently high. When firms
purchase capital without leverage: \( \varphi^{CB}_{0,j} = 0, \forall j \).

A subtle implication of Proposition 5 is that even when firms continue to use
the maximum leverage contract at period 0 and their holding of capital remain unchanged,
the loan to value on their collateralized debt contract may actually increase due to
the change in equilibrium asset prices. The loan to value on the maximum leverage
contract at history 0 is the ratio between the size of the loan \( \pi_{j_0} = \beta [pq_U + (1 - p) q_D] \)
and the price of capital \( q_0 \). The anticipation of central bank interventions increases
both \( q_0 \) and \( q_D \), the former reduces LTV on debt contracts and the latter increases it.
For broad parameterizations, the proportional increase in \( q_D \) typically outweigh the

\(^{12}\) For more details, see Lemma 3
increase in $q_0$, so the net effect is often an increase in the period 0 leverage of the firm. The proposition shows that when the economy is already highly leveraged in normal times, the anticipation of central bank interventions during any potential downturns may indeed increase leverage further. However firms may be borrowing more just to purchase the same amount of capital, because the price of capital is higher when central bank interventions are anticipated. Therefore the increase in leverage during normal times does not always translate directly into greater productive inefficiencies.

1.5 Concluding Remarks

A salient feature of the “central bank intervention” examined in this paper is the potential need for recourse to taxpayer funds, when losses are incurred on the loans extended. In this sense, the intervention is a joint public sector effort - similar to the style of the programs we are seeing during the current pandemic. For instance, in both the Primary and Secondary Market Corporate Credit Facility, the US Treasury provides the equity investment in the facility, and the Federal Reserve lends against the assets held in the facility.

A second point is that although every debt contract is collateralized by a durable asset in the model, in practice a substantial portion of the underlying bonds/loans in the credit facilities may not be explicitly secured. Nevertheless, such loans are still implicitly backed by the value of the tangible assets of the firm. So a high-LTV collateralized debt contract in the model simply corresponds to a loan with a higher degree of credit risk to the lender. The central message of the paper - that central banks should take on greater credit risk when the downturn is severe - remains unchanged regardless of whether the loan is contractually collateralized. The analytical framework presented here highlights the role of defaults and limited enforcement of repayment in equilibrium outcomes. It can be extended into more general settings
where, instead of posting collateral explicitly, promises on debt contracts are backed by the assets of the firm.\textsuperscript{13}

Third, the central bank in the model lends directly to firms that engage in production, whereas traditionally central banks preferred to lend to financial intermediaries. Extending the model to encompass a third sector functioning as the financial intermediary is an interesting direction for future research. Nevertheless, current credit facilities like the Main Street Lending Program in the US demonstrate a policy shift towards more direct lending to the real economy.

Most importantly, the paper emphasizes that central banks have much more than just interest rates in their policy toolkit. Collateral requirements (and risk appetites) are also an important part of the transmission mechanism. Under the Bagehot rule and as standard practice during normal times, collateral requirements at central banks are set mechanically to ensure the central bank never takes on significant credit risk. This leaves the risk-free interest rate as the main policy instrument. In my model - as in real life - variations in collateral requirements allow central banks to intervene across the credit surface, and to provide support to riskier borrowers who may be shut out of the credit markets during a downturn. As such, collateral requirements in central bank lending should play an integral role in policy discussions.

\textsuperscript{13}In a separate working paper, Du (2020), I analyze the collateral equilibrium in economies with more than two states of nature when debt contracts can be ordered by seniority and backed by financial assets. For instance, with three states of nature, borrowers can issue both senior secured debt and junior subordinated debt, where the junior subordinated debt is backed by the residual value of the firm after senior creditors are repaid. I show that any equilibrium in this economy is equivalent to another equilibrium where the senior tranche never defaults, and the junior tranche only defaults in the worst state of the world.
References


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1.6 Appendices

First-order conditions to agents’ optimization

The Lagrangian for the firm’s optimization problem (eqns 1.4 to 1.6) is given by:

\[ L = E_t \{ \sum_{t=0}^{\infty} \beta^t u(x_t) \]

\[ - \sum_{t=0}^{\infty} \beta^t \gamma_t \left[ (x_t - a_t K_{t-1} - e_t) + q_t (K_t - K_{t-1}) + \sum_{j \in J} \varphi_{j,t} \min \{ j, q_t \} - \sum_{j \in J} \varphi_{j,t} \pi_{j,t} \right] \]

\[ - \sum_{t=0}^{\infty} \beta^t \lambda_t \left[ \sum_{j} \max (\varphi_{j,t}, 0) - K_t \right] \]

where \( \gamma_t \) denotes the firm’s marginal utility of income at period \( t \), and \( \lambda_t \) denotes the marginal utility from relaxing the firm’s collateral constraint.

The first-order conditions for the firm can be derived as follows (omitting the notation \( (s^t) \) for brevity):

\[ \frac{\partial L_t}{\partial x_t} = \beta^t u'(x_t) - \beta^t \gamma_t \leq 0 \quad (1.28) \]

\[ \frac{\partial L_t}{\partial K_t} = -\beta^t \gamma_t q_t + E_t [\beta^{t+1} \gamma_{t+1} (a_{t+1} + q_{t+1})] + \beta^t \lambda_t \leq 0 \quad (1.29) \]

\[ \frac{\partial L_t}{\partial \varphi_{j,t}} = \begin{cases} 
\beta^t \gamma_t \pi_{j,t} - E_t [\beta^{t+1} \gamma_{t+1} \delta_{j,t+1}] - \beta^t \lambda_t \leq 0 & \text{if } \varphi_{j,t} \geq 0 \\
\beta^t \gamma_t \pi_{j,t} - E_t [\beta^{t+1} \gamma_{t+1} \delta_{j,t+1}] \geq 0 & \text{if } \varphi_{j,t} \leq 0 \end{cases} \forall j \quad (1.30) \]

\[ x_t \frac{\partial L_t}{\partial x_t} = 0 \quad (1.31) \]

\[ K_t \frac{\partial L_t}{\partial K_t} = 0 \quad (1.32) \]

\[ \varphi_{j,t} \left( \frac{\partial L_t}{\partial \varphi_{j,t}} \right) = 0 \quad \forall j \quad (1.33) \]
Correspondingly, the first-order conditions for the household are given by:

\[
\frac{\partial \tilde{L}_t}{\partial \tilde{x}_t} = \tilde{\beta}^t u' (\tilde{x}_t) - \tilde{\beta}^t \tilde{\gamma}_t \leq 0 \quad (1.34)
\]

\[
\frac{\partial \tilde{L}_t}{\partial \tilde{K}_t} = -\tilde{\beta}^t \tilde{\gamma}_t \tilde{q}_t + E_t \left[ \tilde{\beta}^{t+1} \tilde{\gamma}_{t+1} \left( G' (\tilde{K}_{t+1}) + q_{t+1} \right) \right] + \tilde{\beta}^t \tilde{\lambda}_t \leq 0 \quad (1.35)
\]

\[
\frac{\partial \tilde{L}_t}{\partial \tilde{\varphi}_{j,t}} = \begin{cases} 
\tilde{\beta}^t \tilde{\gamma}_t \tilde{\pi}_{j,t} - E_t \left[ \tilde{\beta}^{t+1} \tilde{\gamma}_{t+1} \delta_{j,t+1} \right] - \tilde{\beta}^t \tilde{\lambda}_t \leq 0 & \text{if } \tilde{\varphi}_{j,t} \geq 0 \\
\tilde{\beta}^t \tilde{\gamma}_t \tilde{\pi}_{j,t} - E_t \left[ \tilde{\beta}^{t+1} \tilde{\gamma}_{t+1} \delta_{j,t+1} \right] \geq 0 & \text{if } \tilde{\varphi}_{j,t} \leq 0 
\end{cases} \quad \forall j \quad (1.36)
\]

\[
\tilde{x}_t \frac{\partial \tilde{L}_t}{\partial \tilde{x}_t} = 0 \quad (1.37)
\]

\[
\tilde{K}_t \frac{\partial \tilde{L}_t}{\partial \tilde{K}_t} = 0 \quad (1.38)
\]

\[
\tilde{\varphi}_{j,t} \left( \frac{\partial \tilde{L}_t}{\partial \tilde{\varphi}_{j,t}} \right) = 0 \quad \forall j \quad (1.39)
\]

**Household behavior**

In this subsection I provide a sketch of the proof for Lemma 1, as well as further discussions on its implications.

Lemma 1 follows quite naturally from assumptions A1-A4. In particular, assumption A4.3 ensures that households always have sufficient endowments to consume after any desired capital purchases, current lending and repayment on existing debt obligations. Households would not borrow from firms in order purchase capital on leverage because (1) households are well endowed in every history (A4.3); and (2) household production exhibits diminishing return to scale (A3.2).

Households are always indifferent between consumption and holding capital. Suppose not, then households are consuming yet strictly prefer holding capital. By linearity of preferences, households must be holding the entire stock of capital. But by assumption \( G' (\tilde{K}) = a_D \), so the price of capital must be so low that firms also want to hold capital, thus violating the market clearing condition for equilibrium. Similarly, if households strictly prefers consumption over holding capital, their marginal
productivity is $G' \left(0\right) = a_U$, and the price of capital would be too high for firms to want to hold capital either in equilibrium, leading to another contradiction.

Lastly, when households lend (i.e. purchase debt contract $j$), they do so at a price that equates their marginal utility from lending to their marginal utility of consumption. Consequently, both the price of capital $q$ (equation 1.16) and the price of loans $\pi_j$ (equation 1.17) can be derived from the household’s first-order conditions.

A consequence of these two asset pricing equations is that a strictly positive down-payment is always required when purchasing capital on leverage.

**Corollary 1.** *Down-payments are strictly positive in every history and for every contract $j$.*

\[
q_t - \pi_{j,t} = E_t \left[ \beta \left( G' \left( \tilde{K}_t \right) + q_{t+1} - \min \left\{ j, q_{t+1} + 1 \right\} \right) \right] \\
\geq \beta G' \left( \tilde{K}_t \right) > 0 \quad \forall j, t, s^t
\]

Specifically, the down-payment on the maximum leverage contract $\bar{j} := \max_{s^{t+1}} \{q_{t+1}(s^{t+1})\}$ is given by:

\[
q_t - \pi_{\bar{j},t} = \beta G' \left( \tilde{K}_t \right) > 0
\]

Corollary 1 shows that the loan will never exceed the price of the capital good posted as collateral. Intuitively, this is because the price of the capital good today reflects both its price next period as well as its expected production next period, but the output from production cannot be collateralized, so the price of the loan reflects only the price of the capital good next period. When the firm defaults, it hands over only the capital good, and not the output from its production.

For the maximally leveraged contract, $\bar{j} = \max_{s^{t+1}} q_{t+1}(s^{t+1})$, we have: $q_t - \pi_{\bar{j}} = \beta G' \left( \tilde{K}_t \right)$. The down-payment on this maximum leverage contract is equal to the value of foregone production for households, and depends on households’ preferences, production function, and capital holding.
Optimal Leverage for Firms (Proof for Lemma 3)

The first part of Lemma 3 states that we can restrict attention to \( j \in J = [\hat{j}, \bar{j}] \).
Recall \( \hat{j} := \min_{s,t+1} (q_t (s^{t+1})) \) is the max-min leverage contract. When the firm is borrowing using contract \( \hat{j} \), it is promising to repay an amount equal to the lowest possible price of the collateral next period. Since delivery on contract \( \hat{j} \) is given by \( \delta_{\hat{j},t+1} = \min \{ \hat{j}, q_{t+1} \} = \hat{j} \), the contract is risk-free and will be priced at \( \pi_{\hat{j},t} = \beta \hat{j} \) with implied interest rate \( 1 + r_{\hat{j},t} := \frac{\hat{j}}{\pi_{\hat{j},t}} = \frac{1}{\beta} \). Selling any other debt contracts \( j < \hat{j} \) will also incur the same risk-free interest rate but raise a lower amount of funds at time \( t \): \( \pi_{j,t} = \beta j < \pi_{\hat{j},t} \quad \forall j < \hat{j} \). Since every contract \( j \) must be backed by one unit of collateral, the firm will always choose \( \hat{j} \) over any \( j < \hat{j} \) when the collateral constraint is binding.

Suppose instead the firm is borrowing using contract \( \bar{j} := \max_{s,t+1} (q_t (s^{t+1})) \). This is the maximum leverage contract, because with any promise \( j > \bar{j} \) the firm will always default and the promise is no longer credible. Consequently any contract \( j > \bar{j} \) generates the same expected delivery as \( \bar{j} \) and will be priced identically. It is therefore possible to restrict attention to \( j \leq \bar{j} \) without loss of generality.

The second and third part of Lemma 3 state that if the firm’s marginal utility of income is high today relative to tomorrow in the Up-state, then the firm will choose the maximum leverage contract; and conversely the max-min leverage contract if the inequality is reversed. To see this formally, note that for \( j \in J = [\hat{j}, \bar{j}] \) and \( \lambda_t > 0 \):

\[
\frac{\lambda_t}{\gamma_t} = \max_{j \in J} \left\{ E_t \left[ \frac{\beta \gamma_{t+1}}{\gamma_t} - \frac{\beta_t \gamma_{t+1}}{\gamma_t} \right] \delta_{j,t+1} \right\} \\
= \max_{j \in J} \left\{ p \left( \frac{\beta U_{t+1}}{\gamma_t} - \frac{\beta U_{t+1}}{\gamma_t} \right) j + (1 - p) \left( \frac{\beta D_{t+1}}{\gamma_t} - \frac{\beta D_{t+1}}{\gamma_t} \right) q_{t+1} \right\} \tag{1.42}
\]
so collateral value is maximized for \( j_t = \bar{j}_t \) when \( \frac{\hat{\beta}_U}{\hat{\gamma}_t} > \frac{\hat{\beta}_U}{\hat{\gamma}_{t+1}} \); and for \( j_t = \bar{j}_t \) when the inequality is reversed. Setting \( \tilde{\beta} = \beta \) and \( \tilde{\gamma}_t = \frac{\hat{\gamma}_U}{\gamma_{t+1}} = 1 \) simplifies the condition to \( \gamma_t > \gamma_{t+1} \Rightarrow j_t^* = \bar{j}_t \) as required.

**Deterministic Steady States**

There are two possible deterministic steady states. One where the firm’s productivity is known to be \( a_U \) permanently, and another where it is \( a_D \) permanently. Let’s denote these two steady states by \( U^\infty \) and \( D^\infty \) respectively.

In the Up-steady-state \( U^\infty \), the firm will consume and hold the entire stock of capital (as per the first-best benchmark). In the absence of uncertainty, there is only one risk-free collateralized debt contract available to trade (with promised repayment \( \tilde{j} = \tilde{j} = q_{U^\infty} \)) and firms are indifferent between purchasing the capital with and without leverage. In the following lemma, I summarize the equilibrium outcomes in \( U^\infty \) when the firm purchases capital on leverage.

**Lemma 6. [The Up-steady-state]** In the deterministic steady state \( U^\infty \) where \( a = a_U \), prices and allocations in the collateral equilibrium are given by:

1. **Capital and contract prices:** \( q_{U^\infty} = \frac{\beta}{1-\beta} a_U; \) \( \pi_{j=q_{U^\infty}} = \beta q_{U^\infty} \)
2. **Capital holdings:** \( \tilde{K}_{U^\infty} = \tilde{K}_{fb} = G^{i-1} (a_U); \) \( K_{U^\infty} = \tilde{K} - \tilde{K}_{U^\infty} \)
3. **Contract trades:** \( \varphi_{j=q_{U^\infty}} = K_{U^\infty} = -\tilde{\varphi}_{j=q_{U^\infty}} \)
4. **Consumption:** \( \bar{x}_{U^\infty} = \bar{e}_{U^\infty} + G \left( \tilde{K}_{U^\infty} \right) + \beta a_U \tilde{K}_{U^\infty}; \) \( x_{U^\infty} = (1-\beta) a_U K_{U^\infty} \)

In the Down-steady-state \( D^\infty \), the household will end up holding the entire stock of capital. The firms that are shown to be permanently unproductive exit the market.

**Lemma 7. [The Down-steady-state]** In the deterministic steady state \( D^\infty \) where \( a = a_D \), prices and allocations in the collateral equilibrium are given by:
1. Capital and contract prices: \( q_{D^\infty} = \frac{\beta}{1-\beta} a_D \); \( \pi_{j=q_{D^\infty}} = \beta q_{D^\infty} \)

2. Capital holdings: \( \tilde{K}_{D^\infty} = \bar{K} \); \( K_{D^\infty} = 0 \)

3. Contract trades: \( \varphi_{j=q_{D^\infty}} = 0 = -\tilde{\varphi}_{j=q_{D^\infty}} \)

4. Consumption: \( \tilde{x}_{D^\infty} = \tilde{e}_{D^\infty} + a_D \); \( x_{D^\infty} = 0 \)

I omit proofs for Lemma 6 and 7 as they can both be readily derived from the relevant first-order conditions.

It is also clear that neither steady states will be reached immediately upon the resolution of uncertainty in the model. Even when the firm is known to be permanently productive at time \( t \), \( (a_{t+\tau} = a_\U \ \forall \tau \geq 0) \), in general the firm might not have enough liquid wealth to purchase \( K_{U^\infty} \) and consume \( x_{U^\infty} \). A period of transition is required whereby the firm gradually builds up its liquid wealth and capital stock towards the steady state. During this transition towards deterministic Up-steady-state \( U^\infty \), it is possible to show that the firm will use leverage in order to accumulate capital faster. The transition to the Down-steady-state \( D^\infty \) is straightforward. Once the firm is known to be permanently unproductive at history \( s^2 = DD \), the firm will exit the market by handing over any collateral, liquidating its remaining capital stock, and consuming the entirety of its net worth. The deterministic steady state \( D^\infty \) will then be reached immediately in the following period.

**Proof of proposition 1 [Productive inefficiency and leverage during the downturn]**

For the first part of Proposition 1 I show that firms will not begin history \( s^1 = D \) with enough liquid wealth to purchase the first-best benchmark level of capital \( K_D^{fb} \) given equilibrium prices for capital and debt contracts.

1. Suppose to the contrary that \( K_D \geq K_D^{fb} \), then firm’s liquid wealth at history \( D \) must be high enough to at least purchase \( K_D^{fb} \) using the maximum leverage
debt contract \( j = q_{DU} \) (since the maximum leverage contract demands the lowest down-payment):

\[
\begin{align*}
  w_D &\geq (q_D - \pi_D j) K^{fb}_D \\
  &= \beta g' \left( K^{fb}_D \right) K^{fb}_D \quad \text{given corollary 1} \\
  &= \beta E[a] K^{fb}_D \quad \text{by definition of } K^{fb}_D \text{ (eqn 1.13)} \\
  &> \beta a_D K^{fb}_D
\end{align*}
\]

2. If the firm used any leverage debt contracts in period 0 \((j \in J = [q_D, q_U])\), then during the downturn it will surrender the collateral posted and retain only the products of its capital holding, so \(w_D = a_D K_0\). For the firm to be able to afford \(K^{fb}_D\) in history \(D\), we must have:

\[
K_0 \geq \beta K^{fb}_D \\
= \beta K^{fb}_0 \quad \text{by definition of } K^{fb}_i \text{ (eqn 1.13)}
\]

However this level of capital holding in period 0 cannot be achieved given the firm’s starting endowment even if it used maximum leverage in period 0:

\[
K_0 \leq \frac{e_0}{(q_0 - \pi_{0,3})} = \frac{e_0}{\beta g' \left( K_0 \right)} \quad \text{given corollary 1} \\
\leq \frac{e_0}{\beta a_D} \quad \text{by assumption A3.2} \\
< \frac{\beta a_D K^{fb}_0}{\beta a_D} = \beta K^{fb}_0 \quad \text{by assumption A4.2}
\]

So we reach a contradiction \(K_0 < \beta K^{fb}_0 \leq K_0\).

3. Suppose instead the firm purchased capital without leverage in period 0. Then \(w_D = (a_D + q_D) K_0\), where \(K_0\) is bounded above by \(K_0 \leq \frac{e_0}{q_0} < \frac{\beta^2 a_D K^{fb}_0}{q_0}\).
Consequently for firm to purchase at least $K^f_D$ in history $D$ we need:

\[
(a_D + q_D) \frac{\beta^2 a_D K^f_D}{q_0} \geq w_D > \beta a_D K^f_D
\]

\[
\Leftrightarrow \frac{\beta (a_D + q_D)}{q_0} > 1
\]

but if $\frac{\beta (a_D + q_D)}{q_0} > 1$ then the household would also want to purchase more capital instead of consuming, leading to a contradiction with respect to Lemma 1. Intuitively, for the firm to transfer enough resources into state $D$ to purchase $K^f_D$ without using leverage in period 0, the price of capital in the downturn $q_D$ must be sufficiently high relative to the initial price $q_0$. But in such situations, the household would also like to purchase more capital in period 0, which would push $q_0$ beyond the level required for the firm.

To prove the second part of Proposition 1, I show that when firms hold less capital than under the first-best benchmark, they will prefer to purchase capital using leverage than without leverage.

**Lemma 8.** If $K_t(s^t) < K^f_t(s^t)$, then $\sum_{j \in J(s^t)} \max \{\varphi_{t,j}(s^t), 0\} = K_t(s^t)$, $\forall t \geq 0$ and $\forall s^t \in S^t$.

**Proof.** It is sufficient to show that the maximum leverage contract $\bar{j}_t$ is preferable to purchasing capital without leverage, when $K_t < K^f_t$. The expected rate of return from buying capital with contract $\bar{j}_t$ at time $t$ is given by:

\[
\gamma_{t,j} := \frac{E_t[\beta \gamma_{t+1}(a_{t+1})]}{q_t - \beta E_t[q_{t+1}]}
\]

\[
= \frac{\beta E_t[\gamma_{t+1}a_{t+1}]}{\beta G'(\tilde{K}_t)} \text{ by corollary 1}
\]

\[
= \frac{E_t[\gamma_{t+1}a_{t+1}]}{G'(\tilde{K}_t)}
\]
The expected rate of return from buying capital without leverage is given by:

\[
\gamma_{t,\text{no}} = \frac{\mathbb{E}_t [\beta \gamma_{t+1} (a_{t+1} + q_{t+1})]}{\beta \mathbb{E}_t [G' \left( \tilde{K}_t \right) + \mathbb{E}_t [q_{t+1}]]} = \frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1} + \mathbb{E}_t [\gamma_{t+1} q_{t+1}]]}{\mathbb{E}_t [G' \left( \tilde{K}_t \right) + \mathbb{E}_t [q_{t+1}]]}
\]

Since \( \frac{a}{b} > \frac{c}{d} \Leftrightarrow \frac{a}{b} > \frac{c}{d} \forall a, b, c, d > 0 \), we have \( \gamma_{t,j} > \gamma_{t,\text{no}} \) if and only if \( \frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1}]}{\mathbb{E}_t [q_{t+1}]} > \frac{\mathbb{E}_t [\gamma_{t+1} q_{t+1}]}{\mathbb{E}_t [a_{t+1}]} \). Furthermore, since \( K_t < K_t^{fb} \Leftrightarrow \tilde{K}_t > \tilde{K}_t^{fb} \), we know \( G' \left( \tilde{K} \right) < \mathbb{E}_t [a_{t+1}] \) and \( \frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1}]}{G' \left( \tilde{K} \right)} > \frac{\mathbb{E}_t [\gamma_{t+1} q_{t+1}]}{\mathbb{E}_t [a_{t+1}]} \). Therefore a sufficient condition for \( \gamma_{t,j} > \gamma_{t,\text{no}} \) is:

\[
\frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1}]}{\mathbb{E}_t [q_{t+1}]} \equiv \frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1}]}{\mathbb{E}_t [q_{t+1}]} \geq \frac{\mathbb{E}_t [\gamma_{t+1} q_{t+1}]}{\mathbb{E}_t [q_{t+1}]} \equiv \frac{\mathbb{E}_t [\gamma_{t+1} a_{t+1}]}{\mathbb{E}_t [q_{t+1}]} 
\]

With a little algebra we can show that the sufficient condition captured in the inequality above is equivalent to:

\[
\frac{a_U}{a_D} \geq \frac{q_U^{t+1}}{q_D^{t+1}}
\]

Since the price of capital is always given by households’ asset pricing equation, it is bounded above and below by its value in the deterministic Up and Down steady states respectively \( q_t \in \left[ \frac{\beta}{1 - \beta} a_D, \frac{\beta}{1 - \beta} a_U \right] \), \( \forall t \geq 0, \forall s^t \in S^t \). Therefore we have

\[
\frac{a_U}{a_D} \geq \frac{q_U^{t+1}}{q_D^{t+1}} \Rightarrow \gamma_{t,j} > \gamma_{t,\text{no}}
\]

as required.

Lemma 8 concludes the proof for Proposition 1. In summary, during the downturn, firms hold too little capital relative to the first-best benchmark. Consequently they are
in expectations more productive than households and would like to utilize leveraged debt contracts to increase their holding of capital. But a combination of fallen liquid wealth and worsened credit conditions constrain the firm’s purchasing power. These two financial frictions work together to exacerbate the downturn.

Proof to Proposition 2 [Optimal leverage during the downturn]

From Lemma 3 we know that \( \bar{J}_D \) is weakly preferred to \( \underline{J}_D \) if and only if firms’ marginal utility of income during the downturn is weakly greater than its marginal utility of income during the subsequent recovery: \( \gamma_D \geq \gamma_{DU} \). So to determine the optimal debt contract for firms at history \( D \), we need to sign the difference between \( \gamma_D \) and \( \gamma_{DU} \), which is given by:

\[
\gamma_D - \gamma_{DU} = E \left[ \beta \gamma_{t+1} a_{t+1} \right] - \gamma_{DU} - \gamma_D \\
= \frac{E \left[ \gamma_{t+1} a_{t+1} \right]}{G' \left( \bar{K}_D \right)} - \gamma_{DU} \\
= \frac{\left[ E \left[ \gamma_{t+1} a_{t+1} \right] - \gamma_{DU} G' \left( \bar{K}_D \right) \right]}{G' \left( \bar{K}_D \right)} \\
= \frac{\left[ p \gamma_{DU} a_U + (1 - p) \gamma_{DD} a_D - \gamma_{DU} G' \left( \bar{K}_D \right) \right]}{G' \left( \bar{K}_D \right)} \\
= \frac{\left( (1 - p) a_D - \gamma_{DU} \left( G' \left( \bar{K}_D \right) - p a_U \right) \right)}{G' \left( \bar{K}_D \right)} \quad \text{since } \gamma_{DD} = 1
\]
First, when $K_D = K_D^{fb}$ (and $\tilde{K}_D = \tilde{K}_D^{fb}$), we have $\gamma_D - \gamma_{DU} \leq 0$:

$$\gamma_D - \gamma_{DU} = \frac{\left[(1-p)a_D - \gamma_{DU}\left(G'(\tilde{K}_D^{fb}) - p a_U\right)\right]}{G'(\tilde{K}_D^{fb})} \leq 0$$

since $\gamma_{DU} \geq 1$

$$\gamma_D - \gamma_{DU} = \frac{\left[(1-p)a_D - \left(G'(K_D^{fb}) - p a_U\right)\right]}{G'(K_D^{fb})}$$

since $G'(K_D^{fb}) = E[a]$

$$\gamma_D - \gamma_{DU} = 0$$

Second, when $K_D = 0$ (and $\tilde{K}_D = 1$), we have $\gamma_D - \gamma_{DU} > 0$

$$\gamma_D - \gamma_{DU} = \frac{\left[(1-p)a_D - \gamma_{DU}\left(G'(1) - p a_U\right)\right]}{G'(\tilde{K}_D^{fb})}$$

$$\gamma_D - \gamma_{DU} = \frac{\left[(1-p)a_D - \gamma_{DU}\left(a_D - p a_U\right)\right]}{G'(\tilde{K}_D^{fb})} > 0$$

since $p \geq \frac{a_D}{a_U}$ by assumption A3.1

Since $G \in C^2$, by the intermediate value theorem $\exists \hat{K}_D \in (0, K_D^{fb})$ s.t. $\gamma_D - \gamma_{DU} = 0$ and the firm is indifferent between the maximum leverage contract and the max-min leverage contract. This concludes the proof.

**Proof of proposition 3 [Unanticipated Intervention at D]**

When firms find it optimal to use the maximum leverage contract $j_D^* = \bar{j}_D$, they will purchase as much capital as they can with their available liquid wealth $w_D$. So firms’ capital holding is given by $K_D = \frac{w_D}{q_D - \pi_D - \pi_{DU}} < K_D^{fb}$ (inequality by Proposition 1).

Suppose the central bank subsidizes the maximum leverage contract $\bar{j}_D$. This will allow the firms to hold a larger stock of capital in equilibrium, and the equilibrium price for capital will rise both immediately and in any ensuing recovery period: $\tilde{q}_D^{CB} >$
\( q_D \) and \( \overline{q}_{DU}^{CB} > q_{DU} \). The required level of subsidy to achieve the first-best benchmark level of capital holding by firms is thus given by:

\[
K^{fb}_{D} = \frac{w_D}{\overline{q}_{D}^{CB} - (1 + \chi^{*}_{j_D=q_D^{CB}}) \pi_{j_D=q_D^{CB}}} = \frac{w_D}{\overline{q}_{D}^{CB} - \pi_{j_D=q_D^{CB}}} 
\]

(1.43)

It remains to show that this subsidized loan entails a positive interest rate \( \pi_{j_D=q_D^{CB}}^{CB} \leq \overline{q}_{DU}^{CB} \), which follows immediately given \( K^{fb}_{D} > 0, \ w_D > 0 \) and \( \overline{q}_{DU}^{CB} \geq \overline{q}_{D}^{CB} \). The last condition \( \overline{q}_{DU}^{CB} \geq \overline{q}_{D}^{CB} \) claims that the price of capital will be (weakly) higher during the recovery (history \( DU \)) than during the downturn (history \( D \)). This is equivalent to showing that firms will hold a higher proportion of capital in the recovery: \( K_{DU} > K^{fb}_{D} \). We do this in two steps. First, observe that in history \( DU \) the uncertainty surrounding firms' productivity has been fully resolved and the economy is transitioning towards the Up-steady-state. Second, firms use leveraged debt contracts along the transition path to build up capital faster\(^{14}\) so \( K_{DU} = \frac{w_{DU}}{q_{DU} - \pi_{j=q_{DU}}} = \frac{a_U K^{fb}_{D}}{\beta G(K_{DU})} \geq \frac{a_U K^{fb}_{D}}{\beta a_U} = \frac{1}{\beta} K^{fb}_{D} > K^{fb}_{D} \) as required.

For the second part of the proposition, suppose the central bank subsidizes the max-min leverage contract \( j_{D} \). The equilibrium price for capital will be (weakly) higher during the downturn: \( \overline{q}_{D}^{CB} \geq q_D \); and remains unchanged if the productivity shock turns out to be permanent and the households hold the entire stock of capital: \( \overline{q}_{DD}^{CB} = q_{DD} \). The required level of subsidy to achieve the first-best benchmark

\(^{14}\)Note that all debt contracts along the transition path are riskless because all uncertainty has already been resolved.
becomes:

\[
K_{fb}^D = \frac{w_D}{q_D^{CB} - \left(1 + \chi_{jD=q_{DD}}^*\right)\pi_{jD=q_{DD}}^D} = \frac{w_D}{q_D^{CB} - \pi_{jD=q_{DD}}^D} \tag{1.44}
\]

Consequently, for sufficiently low \(w_D\) and sufficiently large gap in the price of capital between periods \(q_D^{CB} - q_{DD}\), both of which depend in part on the specifications of the initial endowment for firms \(e_0\) and on the household’s production function \(G(\cdot)\), the size of the central bank loan required to achieve the first-best outcome may exceed the promised value of repayment: \(\pi_{jD=q_{DD}}^{CB} > q_{DD}\), thus violating the effective-lower-bound condition.

**Proof to Proposition 4 [The Announcement Effect]**

When the central bank optimally provides credit support in history \(DU\), \(\chi_{jDU=q_{DUU}}^* > 0\), we know that the firms’ capital holding at \(DU\) is raised to the first-best benchmark \(K_{DUU}^f\). Consequently, the price of capital rises relative to the scenario where the central bank is only expected to intervene at \(D\): \(q_{DU}^{An} > q_{DU}^{CB}\). The price of capital at history \(DD\) remains unchanged: \(q_{DD}^{An} = q_{DD}^{CB}\) because the firms will always default and the households hold the entire stock of capital at \(DD\). Higher capital prices during the recovery eases credit conditions in the private market at history \(D\). For the maximum leverage contract: \(\pi_{jD=q_{DU}^{An}}^D > \pi_{jD=q_{DU}^{CB}}^D\), so the firms can secure a larger loan against the same unit of capital. The improved access to credit leads to an immediate increase in capital prices: \(q_{D}^{An} > q_{D}^{CB}\). With and without the announcement, the central bank will set subsidy, \(\chi_{D}^{An}\) and \(\chi_{jD}^{CB}\) respectively, such that firms hold the first-best level of...
capital at history $D$:

$$q^C_B - (1 + \chi^*_j) \pi_{jD=q^C_B} = \frac{w^D}{K^*_{D}} = q^A_n - (1 + \chi^*_j) \pi_{jD=q^A_n}$$

$$\Leftrightarrow q^A_n - q^C_B = (1 + \chi^*_j) \pi_{jD=q^A_n} - (1 + \chi^*_j) \pi_{jD=q^C_B} > 0 \quad (1.45)$$

where $w^D = a_D K_0$ because the firm used leverage at history $0$.

Since agents’ capital holdings are unchanged, from households’ asset pricing equations (see equation 1.41) we know that the net down-payment required on the private market for maximum leverage contracts also remains unchanged, despite the changes in the price of capital and the size of the loan:

$$q^C_B - \pi_{jD=q^C_B} = \beta G' \left( \tilde{K}^f_{D} \right) = q^A_n - \pi_{jD=q^A_n} \quad (1.46)$$

With a little algebra, combining equations 1.45 and 1.46 above yields: $\frac{\chi^A_n}{\chi^*_j} = \frac{q^A_n - q^C_B}{\pi_{jD=q^A_n} - \pi_{jD=q^C_B}} < 1$ as required.

To see why the balancing tax transfers become a mean-preserving spread, note that:

$$E \left[ T_2 | \chi^*_j \right] - E \left[ T_2 | \chi^A_n \right]$$

$$= \left\{ \frac{1}{\beta} (1 + \chi^*_j) \pi_{jD=q^C_B} - E \left[ q^C_B \right] \right\} - \left\{ \frac{1}{\beta} (1 + \chi^*_j) \pi_{jD=q^C_B} - E \left[ q^A_n \right] \right\}$$

$$= \frac{1}{\beta} \left\{ (1 + \chi^*_j) \pi_{jD=q^C_B} - (1 + \chi^*_j) \pi_{jD=q^C_B} \right\} + \left\{ E \left[ q^A_n - q^C_B \right] \right\}$$

$$= - \frac{1}{\beta} \left\{ q^A_n - q^C_B \right\} + \left\{ E \left[ q^A_n - q^C_B \right] \right\}$$

$$= - \frac{1}{\beta} \left\{ \beta E \left[ G' \left( \tilde{K}^f_{D} \right) + q^A_n \right] - \beta E \left[ G' \left( \tilde{K}^f_{D} \right) + q^C_B \right] \right\} + \left\{ E \left[ q^A_n - q^C_B \right] \right\}$$

$$= 0$$
Lastly, identical liquidation value \( q_{DD}^{AD} = q_{DD}^{CB} \) at history \( DD \) but larger public liability given announcement \( \frac{1}{\beta} \left( 1 + \chi_{jD}^{AD} \right) \pi_{jD = q_{DU}^{AN}} > \frac{1}{\beta} \left( 1 + \chi_{jD}^{*} \right) \pi_{jD = q_{DU}^{CB}} \) means a larger shortfall at \( DD \): \( T_{DD} | \chi_{jD}^{AN} > T_{DD} | \chi_{jD}^{*} \); and correspondingly a larger windfall at \( DU \).
2 The Collateral Rule - Theory for the Credit Default Swap Market

2.1 Introduction

The empirical results in Capponi et al. (2020) demonstrate that extreme tail risk measures have a higher explanatory power for observed collateral requirements than the standard Value-at-Risk rule. This finding lends support to models where the collateral rule is determined endogenously, like Fostel and Geanakoplos (2015). A key prediction of Fostel and Geanakoplos (2015) is that in a binomial economy (i.e. when there are two states of nature only), any collateral equilibrium is equivalent to one in which there is no default - that is, where collateral covers the most extreme losses. But linking the empirical results on the CDS market to this model faces two main limitations: (1) the conclusions only hold if there are two states of nature; and (2) defaults on CDS obligation, although rare, do arise in practice.

We develop a new model of endogenous collateral that is specifically suited for the CDS market: it features a continuum of states (as opposed to just two), and non-zero default probability in equilibrium. Trade in this model occurs because of differences in beliefs. The model presented in this paper is a natural adaptation of Simsek (2013), where belief disagreements are central to asset prices and endogenous margin requirements. But unlike Simsek (2013) who considers standard debt contracts and short selling, our model is specialized to an economy where the only contracts available for trading are state contingent promises (CDSs) backed by risk-free collateral (cash). Thus the model presented here is not a nested version of Simsek (2013), but an extension of the framework to the CDS market.

In our model, optimists naturally sell insurance (CDS protection) to pessimists, and pessimists require that the sellers post collateral (in the form of cash). The

\footnote{This is joint work with Agostino Capponi and Stefano Giglio.}
amount of cash required to collateralize the CDS contract (the collateralization rate) arises endogenously in the model. We show that what matters for the level of collateralization is not the extent of the disagreement between market participants per se, but the nature of their disagreement. In particular, when the optimists become more optimistic, the level of collateralization falls; but when the pessimists attach a larger weight to negative tail events, the level of collateralization rises. In both cases the level of disagreement between participants widened, but the change in the collateral requirement went in opposite directions.

Most importantly, the model can generate the high margin requirements and low default probabilities we observe in the data, when the clearing house (i.e. the pessimist in the model) places a greater emphasis on extreme tail risks. Moreover, whilst the collateral requirement that arises in such an equilibrium may be viewed as onerous by the clearing members (the optimist in the model), it may nevertheless not be enough to fully prevent default when viewed from the clearinghouse’s perspective.

### 2.2 Model Setup

Consider an economy with two periods $t = \{0, 1\}$ and two risk-neutral agents: one optimist and one pessimist. All agents trade in period $t = 0$ and consume in period $t = 1$. Uncertainty is captured by a continuum of states $s \in S = [s_{\min}, s_{\max}]$ realized in period $t = 1$, with $s_{\min}$ normalized to zero for simplicity. The pessimist, denoted by $i = 0$, holds prior beliefs over $S$ given by the distribution $F_0$ and corresponding density $f_0$. The optimist, denoted by $i = 1$, has prior beliefs characterized by the distribution $F_1$ and density $f_1$. The optimist has a higher expectation than the pessimist on the state in period 1, i.e., $E_1[s] > E_0[s]$. These prior beliefs are common knowledge for all agents.

At the start of period $t = 0$, each agent $i = \{0, 1\}$ is endowed with $n_i$ units of the numeraire consumption good which can be safely stored without depreciation for
consumption at period $t = 1$. I assume that the only other asset available is cash, which also yields one unit of the consumption good in period $t = 1$, but - unlike the the consumption good - cash can be used as collateral in CDS contracts. At $t = 0$, the entire endowment of cash (normalized to 1) is held by an un-modeled third party, who can sell cash in exchange for the numeraire consumption good at the equilibrium price $p$. The price of the consumption good is normalized to 1.

The optimist and pessimist have identical (linear) preferences over the consumption good, so trading between the two is driven purely by differences in beliefs. I assume that the only class of financial contracts available to trade is a simple CDS contract. Recall that the payoff of a CDS contract is zero if there is no default of the underlying (in our model, when $s$ high), and $1 - R$ in the case of default, where $R$ is the state-contingent recovery rate of the underlying bond. Since the recovery $R$ worsens as the fundamentals of the underlying deteriorate, the payoff of the CDS becomes larger as the state $s$ becomes worse. I model the promised payoff of a CDS in a simple way, i.e., as $s_{\text{max}} - s$. We can think of the case $s = s_{\text{max}}$ as the event in which the underlying bond does not default, so that the CDS does not pay anything; $0 < s < s_{\text{max}}$ as the intermediate case in which the underlying bond defaults, but there is positive recovery, so that the the CDS pays off some amount; and $s = 0$ as the extreme case of zero recovery, where the payoff of the CDS is maximal (and equal to $s_{\text{max}}$).

To enforce payment of the promise, the seller of CDSs needs to post some amount of collateral. Following the endogenous collateral literature, consider the family of CDS contracts $B^{\text{CDS}} = \{[s_{\text{max}} - s]_{s \in S}, \gamma\}$, each composed of a promise of $(s_{\text{max}} - s)$ units of the consumption good in state $s$ at $t = 1$, backed by $\gamma$ units of cash as collateral. Denote by $q(\gamma)$ the $t = 0$ price of such a CDS contract with collateral level $\gamma$. In general, multiple CDS contracts may coexist in equilibrium: they have the same promised payment $(s_{\text{max}} - s)$, but different amounts of collateral posted $\gamma$, and
as a consequence – trade for different prices $q(\gamma)$. Thus we can index the different CDS contracts in $B^{CDS}$ by $\gamma$.

Since the promised payment on a CDS contract is enforceable only through the potential of seizing the collateral, the actual delivery on each contract in state $s$ is given by the minimum of the promised payment and the value of the collateral in that state: $\delta(s, \gamma) := \min \{s^{max} - s, \gamma\}$. In other words, in any state $\tilde{s}$ such that $s^{max} - \tilde{s} > \gamma$, the seller of the CDS contract would default on her promise, and the buyer would only receive the value of the collateral $\gamma$.

Denote by $\mu^+_i, \mu^-_i$, respectively, agent $i$’s long and short positions on CDS contracts (where a long position means that the agent has purchased the corresponding CDS contract). Let $a_i \in \mathbb{R}_+$ denote agent $i$’s holding of the numeraire consumption good; and $c_i \in \mathbb{R}^+$ be her cash holdings. Then agent $i$’s budget constraint can be written as:

$$a_i + p c_i + \int_{\gamma \in B^{CDS}} q(\gamma) d\mu^+_i \leq n_i + \int_{\gamma \in B^{CDS}} q(\gamma) d\mu^-_i,$$

(2.1)

where the left hand side represents the total value of the agent’s portfolio, comprised of her holding of the numeraire good $a_i$, the value of her cash holding $p c_i$ and her long-position in CDS contracts $\int_{\gamma \in B^{CDS}} q(\gamma) d\mu^+_i$. The right hand side represents the total value of the funding available to the agent, comprising of her endowment of the consumption good $n_i$ and the amount she can raise by shorting CDS contracts, $\int_{\gamma \in B^{CDS}} q(\gamma) d\mu^-_i$. If an agent $i$ has a short position on the CDS contracts, then she is also subject to the collateral constraint:

$$\int_{\gamma \in B^{CDS}} \gamma d\mu^-_i \leq c_i,$$

(2.2)

which means that agent $i$ must have sufficient cash holdings to satisfy the collateral requirements for the CDS contracts sold. In contrast, the purchaser of the CDS contracts (the party who is long) is not subject to any collateral requirements.
The optimization problem for each agent $i$ is given by:

$$\max_{(a_i, c_i, \mu^+_i, \mu^-_i) \in \mathbb{R}^4_+} a_i + c_i + E_i \left[ \int_\gamma \min \{s^{\text{max}} - s, \gamma \} \, d\mu^+_i \right] - E_i \left[ \int_\gamma \min \{s^{\text{max}} - s, \gamma \} \, d\mu^-_i \right]$$

subject to the budget constraint (2.1) and the collateral constraint (2.2).

**Definition 1.** A **collateral equilibrium** is a set of portfolio choices $(\hat{a}_i, \hat{c}_i, \hat{\mu}^+_i, \hat{\mu}^-_i)_{i \in \{0,1\}}$ and a set of prices $(p \in \mathbb{R}_+, q : \gamma \to \mathbb{R}_+)$ such that the portfolio choices solve the optimization problem (2.3) of each agent $i \in \{0,1\}$; and the prices are such that the market for cash clears $\sum_{i \in \{0,1\}} \hat{c}_i = 1$, and the CDS markets clear $\sum_{i \in \{0,1\}} \hat{\mu}^+_i = \sum_{i \in \{0,1\}} \hat{\mu}^-_i$.

We will show that although the entire family of CDS contracts will be priced, only one contract will be traded in equilibrium. This result is line with the literature on collateral equilibrium (Fostel and Geanakoplos (2015); Simsek (2013)). Furthermore, the collateral level $\gamma$ of the actively traded CDS contract, and the price of cash $p$, will be determined endogenously. The equilibrium level of collateral requirement $\gamma$ will depend on the nature of belief differences between the optimist and the pessimist.

### 2.3 Existence and Uniqueness of the Collateral Equilibrium

Following the approach in Simsek (2013), it is possible to show that (under suitable assumptions over initial endowments and beliefs) the collateral equilibrium exists, is unique, and is equivalent to a principal-agent equilibrium where the optimist chooses her cash holdings and the optimal CDS contract to sell, subject to the pessimist’s participation constraint. To this end, we impose the following assumptions on initial endowments and prior beliefs, that parallel similar assumptions in Simsek (2013):
Assumption A1: [Restriction on Initial Endowments]

\[ n_1 < \frac{E_{1}\left[s\right]}{s_{\text{max}}} \]  \hspace{1cm} (2.4)

and \[ n_0 > \frac{E_{0}\left[s_{\text{max}} - s\right]}{E_{1}\left[s_{\text{max}} - s\right]} - n_1 \]  \hspace{1cm} (2.5)

The first inequality ensures that the optimist’s initial endowment is not large enough to purchase the entire supply of cash in the economy with her own resources alone (that is, she will need to raise some more of the numeraire consumption good by selling CDS contracts). The second inequality ensures that the initial endowment of the pessimist \((n_0)\) is large enough that the pessimist will always have some residual consumption after paying for the CDS.

Since the pessimist is risk neutral, this implies that her expected return on any CDS contracts purchased must also be equal to 1 in equilibrium. Thus the equilibrium price of a CDS contract with collateral \(\gamma\) must be given by:

\[ q(\gamma) = E_0 \left[ \min (s_{\text{max}} - s, \gamma) \right]. \]  \hspace{1cm} (2.6)

This pricing equation serves as a convenient characterization of the pessimist’s participation constraint.

We can now formulate the optimist’s problem as choosing the level of cash holdings \(c_1\), and the CDS contract \(\gamma\) to sell, so as to maximize the expected payoffs, subject to the pessimist’s participation constraint of achieving an expected return of 1 on the

\(^{16}\)For a more detailed discussion of this, see Appendix 2.7.

\(^{17}\)More specifically, this inequality implies that the sum of the endowments needs to be greater than the maximum price cash can take in equilibrium (which I will show is bounded from above by \(\left(\frac{E_0[s_{\text{max}} - s]}{E_1[s_{\text{max}} - s]}\right)\)).
CDS contract sold:

\[
\max_{(c_1,\gamma) \in \mathbb{R}_+^2} c_1 - \frac{c_1}{\gamma} E_1 [\min \{s^{\max} - s, \gamma\}] \\
\text{s.t. } pc_1 = n_1 + \frac{c_1}{\gamma} E_0 [\min \{s^{\max} - s, \gamma\}]
\] (2.7)

This leads us to define the principal-agent equilibrium as follows:

**Definition 2.** A principal-agent equilibrium is a pair of optimist’s portfolio choices \((c_1^*, \gamma^*)\) and price for cash \((p^*)\) such that the optimist’s portfolio solves her optimization problem (2.7), and the market for cash clears: \(c_1^* = 1\).

In order to show equivalence between the principal-agent equilibrium outlined here and the collateral equilibrium defined previously, further restrictions on the nature of belief differences are required:

**Assumption A2: [Restrictions on Prior Beliefs]** The probability densities of the optimist’s and the pessimist’s beliefs satisfy the monotone likelihood ratio property:

\[
\frac{f_1(s_1)}{f_0(s_1)} > \frac{f_1(s_0)}{f_0(s_0)} \text{ for every } s_1 > s_0
\] (2.8)

Note that this assumption implies: (1) first-order stochastic dominance: \(F_1(s) < F_0(s), \forall s \in (s^{\min}, s^{\max})\); (2) monotone hazard rate: \(\frac{f_1(s)}{1 - F_1(s)} < \frac{f_0(s)}{1 - F_0(s)}, \forall s \in (s^{\min}, s^{\max})\); and (3) \(\frac{f_0(s)}{F_0(s)} < \frac{f_1(s)}{F_1(s)} \forall s \in (s^{\min}, s^{\max})\) (which in turn implies \(\frac{d}{ds} \frac{F_0(s)}{F_1(s)} < 0\)).

We can then prove the following Proposition:

**Proposition 6. [Existence, Uniqueness, and Equivalence of Equilibria]** Under Assumptions A1 and A2:

1. There exists a unique principal-agent equilibrium \((p^*, (c_1^*, \gamma^*))\) s.t. \(p^* > 1\).
2. There exists a collateral general equilibrium, \((\hat{a}_i, \hat{c}_i, \hat{\mu}^+_i, \hat{\mu}^-_i)_{i \in \{0,1\}}\), whereby the optimist sells CDS to the pessimist (i.e. \(\hat{\mu}^+_1 = \hat{\mu}^-_0 = 0\)), and only a single CDS contract is actively traded (i.e. \(\hat{\mu}^+_0\) is a measure that puts weight only at one contract \(\hat{\gamma} \in \mathcal{B}^{CDS}\)). This collateral equilibrium is unique in the sense that the price of cash \(\hat{p}\) and the price of the traded CDS contract \(q(\hat{\gamma})\) are uniquely determined.

3. The collateral equilibrium and the principal-agent equilibrium are equivalent:

\[
\hat{p} = p^*, \quad \hat{c}_1 = c_1^*, \quad \hat{\gamma} = \gamma^*
\]

and \(q(\hat{\gamma}) = E_0 \left[ \min \left( s^{max} - s, \gamma^* \right) \right] \)

The detailed proof is reported in the Appendix. Intuitively, because under assumption A1 the pessimist will hold a surplus of the consumption good in equilibrium (he has a larger endowment than what the optimist would want to borrow), he must be indifferent between holding the consumption good (with a sure return of 1) and holding the CDS sold by the optimist. Hence, the optimist effectively holds all the bargaining power when deciding which CDS they should trade, and will only trade in the contract that maximizes the optimist’s expected return. Assumption A2 provides the sufficient conditions for there to exist a unique contract \(\gamma^*\) that solves the optimist’s principal agent problem.

2.4 Characterizing the Equilibrium

In this section, we show that in equilibrium the optimist will wish to sell CDS contracts to the pessimist (so as to bet on the events she thinks are more likely). But, to do so, the optimist must first obtain more units of the numeraire good from the pessimist by selling CDS contracts, in order to purchase the cash required to collateralize the
CDS contracts.\footnote{This mechanism is analogous to a mortgage contract, where the borrower is raising funds from the lender in order to purchase the collateral (i.e. the house) required to back the mortgage.} Because cash is the only asset that can be used as collateral, its equilibrium price will exceed its fundamental value \((p > 1)\), so cash is held exclusively by the optimist in equilibrium.

To formally characterize the equilibrium, first substitute \(c_1 = \left( p - \frac{1}{\gamma} E_0 \min(\bar{s} - s, \gamma) \right) \) from the optimist’s constraint into his objective function. This reduces the dimension of the problem by one, and allows us to restrict attention to choosing only the optimal contract \(\gamma\). The resulting first order condition characterizes the optimal contract choice for given \(p\):

\[\text{Proposition 7. Under assumptions A1 and A2, and fixing a price for cash } p, \text{ the optimal CDS contract, } \bar{s}, \text{ with respect to the optimist’s problem (2.7) is given by the unique solution to:}\]

\[ p = F_0 (s^{\text{max}} - \bar{s}) + (1 - F_0 (s^{\text{max}} - \bar{s})) \frac{E_0 \left[ \frac{s^{\text{max}} - s}{s^{\text{max}} - \bar{s}} \right]}{E_1 \left[ \frac{s^{\text{max}} - s}{s^{\text{max}} - \bar{s}} \right]} =: p^{\text{opt}} (\bar{s}) \quad (2.9) \]

Inverting \(p^{\text{opt}} (\bar{s})\) gives the optimal CDS contract \(\bar{s}\) for the optimist, for given price of cash \(p\). This first order condition effectively determines how much collateralization the optimist (seller) chooses for the CDS contract, since the states \(s \leq s^{\text{max}} - \bar{s}\) are the ones where the optimist is defaulting on the promise of delivering the CDS payment and is instead relinquishing the collateral. Once the equilibrium price of cash \(p^*\) is known (see Eq. (2.11) below), we can use Eq. (2.9) to derive the equilibrium level of collateral \(\gamma^*\) by setting \(p^* = p^{\text{opt}} (\gamma^*)\).

Equation (2.9) also implies that the price of cash is composed of two parts: (a) the pessimist’s assessment of the probability of default on the CDS: \((F_0 (s^{\text{max}} - \bar{s}))\), multiplied by the pessimist’s valuation of cash in the default state (1); plus (b) the pessimist’s assessment of the probability of the no-default state, multiplied by the value of cash to the optimist in the non-default state (each unit of cash allowed the
optimist to borrow $\frac{1}{s}E_0[s^\text{max} - s | s \geq s^\text{max} - \bar{s}]$ from the pessimist, with an expected actual delivery of $\frac{1}{s}E_1[s^\text{max} - s | s \geq s^\text{max} - \bar{s}]$. Given the differences in beliefs, the optimist expects to deliver less than what the pessimist envisages on the CDS contract: $\frac{E_0[s^\text{max} - s | s \geq s^\text{max} - \bar{s}]}{E_1[s^\text{max} - s | s \geq s^\text{max} - \bar{s}]} \geq 1$. Hence, $p \geq 1$, i.e., cash generates collateral value. Consistent with the “Asymmetric Disciplining of Optimism” result in Simsek (2013), the pessimist’s beliefs are used in assigning weights to the default and non-default states. The belief of the optimist enters only in determining the value of collateral in the non-default state.

Further, we show in the proof of Proposition 7 that $p^{opt}(\bar{s})$ is an increasing function of $\bar{s}$. To see this intuitively, denote the optimist’s perceived interest rate on (borrowing through selling) the CDS contract as

$$1 + r_1^{per}(\bar{s}) := \frac{E_1[\min\{s^\text{max} - s, \bar{s}\}]}{E_0[\min\{s^\text{max} - s, \bar{s}\}]}
$$

(2.10)

Under assumption A2, it can be shown that the perceived interest is decreasing in $\bar{s}$ ($\frac{d(1+r_1^{per}(\bar{s}))}{d\bar{s}} < 0$). In other words, given the nature of belief differences, offering a CDS contract with higher margin requirements is more attractive for the optimist. Thus the higher the margins $\bar{s}$ posted, the greater the discrepancy between expected deliveries becomes, and the more attractive selling the CDS is to the optimist. Thus, increasing $\bar{s}$ increases the collateral value of cash.

To close the model, we impose the market clearing condition for cash. The budget constraint in equation (2.7) implies that the optimist’s demand for cash is given by:

$$c_1 = \frac{n_1}{p - \frac{1}{s}E_0[\min\{s^\text{max} - s, \bar{s}\}]}
$$

Since the supply of cash is normalized to 1, we have:

$$p = n_1 + \frac{1}{s}E_0[\min\{s^\text{max} - s, \bar{s}\}] =: p^{mc}(\bar{s})
$$

(2.11)
We show that $p_{mc}(\bar{s})$ is a decreasing function of $\bar{s}$,\(^{19}\) with boundary conditions $p_{mc}(s^{\text{min}}) > p_{opt}(s^{\text{min}})$ and $p_{mc}(s^{\text{max}}) < p_{opt}(s^{\text{max}})$. Hence, the unique intersection between $p_{opt}(\bar{s})$ and $p_{mc}(\bar{s})$ pins down the equilibrium price for cash $p^*$ and margin requirement $\gamma^*$.

**Proposition 8.** Under assumptions A1 and A2, there is a unique principal-agent equilibrium $(c^*_1, \gamma^*, p^*)$ characterized by:

\[
\begin{align*}
    c^*_1 &= 1 \\
    p^* &= p_{mc}(\gamma^*) = p_{opt}(\gamma^*)
\end{align*}
\]

where $p_{opt}(\cdot)$ and $p_{mc}(\cdot)$ are respectively defined in equations (2.9) and (2.11).

**Proof.** See appendix. \qed

### 2.5 Comparative Statics and Illustrative Example

We present a simple numerical example to illustrate the collateral equilibrium and perform comparative statics, before presenting the more general results as propositions.

Suppose that the starting endowments of the numeraire consumption good are $n_0 = 3$ and $n_1 = 0.5$ for the pessimist and the optimist respectively. Let the set of states and the prior beliefs be given by: $\mathcal{S} = [0, 1]$ and

\[
\begin{align*}
    F_0(s) &= s^{\frac{1}{2}} \quad \forall s \in \mathcal{S} \\
    F_1(s) &= s^3 \quad \forall s \in \mathcal{S}
\end{align*}
\]

\(^{19}\) $p_{mc}(\bar{s})$ is downward sloping because $\frac{1}{2}E_0[\min\{s^{\text{max}} - s, \bar{s}\}]$ is decreasing in $\bar{s}$. Intuitively, even though a CDS contract with higher margin requirements demands a higher price (i.e. $E_0[\min\{s^{\text{max}} - s, \bar{s}\}]$ is increasing in $s$), the need to post more margins reduces the number of such contracts the optimist can sell with a single unit of cash as collateral. The overall effect is that posting more margins reduces the total amount that the optimist can borrow, $\frac{1}{2}E_0[\min\{s^{\text{max}} - s, \bar{s}\}]$, and this reduction in the purchasing power of optimists reduces the market clearing price of cash.
Panel A of Figure 2.1 plots the upward sloping cash price schedule \( \{p_{\text{opt}}(\tilde{s})\}_{\tilde{s} \in [0,s_{\text{max}}]} \) (equation (2.9)), and the downward sloping cash price schedule \( \{p_{\text{mc}}(\tilde{s})\}_{\tilde{s} \in [0,s_{\text{max}}]} \) schedule (equation (2.11)), the intersection of which characterizes the equilibrium price of cash \( p^* \) and the equilibrium margin requirement \( \gamma^* \).

Suppose instead that the optimist becomes less optimistic, such that:

\[
\tilde{F}_1(s) = s^{\frac{3}{2}} \quad \forall s \in S
\]

Then we can see from equation (2.9) that the decreased optimism of the optimist shifts the \( p_{\text{opt}}(\tilde{s}) \) schedule downwards and leaves the \( p_{\text{mc}}(\tilde{s}) \) schedule untouched. The resulting equilibrium, as illustrated in panel B of Figure 2.1, is comprised of a lower price for cash \( \tilde{p}^* \) and higher margin requirements \( \tilde{\gamma}^* \).

We generalize this intuition in the following general proposition:

**Proposition 9. [Change in optimist’s beliefs]** Suppose that the optimist’s beliefs becomes ‘less optimistic’ in the sense that it changes from \( F_1 \) to \( \tilde{F}_1 \), s.t. the monotone likelihood ratio condition is satisfied: \( \frac{f_1(s_1)}{\tilde{f}_1(s_1)} > \frac{f_1(s_0)}{\tilde{f}_1(s_0)} \) for every \( s_0, s_1 \in [s_{\text{min}}, s_{\text{max}}] \) and \( s_1 > s_0 \). Then the equilibrium level of collateral (\( \gamma^* \)) increases, and the equilibrium collateral value (\( p^* \)) falls. Conversely, when the optimist’s beliefs becomes ‘more optimistic’ s.t. \( \frac{f_1(s_1)}{\tilde{f}_1(s_1)} < \frac{f_1(s_0)}{\tilde{f}_1(s_0)} \) for every \( s_0, s_1 \in [s_{\text{min}}, s_{\text{max}}] \) and \( s_1 > s_0 \), then the equilibrium level of collateral falls and the equilibrium collateral value rises.

**Proof.** See appendix. \( \Box \)

While the changes in the optimist’s beliefs lead to unambiguous changes in the equilibrium, the same is not true for the pessimist’s beliefs. Since the pessimist’s beliefs feature in the cash price schedule of both the optimist (i.e, \( p_{\text{opt}} \)) and the pessimist (i.e., \( p_{\text{mc}} \)), a given change in the pessimist’s beliefs can either increase or decrease equilibrium collateral levels depending on the initial equilibrium. Intuitively, when the pessimist becomes more pessimistic, she is willing to pay more for any given
CDS contracts (i.e. $E_0 [\min \{ s^{\max} - s, \bar{s} \}]$ increases). This means that the optimist is now both: a) able to raise more funding from selling CDS contracts; and b) willing to pay more for each unit of collateral. The former effect shifts up the market clearing schedule $p^{mc}$, and the latter effect can push up the $p^{opt}$ schedule. Therefore, whether the equilibrium collateral level increases depends on the interaction of these two effects. Moreover, since the $p^{opt}$ schedule also depend on the pessimist’s evaluation of the default probability (i.e. $F_0 (s^{\max} - \bar{s})$), the effect of increased pessimism is not always monotone along $S = [0, 1]$. For low levels of collateral $\bar{s}$, the protection offered may be deemed insufficient, so the new $p^{opt}$ schedule may be lower than before; but for high levels of collateral, the new $p^{opt}$ schedule might be higher. In short, the $p^{opt}$ schedule may pivot as well as shift in response to a change in pessimist’s beliefs, creating further ambiguity in the direction of the equilibrium collateral level $\gamma^*$.

Instead, we show that a sufficient (but not necessary) condition for the equilibrium collateral level to increase is for the pessimist to become more concerned about the tail risks (i.e. to place a larger weight on the default states $\{ s \in S : \gamma < s^{\max} - s \}$) such that the probability density she attaches to all the non-default states are consistently lower than before, i.e., $\tilde{f}_0 (s) < f_0 (s) \ \forall s \in [s^{\max} - \gamma^*, s^{\max}]$.

For illustration, consider a change in pessimist’s beliefs from $F_0$ (eqn. 2.12) to $\tilde{F}_0$:

$$\tilde{F}_0 (s) = s^{\frac{1}{4}} \ \forall s \in S$$

Then the equilibrium collateral level and collateral value both increase (Panel C of Figure 2.1). Note that under the specific beliefs outlined here, the equilibrium level of collateral in Panel C is 0.8251, and the optimist believes default will occur with probability 0.0053. This is consistent with our empirical finding that extreme tail risk measures are important in explaining the high collateral levels and low default

\footnote{Recall from equation (2.9) the $p^{opt}$ is the weighted average of 1 and $\frac{E_0 [s^{\max} - s | s \geq s^{\max} - \bar{s}]}{E_1 [s^{\max} - s | s \geq s^{\max} - \bar{s}]} > 1$, with weights $F_0 (s^{\max} - \bar{s})$ and $(1 - F_0 (s^{\max} - \bar{s}))$ respectively.}
frequencies observed in practice. We state this result more formally in the following proposition.

**Proposition 10.** [**Pessimist’s concern for tail events**]: For any given initial equilibrium \((\gamma^*, p^*)\), if the pessimist becomes “more concerned about tail events” in the sense that her beliefs changes from \(F_0\) to \(\tilde{F}_0\), such that: \(\tilde{f}_0(s) < f_0(s)\) \(\forall s \in [s_{\text{max}} - \gamma^*, s_{\text{max}}]\), then the equilibrium collateral level \((\gamma^*)\) increases.

**Proof.** See Appendix.

Finally, note that the key to high collateral requirements in the model is the nature – rather than the degree – of the belief difference between the two agents. The equilibrium level of collateral increases both when the optimist becomes more pessimistic (which reduces the extent of the disagreement), and when the pessimist becomes more concerned about the tail events (which increases the extent of the disagreement). What the market participant disagree about matters more than how much they disagree per se.

### 2.6 Concluding Remarks

We have shown that under suitable assumptions, a unique collateral equilibrium exists for the trading of CDS contracts between optimists and pessimists. In this equilibrium, the pessimist buys CDS contracts from the optimist. Since selling a CDS requires cash as collateral, the optimist needs to secure more cash from the market. In equilibrium, the price of cash will be higher than the value of the non-collateralizable numeraire consumption good. This endogenous premium on cash reflects its value as collateral.

In the collateral equilibrium, only one CDS contract with a particular level of collateral \(\gamma^*\) will be actively traded. Default on the CDS obligations will, in general, happen with positive probability. The level of collateral \(\gamma^*\), the price of the CDS
contract \( q(\gamma^*) \), and the price of cash \( p^* \) all depend on the belief disagreement between the optimist and the pessimist. In particular, the higher the disagreement about the probability of states in which counterparty defaults may occur (i.e. the left tail of the distribution), the higher the margin requirements \( \gamma^* \) in equilibrium, and the higher the CDS price \( q(\gamma^*) \) the buyer is willing to pay.

The methodological framework presented here can thus generalize the key intuition of Fostel and Geanakoplos (2015), which predicts that observed collateralization levels might be set extremely conservatively in order to cover extreme losses in tail events. But unlike Fostel and Geanakoplos (2015), which restricts attention to economies with two states and the collateral equilibrium without default, the model in this paper allows for a continuum of states, and a low but strictly positive probability of default. Furthermore, we show that as the optimist’s beliefs become ‘less optimistic’, or when the pessimist becomes even more concerned about tail events, the equilibrium level of collateralization rises, such that from the perspective of the market participants the probability of default on CDS contracts tend to zero.

The model can therefore explain the particularly high collateral levels \( (\gamma^*) \) and the low but strictly positive probability of default events that we observe in the data (Capponi et al. (2020)). Viewed through the lens of this endogenous collateral framework, the empirical results point to disagreement about the states of the world in which CDS sellers would default on their obligations to the clearinghouse as the fundamental reason why collateral levels are set so high.


2.7 Appendices

Figure 2.1: Collateral Equilibrium and Comparative Statics

Panel A illustrates the upward-sloping $p_{\text{opt}}(\bar{s})$ schedule (eqn 2.9, blue line) and the downward-sloping $p_{\text{mc}}(\bar{s})$ schedule (eqn 2.11, red line), the intersection of which gives the collateral equilibrium $(\gamma^*, p^*)$. The $p_{\text{opt}}(\bar{s})$ schedule is increasing in the level of collateralization ($\bar{s}$) because the pessimist is willing to pay a higher price for a better collateralized CDS, which in turn increases the collateral value of cash ($p$) for the optimist. The $p_{\text{mc}}(\bar{s})$ schedule is decreasing in the level of collateralization ($\bar{s}$) because even though each unit of a better collateralized CDS is worth more, the scarcity of collateral implies that a fewer number of such contracts can be written and the total amount of funding the optimist can raise from the pessimist is lower. Therefore, in order for the market for collateral (i.e. cash) to clear, its equilibrium price ($p$) must fall as the level of collateralization increases.

Panel B illustrates the case where the optimist becomes 'less optimistic': in the sense that her beliefs change from $F_1(s)$ to $\tilde{F}_1(s)$, where $\frac{f_1(s)}{f_1(s_0)} > \frac{\tilde{f}_1(s)}{\tilde{f}_1(s_0)}$ for every $s_0, s_1 \in [s_{\min}, s_{\max}]$ and $s_1 > s_0$. When the optimist becomes more pessimistic, the $p_{\text{opt}}(\bar{s})$ schedule shifts down to $\tilde{p}_{\text{opt}}(\bar{s})$ (yellow dashed line). Intuitively, this is because as the optimistic and pessimistic beliefs converge, trading in CDSs becomes less attractive for both, so the optimist's demand for cash (the required collateral) falls (weakly) for every level of collateralization. The resulting equilibrium is one where the price of cash (its collateral value) is lower and the level of collateralization is higher.

Panel C illustrates the case where the pessimist becomes more concerned about the tail risks associated with the default states $\{s \in S : s < s_{\max} - \gamma^*\}$. Specifically, the pessimist's beliefs changes from $F_0(s)$ to $\tilde{F}_0(s)$, where $f_0(s) > \tilde{f}_0(s)$ $\forall s \in [s_{\max} - \gamma^*, s_{\max}]$. A greater concern for tail events increases the value of highly collateralized CDS contracts, and decreases the value of less collateralized contracts, so the collateral value of cash for the optimist is higher only when it is used to back highly collateralized CDS contracts (the $p_{\text{opt}}(\bar{s})$ schedule pivots to $\tilde{p}_{\text{opt}}$, yellow upward-sloping dashed line). The total amount of the numeraire good the optimist can raise through selling such CDS contracts also rises, pushing up the market clearing price for cash (the $p_{\text{mc}}(\bar{s})$ schedule shifts up to $\tilde{p}_{\text{mc}}$, purple downward-sloping dashed line). The resulting equilibrium is one where the equilibrium level of collateralization ($\gamma^*$) is higher (see Proposition 10). Note that under the specific beliefs outlined in the illustrative example section of the main text, the equilibrium level of collateral in Panel C is 0.8251, and the optimist believes default will occur with probability 0.0053.
A. Additional discussion of the assumptions

We discuss here in greater detail some of the assumptions in the model. The first inequality (2.4) states that the optimist’s initial endowment is not large enough to purchase the entire supply of cash by issuing only fully collateralized CDS contracts.

To see this, note that the value of one unit of cash to the optimist when selling contract $\gamma$ is given by:

$$1 + \frac{1}{\gamma} [E_0 [\min \{s_{\text{max}} - s, \gamma\}] - E_1 [\min \{s_{\text{max}} - s, \gamma\}]]$$

The total amount of funding the optimist can raise by selling riskless CDS contracts $\gamma = s_{\text{max}}$ (the contract with the maximum collateral) using 1 unit of collateral (i.e. cash) is:

$$\frac{1}{s_{\text{max}}} E_0 [\min \{s_{\text{max}} - s, s_{\text{max}}\}]$$

We assume that the optimist’s endowment is not large enough to finance the purchase of cash using riskless CDS contracts:

$$n_1 + \frac{1}{s_{\text{max}}} E_0 [\min \{s_{\text{max}} - s, s_{\text{max}}\}] < 1 + \frac{1}{s_{\text{max}}} [E_0 [\min \{s_{\text{max}} - s, s_{\text{max}}\}] - E_1 [\min \{s_{\text{max}} - s, s_{\text{max}}\}]]$$

$$n_1 < 1 - \frac{1}{s_{\text{max}}} E_1 [\min \{s_{\text{max}} - s, s_{\text{max}}\}]$$

$$= 1 - \frac{1}{s_{\text{max}}} [s_{\text{max}} - E_1 [s]] = \frac{E_1 [s]}{s_{\text{max}}}$$

Since $n_1 < \frac{E_1 [s]}{s_{\text{max}}} \leq 1$, this in turn ensures that the optimist cannot simply purchase the entire stock of cash (normalized to 1) without borrowing from the pessimist.

B. Proof of Proposition 7

Returning to the optimist’s problem under the principal-agent formulation (2.7), we can substitute the pessimist’s participation constraint into the objective function to
reduce dimension of the choice variable to one:

$$\max_{\gamma \in \mathbb{R}^+} \left( p - \frac{1}{\gamma} E_0 [\min \{s_{\text{max}} - s, \gamma\}] \right) \left[ 1 - \frac{1}{\gamma} E_1 [\min \{s_{\text{max}} - s, \gamma\}] \right]$$ \tag{2.14}$$

$$\Leftrightarrow \max_{\gamma \in \mathbb{R}^+} R_{1}^{\text{CDS}} (\gamma)$$

where we define $R_{1}^{\text{CDS}} (\gamma)$ as the expected return to the optimist who buys one unit of cash and uses it to back the sale of the CDS contract:

$$R_{1}^{\text{CDS}} (\gamma) := \frac{1 - \frac{1}{\gamma} E_1 [\min \{s_{\text{max}} - s, \gamma\}]}{p - \frac{1}{\gamma} E_0 [\min \{s_{\text{max}} - s, \gamma\}]} = \frac{\gamma - E_1 [\min \{s_{\text{max}} - s, \gamma\}]}{p \gamma - E_0 [\min \{s_{\text{max}} - s, \gamma\}]}$$ \tag{2.15}$$

(the numerator is the expected $t = 1$ payoff from purchasing the cash whilst simultaneously selling the CDS; the denominator is the down payment required to purchase the cash).

Note that since $E_i [\min \{s_{\text{max}} - s, \gamma\}] \equiv \gamma F_i (s_{\text{max}} - \gamma) + \int_{s_{\text{max}} - \gamma}^{s_{\text{max}}} F_i (s) \, df_i (s)$, we have

$$\frac{dE_i [\min \{s_{\text{max}} - s, \gamma\}]}{d\gamma} = F_i (s_{\text{max}} - \gamma) - \gamma f_i (s_{\text{max}} - \gamma) - (-1) (s_{\text{max}} - (s_{\text{max}} - \gamma)) f_i (s_{\text{max}} - \gamma)$$

$$= F_i (s_{\text{max}} - \gamma)$$

Therefore the derivative of $R_{1}^{\text{CDS}} (\gamma)$ with respect to $\lambda$ can be expressed as:

$$\frac{dR_{1}^{\text{CDS}} (\gamma)}{d\gamma} = \frac{(1 - F_1 (s_{\text{max}} - \gamma)) (p \gamma - E_0 [\min \{s_{\text{max}} - s, \gamma\}])}{(p \gamma - E_0 [\min \{s_{\text{max}} - s, \gamma\}]^2}$$

$$- \frac{(p - F_0 (s_{\text{max}} - \gamma)) (\gamma - E_1 [\min \{s_{\text{max}} - s, \gamma\}])}{(p \gamma - E_0 [\min \{s_{\text{max}} - s, \gamma\}]^2}$$

$$= \frac{1}{(p \gamma - E_0 [\min \{s_{\text{max}} - s, \gamma\}])} \left[ (1 - F_1 (s_{\text{max}} - \gamma)) - (p - F_0 (s_{\text{max}} - \gamma)) R_{1}^{\text{CDS}} (\gamma) \right]$$
The first order condition for the optimist’s problem simplifies to:

\[
(1 - F_1(s^{\text{max}} - \bar{s})) = (p - F_0(s^{\text{max}} - \bar{s})) \frac{s - E_1[\min \{s^{\text{max}} - s, \bar{s}\}]}{p\bar{s} - E_0[\min \{s^{\text{max}} - s, \bar{s}\}]}
\]

Re-arranging and simplifying notation (let \( F_i := F_i(s^{\text{max}} - \bar{s}) \)) gives:

\[
p = \frac{(1 - F_1) E_0[\min \{s^{\text{max}} - s, \bar{s}\}] + F_0 E_1[\min \{s^{\text{max}} - s, \bar{s}\}] - \bar{s} F_0}{(1 - F_1) \left[ \bar{s} F_0 + \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_0(s) \, ds \right] + F_0 \left[ \bar{s} F_1 + \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_1(s) \, ds \right] - \bar{s} F_0}
\]

\[
= \frac{(1 - F_1) \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_0(s) \, ds}{(1 - F_1) \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_1(s) \, ds}
\]

\[
= F_0 + \frac{(1 - F_1) (1 - F_0) E_0 [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]}{(1 - F_1) E_1 [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]}
\]

\[
= F_0 (s^{\text{max}} - \bar{s}) + (1 - F_0 (s^{\text{max}} - \bar{s})) \frac{E_0 [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]}{E_1 [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]}
\]

\[
=: p^{opt}(\bar{s})
\]

as required.

**Lemma 9.** The price of cash (i.e. the collateral) is strictly increasing in the optimist’s desired level of collateralization: \( \frac{dp^{opt}(\bar{s})}{ds} > 0 \) for \( \bar{s} \in (s^{\text{min}}, s^{\text{max}}) \).

**Proof.** Using the fact that

\[
\frac{d}{ds} E_i [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]
\]

\[
= \frac{d}{ds} \left( \frac{1}{1 - F_i(s^{\text{max}} - \bar{s})} \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_1(s) \, ds \right)
\]

\[
= \frac{\bar{s} f_i(s^{\text{max}} - \bar{s}) (1 - F_i(s^{\text{max}} - \bar{s})) - f_i(s^{\text{max}} - \bar{s}) \int_{s^{\text{max}} - \bar{s}}^{s^{\text{max}}} (s^{\text{max}} - s) f_1(s) \, ds}{(1 - F_i(s^{\text{max}} - \bar{s}))^2}
\]

\[
= \frac{\bar{s} f_i(s^{\text{max}} - \bar{s}) - f_i(s^{\text{max}} - \bar{s}) E_i [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}]}{1 - F_i(s^{\text{max}} - \bar{s})}
\]

\[
= \frac{f_i(s^{\text{max}} - \bar{s})}{1 - F_i(s^{\text{max}} - \bar{s})} (\bar{s} - E_i [s^{\text{max}} - s|s \geq s^{\text{max}} - \bar{s}])
\]

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Differentiating $p^{opt}(\bar{s})$ (and using short-hands: $f_i := f_i(s^{max} - \bar{s})$, $F_i := F_i(s^{max} - \bar{s})$, and $E_i := E_i[s^{max} - s|s \geq s^{max} - \bar{s}]$ yields:

$$
\frac{dp^{opt}(\bar{s})}{ds} = -f_0 + \left[ f_0 E_0 + (1 - F_0) \left( \frac{f_0}{1 - F_0} (\bar{s} - E_0) \right) \right] E_1 - \left[ \frac{f_1}{1 - F_1} (\bar{s} - E_1) \right] (1 - F_0) E_0
$$

$$
= -f_0 + \bar{s} f_0 \frac{1 - F_0}{E_1} E_0 - f_1 \frac{1 - F_0}{1 - F_1} E_0 + f_1 \frac{1 - F_0}{1 - F_1} E_1
$$

$$
= \frac{f_0}{E_1} \left( \frac{\bar{s}}{E_1} - 1 \right) + f_1 \frac{1 - F_0}{1 - F_1} E_1 \left( 1 - \frac{\bar{s}}{E_1} \right)
$$

$$
= \left( \frac{\bar{s}}{E_1} - 1 \right) \left( f_0 - f_1 \int_{s^{max} - \bar{s}}^{s^{max}} (s^{max} - s) f_0 (s) ds \right)
$$

where the first inequality follows from $\bar{s} > E_1[s^{max} - s|s \geq s^{max} - \bar{s}] \forall \bar{s} \in (s^{min}, s^{max})$, and the second inequality holds due to assumption A2:

$$
f_0 (s^{max} - \bar{s}) \int_{s^{max} - \bar{s}}^{s^{max}} (s^{max} - s) f_1 (s) ds - f_1 \int_{s^{max} - \bar{s}}^{s^{max}} (s^{max} - s) f_0 (s) ds
$$

$$
= \int_{s^{max} - \bar{s}}^{s^{max}} (s^{max} - s) [f_0 (s^{max} - \bar{s}) f_1 (s) - f_1 (s^{max} - \bar{s}) f_0 (s)] ds
$$

$$
> 0 \quad \text{since by A2:} \quad \frac{f_1 (s^{max} - \bar{s})}{f_0 (s)} > \frac{f_1 (s^{max} - \bar{s})}{f_0 (s^{max} - \bar{s})} \forall s > (s^{max} - s)
$$

\[\square\]

**Lemma 10.** The optimist’s perceived interest rate on the CDS is strictly decreasing in the level of collateral $\bar{s}$: $\frac{d(1 + r^{per}(\bar{s}))}{d\bar{s}} < 0$ for $\bar{s} \in (s^{min}, s^{max})$.

**Proof.** Recall from equation (2.10) that the perceived interest rate is defined as

$$
1 + r^{per}_1(\bar{s}) := \frac{E_i[\min \{ s^{max} - s, \bar{s} \}]}{E_i[\min \{ s^{max} - s, \bar{s} \}]}.
$$

Differentiating with respect to $\bar{s}$ (using the result $\frac{dE_i[\min \{ s^{max} - s, \bar{s} \}]}{d\bar{s}} = F_i(s^{max} - \bar{s})$) yields:

$$
\frac{d(1 + r^{per}_1(\bar{s}))}{d\bar{s}} = \frac{F_i(s^{max} - \bar{s}) E_0[\min \{ s^{max} - s, \bar{s} \}] - F_0(s^{max} - \bar{s}) E_1[\min \{ s^{max} - s, \bar{s} \}]}{(E_0[\min \{ s^{max} - s, \bar{s} \}])^2}
$$

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Because the denominator in the above expression is always positive, to show that 
\[ \frac{d (1 + p_{1\text{per}}(s))}{ds} < 0 \] it is enough to show that for \( \bar{s} \in (s^\min, s^\max) \), we have 
\[ \frac{E_1[\min\{s^\max - s, \bar{s}\}]}{E_0[\min\{s^\max - s, \bar{s}\}]} > \]
\[ \frac{F_1(s^\max - \bar{s})}{F_0(s^\max - \bar{s})} \]. Proceed as follows:

\[
\frac{E_1[\min\{s^\max - s, \bar{s}\}]}{E_0[\min\{s^\max - s, \bar{s}\}]} = \frac{\bar{s} F_1 (s^\max - \bar{s}) + \int_{s^\max - \bar{s}}^{s^\max} (s^\max - s) f_1 (s) \, ds}{\bar{s} F_0 (s^\max - \bar{s}) + \int_{s^\max - \bar{s}}^{s^\max} (s^\max - s) f_0 (s) \, ds}
\]
\[
> \frac{\frac{F_1(s^\max - \bar{s})}{F_0(s^\max - \bar{s})} \bar{s} F_0 (s^\max - \bar{s}) + \int_{s^\max - \bar{s}}^{s^\max} (s^\max - s) f_0 (s) \, ds}{\bar{s} F_0 (s^\max - \bar{s}) + \int_{s^\max - \bar{s}}^{s^\max} (s^\max - s) f_0 (s) \, ds}
\]
\[
= \frac{F_1 (s^\max - \bar{s})}{F_0 (s^\max - \bar{s})}
\]

where the first inequality follows from Assumption A2 \((f_0 (s) \frac{F_1(s)}{F_0(s)} < f_1 (s))\), and the second inequality follows from Assumption A2 \(\frac{d}{ds} \frac{F_0(s)}{F_1(s)} < 0\).

C. Proof of Proposition 8

In the main body, we argued that the principal-agent equilibrium is given by the intersection of the optimality condition \(p^{\text{opt}}(\bar{s}) = F_0 (s^\max - \bar{s}) + (1 - F_0 (s^\max - \bar{s})) \frac{E_0 [s^\max - s | s > s^\max - \bar{s}]}{E_1 [s^\max - s | s > s^\max - \bar{s}]} \)
derived from the optimist’s optimization problem, and the market clearing condition for cash: \(p^{mc}(\bar{s}) = n_1 + \frac{1}{\bar{s}} E_0[\min\{s^\max - s, \bar{s}\}]\). From the proof to Proposition 7, We have also established that \(p^{\text{opt}}(\bar{s})\) is strictly increasing in \(\bar{s}\) over \(\bar{s} \in (s^\min, s^\max)\). It remains to show that (i) \(p^{mc}(\bar{s})\) is strictly decreasing in \(\bar{s}\) over \(\bar{s} \in (s^\min, s^\max)\); and (ii) the boundary conditions are such that an intersection exists: \(p^{mc}(s^\min) > p^{\text{opt}}(s^\max)\) and \(p^{mc}(s^\max) < p^{\text{opt}}(s^\max)\).

1. Show that \(p^{mc}(\bar{s})\) is strictly decreasing in \(\bar{s}\) over \(\bar{s} \in (s^\min, s^\max)\).
Given \( p^{mc}(\bar{s}) = n_1 + \frac{1}{s} E_0 [\min \{s^{max} - s, \bar{s}\}] \), differentiating with respect to \( \bar{s} \) yields:

\[
\frac{dp^{mc}(\bar{s})}{d\bar{s}} = \frac{F_0 \left( s^{max} - \bar{s} \right) \bar{s} - E_0 \left[ \min \{s^{max} - s, \bar{s}\} \right]}{\bar{s}^2}
\]

\[
= \frac{\bar{s} F_0 \left( s^{max} - \bar{s} \right) - \left( \bar{s} F_0 \left( s^{max} - \bar{s} \right) + \int_{s^{max} - \bar{s}}^{s^{max}} s dF_0 \right)}{\bar{s}^2}
\]

\[
= -\frac{\int_{s^{max} - \bar{s}}^{s^{max}} s dF_0}{\bar{s}^2} < 0 \quad \forall \bar{s} \in (s^{min}, s^{max})
\]

2. Consider the boundary conditions for \( p^{mc}(\bar{s}) \) and \( p^{opt}(\bar{s}) \):

(a) For \( \bar{s} = s^{max} \), we have:

\[
p^{mc}(s^{max}) = n_1 + \left( \frac{s^{max} - E_0 [s]}{s^{max}} \right)
\]

\[
< \frac{E_1 [s]}{s^{max}} + \left( \frac{s^{max} - E_0 [s]}{s^{max}} \right) \quad \text{by assumption A1}
\]

\[
= 1 + \frac{E_0 [s^{max} - s] - E_1 [s^{max} - s]}{s^{max}}
\]

\[
< 1 + \frac{E_0 [s^{max} - s] - E_1 [s^{max} - s]}{E_1 [s^{max} - s]}
\]

\[
= E_0 \frac{[s^{max} - s]}{E_1 [s^{max} - s]} = p^{opt}(s^{max}).
\]

(b) For \( \bar{s} = s^{min} \equiv 0 \), using L’Hospital’s rule, we have:

\[
\lim_{\bar{s} \downarrow 0} p^{mc}(\bar{s}) = n_1 + \lim_{\bar{s} \downarrow 0} \frac{d}{d\bar{s}} \left( E_0 [\min \{s^{max} - s, \bar{s}\}] \right)
\]

\[
= n_1 + \lim_{\bar{s} \downarrow 0} \frac{F_0 (s^{max} - \bar{s})}{1} = n_1 + 1
\]
and

\[
\lim_{\bar{s} \downarrow 0} p_{opt}^{\bar{s}} = \lim_{\bar{s} \downarrow 0} \left[ F_0(s_{max} - \bar{s}) + (1 - F_0(s_{max} - \bar{s})) \frac{E_0[s_{max} - s | s \geq s_{max} - \bar{s}]}{E_1[s_{max} - s | s \geq s_{max} - \bar{s}]} \right]
\]

\[
= \lim_{\bar{s} \downarrow 0} F_0(s_{max} - \bar{s}) + \lim_{\bar{s} \downarrow 0} \frac{1}{1 - F_1(s_{max} - \bar{s})} \int_{s_{max} - \bar{s}}^{s_{max}} (s_{max} - s) f_0(s) ds \\
= 1 + \lim_{\bar{s} \downarrow 0} (1 - F_1(s_{max} - \bar{s})) \lim_{\bar{s} \downarrow 0} \frac{d}{ds} \int_{s_{max} - \bar{s}}^{s_{max}} (s_{max} - s) f_0(s) ds \\
= 1 + 0 \times \lim_{\bar{s} \downarrow 0} \frac{s f_0(s_{max} - \bar{s})}{s f_1(s_{max} - \bar{s})} = 1 + 0 \times \frac{f_0(s_{max})}{f_1(s_{max})}
\]

so

\[
p_{mc}(s_{min}) = n_1 + 1 > 1 = p_{opt}^{s_{min}}
\]

3. Because \(p_{opt}^{\gamma}\) and \(p_{mc}^{\gamma}\) are both continuous functions, by the intermediate value theorem they intersect at some interior point \(\gamma^* \in (s_{min}, s_{max})\) and \(p^* \in [1, \frac{E_0[s_{max} - s]}{E_1[s_{max} - s]}]\). Since \(p_{mc}^{\bar{s}}\) is strictly decreasing, and \(p_{opt}^{\bar{s}}\) is strictly increasing, the intersection is unique.

D. Proof of Proposition 6

The existence of a unique Principal-Agent equilibrium has been shown in the proof of propositions 7 and 8. In this section we show that a collateral general equilibrium as defined in the main body exists and is equivalent to the principal-agent equilibrium.

1. Step 1: Simplifying observations for solving the collateral general equilibrium:

(a) Without loss of generality, we can show that the equilibrium price of cash satisfies \(\hat{p} \in \left[1, 1 + \frac{E_1[s] - E_0[s]}{s_{max}}\right]\). Since cash guarantees a safe return of 1, its equilibrium price will never fall below 1. The optimist attach a higher value to cash, because by using cash as collateral in selling CDS
contracts $\gamma$, an optimist also gains the difference in the expected delivery 
$\frac{1}{\gamma} [E_0 [\min \{s^{\max} - s, \gamma\}] - E_1 [\min \{s^{\max} - s, \gamma\}]]$. This difference in beliefs is fully exploited when the CDS is fully collateralized at $\gamma = s^{\max}$.

So the maximum price an optimist is willing to pay for cash is equal to: 

$$1 + \frac{1}{\gamma^{\max}} [E_0 [\min \{s^{\max} - s, s^{\max}\}] - E_1 [\min \{s^{\max} - s, s^{\max}\}]] = 1 + \frac{[E_1[s] - E_0[s]]}{s^{\max}},$$

at which point the optimist would also weakly prefer to hold the illiquid asset instead. $\frac{[E_1[s] - E_0[s]]}{s^{\max}}$ can be interpreted as the upper bound on the equilibrium collateral value for cash. For the rest of the proof, we restrict attention to the more interesting cases where $\hat{p}$ is strictly less than 

$1 + \frac{[E_1[s] - E_0[s]]}{s^{\max}}$.

(b) Note that each agent’s optimization problem (2.3) is linear in the objective variables, thus their value functions will take the form $v_i n_i$, where $v_i$ denotes the return on agent $i$’s endowment $n_i$.

(c) Since the agents can always just hold their endowment of the illiquid asset or buy cash in order to sell the fully collateralized CDS contract $\gamma = s^{\max}$, we must have:

$$v_i \geq \max \left\{ 1, \frac{1}{\gamma^{\max}} E_i [\min \{s^{\max} - s, s^{\max}\}] \right\} \quad \forall i = \{0, 1\} \quad (2.16)$$

In the above expression, 1 represents the rate of return on the illiquid asset; and $\left( \frac{1}{\gamma^{\max}} E_i [\min \{s^{\max} - s, s^{\max}\}] \right)$ represents the expected rate of return on buying cash to use as collateral in selling the CDS contract $\gamma = s^{\max}$. For the latter, the expected payoff in the second period is 1 (from the cash) minus $\frac{1}{\gamma^{\max}} E_i [\min \{s^{\max} - s, s^{\max}\}]$ (the expected delivery on the CDS contract $\gamma$). The down-payment on this transaction is $(p - \frac{1}{\gamma^{\max}} E_0 [\min \{s^{\max} - s, s^{\max}\}])$, \footnote{Note that even though $\frac{1}{\gamma} [E_0 [\min \{s^{\max} - s, \gamma\}] - E_1 [\min \{s^{\max} - s, \gamma\}]]$ is maximized at $\gamma = s^{\max}$. This is does not mean the optimist will always want to sell the fully collateralised CDS at any $p$. The $p^{opt}(s)$ curve plots the optimal collateral level $s$ for the optimist at any given price $p$. (Derived from the interior solution to max. $R$, where $R := \frac{1 - \frac{1}{\gamma} E_1[\min\{s^{\max} - s, \gamma\}]}{p - \frac{1}{\gamma} E_0[\min\{s^{\max} - s, \gamma\}]}$.}
where \( p \) is the price paid for the unit of cash, and \( \frac{1}{s_{\text{max}}} E_0 \min \{ s_{\text{max}} - s, s_{\text{max}} \} \) is the amount raised from selling the CDS to the pessimist who values it the most.

(d) Summing over the two agents’ budget constraints (2.1), and imposing the market clearing conditions in equilibrium (holdings of CDS contracts must cancel out and the sum of total cash holdings is normalized to 1) yields:

\[
a_0 + a_1 + \hat{p} \times 1 = n_0 + n_1
\]

Recall that by Assumption A1 \( n_0 + n_1 > 1 + \frac{E_1[s_{\text{max}}-s]}{E_1[s_{\text{max}}-s]} \), so \( \hat{p} \in [1, 1 + \frac{E_1[s_{\text{max}}-s]}{E_1[s_{\text{max}}-s]}] \) implies that \( a_0 + a_1 > 0 \) (i.e. one or more agents must hold the illiquid asset in equilibrium).

(e) Since \( p < 1 + \frac{E_1[s_{\text{max}}-s]}{s_{\text{max}}} \), we know from equation (2.16) that \( v_1 > 1 \). Therefore, the pessimist must be the one holding the illiquid asset in the collateral equilibrium, which gives us \( v_0 = 1 \).

(f) Lastly, without loss of generality, CDS contracts with \( \gamma > s_{\text{max}} \) will not be used in equilibrium (such contracts tie down a larger amount of collateral without a compensating increase in price).

2. Agents’ bid and ask prices for CDS contracts:

(a) An agent’s bid price is the price that would make her indifferent between buying the CDS contract and simply receiving the equilibrium value per net worth \( v_i \), so:

\[
q_{0}^{\text{bid}} (\gamma) = E_0 \frac{\min \{ s_{\text{max}} - s, \gamma \}}{v_0} = E_0 \min \{ s_{\text{max}} - s, \gamma \}
\]

\[
> q_{1}^{\text{bid}} (\gamma) = E_1 \frac{\min \{ s_{\text{max}} - s, \gamma \}}{v_1}
\]
(b) An agent’s ask price for the CDS contract $\gamma$ is the price that would make the trader indifferent between taking a negative position in the CDS $\gamma$ and simply receiving the equilibrium value $v_i$, so:

$$v_0 = 1 - \frac{1}{\gamma} E_0 \left[ \min \{s_{\text{max}} - s, \gamma \} \right] \frac{p}{p - \frac{1}{\gamma} q_{0}^{\text{ask}}(\gamma)} = 1$$

$$v_1 = 1 - \frac{1}{\gamma} E_1 \left[ \min \{s_{\text{max}} - s, \gamma \} \right] \frac{p}{p - \frac{1}{\gamma} q_{1}^{\text{ask}}(\gamma)} > 1 \quad (2.18)$$

(c) Market clearing for CDS contracts $(\hat{\mu}_0^+ + \hat{\mu}_0^- = \hat{\mu}_1^- + \hat{\mu}_0^-)$ implies:

$$\min_i q_i^{\text{ask}}(\gamma) \geq q(\gamma) \geq \max_i q_i^{\text{bid}}(\gamma) \quad \forall \gamma$$

(Suppose $\max_i q_i^{\text{bid}}(\gamma) > q(\gamma)$, then some buyer wants to buy an infinite amount, but seller can only sell a finite amount due to the collateral constraint. It also cannot occur that $q(\gamma) > \max \{\min_i q_i^{\text{ask}}(\gamma), \max_i q_i^{\text{bid}}(\gamma)\}$, so we must have $\min_i q_i^{\text{ask}}(\gamma) \geq q(\gamma)$)

(d) A CDS contract is traded in positive quantities only if:

$$q_i^{\text{ask}}(\hat{\gamma}) = q(\hat{\gamma}) = q_j^{\text{bid}}(\hat{\gamma}) \quad \text{for some } \{i, j\} = \{0, 1\}$$

(e) Claim the pessimist’s ask prices are always higher than optimist’s bid prices:

$$q_0^{\text{ask}}(\gamma) = \gamma (p - 1) + E_0 \left[ \min \{s_{\text{max}} - s, \gamma \} \right] \quad \text{from eqn (2.18)}$$

$$> E_0 \left[ \min \{s_{\text{max}} - s, \gamma \} \right] \quad \text{since } p > 1$$

$$= q_0^{\text{bid}}(\gamma) > q_1^{\text{bid}}(\gamma) \quad \text{from eqn (2.17)}$$
so there are no traded CDS contracts in which the optimist buy and the pessimist sell.

(f) The equilibrium prices of CDS contracts are therefore:

\[
\begin{align*}
q(\hat{\gamma}) &= q_{0}^{\text{bid}}(\hat{\gamma}) = q_{1}^{\text{ask}}(\hat{\gamma}) \quad \text{for each } \hat{\gamma} \text{ with positive trade} \\
q(\gamma) &\in \left[\max, q_{0}^{\text{bid}}(\gamma), \min, q_{1}^{\text{ask}}(\gamma)\right] \quad \text{for each } \gamma
\end{align*}
\]  

(2.19)

3. Characterize the equilibrium in CDS markets for a given price for cash \( p \in \left[1, \frac{E_{0}[s_{\text{max}} - s]}{E_{1}[s_{\text{max}} - s]}\right] \)

- The optimist faces quasi-equilibrium prices for all CDS contracts (even those that are not positively traded in equilibrium):

\[
\bar{q}(\gamma) = q_{0}^{\text{bid}}(\gamma) = E_{0}[\min\{s_{\text{max}} - s, \gamma\}]
\]  

(2.20)

- Given these quasi-equilibrium prices, optimists solve the following optimization problem:

\[
v_{1}n_{1} = \max_{c_{1} \geq 0, \mu_{1}} c_{1} - \int_{\gamma \in \mathcal{B}_{\text{CDS}}} \frac{1}{2}E_{1}[\min\{s_{\text{max}} - s, \gamma\}] d\mu_{1}^{-}
\]  

(2.21)

s.t. \( p c_{1} - \int_{\gamma \in \mathcal{B}_{\text{CDS}}} \frac{1}{2}E_{0}[\min\{s_{\text{max}} - s, \gamma\}] d\mu_{1}^{-} = n_{1} \quad \text{[budget constraint]}
\]  

(2.22)

\[
\int_{\gamma \in \mathcal{B}_{\text{CDS}}} \frac{1}{2}d\mu_{1}^{-} \leq c_{1} \quad \text{[collateral constraint]}
\]  

(2.23)

- Since \( v_{1} > 1 \), the collateral constraint binds in equilibrium. Let \( \lambda_{1} \) denote the Lagrangian multiplier for the collateral constraint. \( v_{1} \) will correspond to the multiplier for the budget constraint.
• The FOCs for $c_1$ and $\mu_1$ yields:

$$1 + \lambda_1 = v_1 p$$

$$v_1 \frac{1}{\gamma} E_0 \left[ \min \{s^{\text{max}} - s, \gamma \} \right] \leq \frac{1}{\gamma} E_1 \left[ \min \{s^{\text{max}} - s, \gamma \} \right] + \lambda_1 \quad \text{with equality only if } \gamma \in \text{supp} (\mu_1^-)$$

• Combining the FOCs yield:

$$v_1 p = 1 + \lambda_1$$

$$\geq 1 + v_1 \frac{1}{\gamma} E_0 \left[ \min \{s^{\text{max}} - s, \gamma \} \right] - \frac{1}{\gamma} E_1 \left[ \min \{s^{\text{max}} - s, \gamma \} \right]$$

$$\Rightarrow v_1 \geq \frac{1 - \frac{1}{\gamma} E_1 \left[ \min \{s^{\text{max}} - s, \gamma \} \right]}{p - \frac{1}{\gamma} E_0 \left[ \min \{s^{\text{max}} - s, \gamma \} \right]} =: R_1^{CDS} (\gamma) \quad \text{with equality only in } \gamma \in \text{supp} (\mu_1^-)$$

• As per the proof of Proposition 7, $R_1^{CDS} (\gamma)$ has a unique maximum characterized by $p^{\text{opt}} (\gamma = \tilde{s})$. So again the unique collateral-GE is pinned down by the intersection between $p^{\text{opt}} (\tilde{s})$ and the market clearing condition for cash: $p^{mc} (\tilde{s}) = n_1 + \frac{1}{\gamma} E_0 \left[ \min \{s^{\text{max}} - s, \tilde{s} \} \right]$, s.t. the equilibrium collateral level $\hat{\gamma}$ and the price of cash $\hat{p}$ satisfies:

$$\hat{p} = p^{\text{opt}} (\hat{\gamma}) = p^{mc} (\hat{\gamma})$$

4. It follows that the unique general equilibrium is equivalent to the principal-agent equilibrium.

E. Proof of Proposition 9

We prove the case where the optimist becomes 'more pessimistic'. Let $g_1 (s) := \frac{f_1 (s)}{1 - F_1 (s^{\text{max}} - \bar{s})}$ for all $s \in [s^{\text{max}} - \bar{s}, s^{\text{max}}]$, $\forall \bar{s} \in [s^{\text{min}}, s^{\text{max}}]$ and $\bar{g}_1 (s) := \frac{f_1 (s)}{1 - F_1 (s^{\text{max}} - \bar{s})}$ for all $s \in [s^{\text{max}} - \bar{s}, s^{\text{max}}]$. Then by Assumption A2 $g (\cdot)$ and $\bar{g} (\cdot)$ must also satisfy the monotone likelihood ratio condition: $\frac{d}{ds} \left( \frac{g_1 (s)}{\bar{g}_1 (s)} \right) > 0 \quad \forall s \in S \Rightarrow \frac{d}{ds} \left( \frac{g_1 (s)}{\bar{g}_1 (s)} \right) > 0 \quad \forall s \geq (s^{\text{max}} - \bar{s})$. 
This in turn implies \( \tilde{E}_1 [s \mid s > s_{max} - \tilde{s}] < E_1 [s \mid s \in s_{max} - \bar{s}] \forall \bar{s} \in (s_{min}, s_{max}) \), so the upward sloping \( p^{opt} \) curve shifts down when the optimist becomes 'more pessimistic'. The converse of the proposition follows from the same logic.

\[ \text{F. Proof of Proposition 10} \]

We show that a sufficient (but not necessary) condition for the equilibrium collateral level to increase is for the pessimist to attach sufficiently larger probability weights to the default states.

Let \( F_0 (s) \) and \( \tilde{F}_0 (s) \) denote the initial and the new beliefs of the pessimist respectively. For brevity, let \( \tilde{E}_0 [x] := E_{\tilde{F}_0} [x]; \tilde{p}^{opt} (\tilde{s}) := \tilde{F}_0 (s_{max} - \tilde{s}) + \left(1 - \tilde{F}_0 (s_{max} - \tilde{s}) \right) \frac{\tilde{E}_0 [s_{max} - s \mid s] > s_{max}}{\tilde{E}_1 [s_{max} - s \mid s] > s_{max}} \) and \( \tilde{p}^{mc} (\tilde{s}) := n_1 + \frac{1}{\tilde{s}} \tilde{E}_0 [\min \{s_{max} - s, \tilde{s}\}] \). Define, respectively, the initial and the new equilibrium collateral levels \( \gamma^* \) and \( \gamma^{**} \) implicitly as:

\[ p^{opt} (\gamma^*) = \tilde{p}^{mc} (\gamma^*) \]
\[ \tilde{p}^{opt} (\gamma^{**}) = \tilde{p}^{mc} (\gamma^{**}) \]

Then given \( \tilde{p}^{mc} \) is strictly decreasing, \( p^{opt} \) is strictly increasing, and the two curves intersects within \( (s_{min}, s_{max}) \) (see proofs for Propositions 7 and 8), we have \( \gamma^{**} > \gamma^* \) iff:

\[ \tilde{p}^{mc} (\gamma^*) > \tilde{p}^{opt} (\gamma^*) \]
\[ \Leftrightarrow \tilde{p}^{mc} (\gamma^*) - \tilde{p}^{mc} (\gamma^*) > \tilde{p}^{opt} (\gamma^*) - \tilde{p}^{opt} (\gamma^*) \]
With a little bit of algebra, we can show that:

\[
\begin{aligned}
\tilde{p}^{\text{mc}}(\gamma^*) - \tilde{p}^{\text{mc}}(\gamma^*) &= \frac{1}{\gamma^*} \left[ \tilde{E}_0 \left[ \min \left\{ s^{\text{max}} - s, \gamma^* \right\} \right] - E_0 \left[ \min \left\{ s^{\text{max}} - s, \gamma^* \right\} \right] \right] \\
&= \frac{1}{\gamma^*} \left[ \gamma^* \tilde{F}_0 + \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, f_0(s) \, ds - \gamma^* F_0 - \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, f_0(s) \, ds \right] \\
&= \left( \tilde{F}_0 - F_0 \right) + \frac{1}{\gamma^*} \left[ \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, \tilde{f}_0(s) \, ds - \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, f_0(s) \, ds \right]
\end{aligned}
\]

and

\[
\begin{aligned}
\tilde{p}^{\text{opt}}(\gamma^*) - p^{\text{opt}}(\gamma^*) &= \tilde{F}_0 + \left( 1 - \tilde{F}_0 \right) \frac{\tilde{E}_0 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]}{E_1 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]} \\
&\cdots \left[ F_0 + \left( 1 - F_0 \right) \frac{E_0 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]}{E_1 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]} \right] \\
&= \left( \tilde{F}_0 - F_0 \right) + \frac{\left[ \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, \tilde{f}_0(s) \, ds - \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, f_0(s) \, ds \right]}{E_1 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]}
\end{aligned}
\]

Taken together, we get:

\[
\begin{aligned}
\left[ \tilde{p}^{\text{mc}}(\gamma^*) - p^{\text{mc}}(\gamma^*) \right] - \left[ \tilde{p}^{\text{opt}}(\gamma^*) - p^{\text{opt}}(\gamma^*) \right]
&= \left[ \frac{1}{\gamma^*} \frac{1}{E_1 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]} \right] \left[ \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, \tilde{f}_0(s) \, ds - \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, f_0(s) \, ds \right] \\
&= \left[ \frac{1}{E_1 \left[ s^{\text{max}} - s | s > s^{\text{max}} - \gamma^* \right]} - \frac{1}{\gamma^*} \right] \left[ \int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, \left( f_0(s) - \tilde{f}_0(s) \right) \, ds \right]
\end{aligned}
\]

(2.24)

Therefore, whether the collateral requirement in the new equilibrium increases

\((\gamma^{**} > \gamma^*)\)

depends on whether the second term

\(\int_{s^{\text{max}} - \gamma^*}^{s^{\text{max}}} (s^{\text{max}} - s) \, \left( f_0(s) - \tilde{f}_0(s) \right) \, ds \)

is positive, for which a sufficient condition is that

\(f_0(s) > \tilde{f}_0(s) \quad \forall s \in [s^{\text{max}} - \gamma^*, s^{\text{max}}].\)
3 From Binomial No-Default to Multinomial Max-Min

3.1 Introduction

The theoretical model of collateral equilibrium presented in Chapter 1 draws from the analytical framework first developed in Geanakoplos (1997). Fostel and Geanakoplos (2015) went on to provide a complete characterization of the collateral equilibrium in models with two states of nature in every period (i.e. “binomial models”) with a finite number or a continuum of heterogeneous agents, and where debt contracts are backed by financial assets. In contrast, the model in Chapter 2 draws from Simsek (2013), which characterize the collateral equilibrium in models with a continuum of states, but only two agents. A natural next step is therefore to generalize the insights of the endogenous leverage framework to models with more than two agents and more than two states.

The Binomial No-Default Theorem of Fostel and Geanakoplos (2015) states that “in binomial economies with financial assets serving as collateral, any equilibrium is equivalent in real allocations and prices to another equilibrium in which there is no default”. In this Chapter, I extend this theorem to economies with more than two states of nature when debt can be ordered by seniority. For instance, with three states of nature, borrowers can issue both senior secured debt and junior unsecured debt. The senior secured debt is explicitly backed by the risky financial asset held by the firm, whereas the junior unsecured debt is implicitly backed by the residual value of the firm after the senior creditors satisfy their claims. The Multinomial Max-Min theorem I prove states that any equilibrium is equivalent to another equilibrium where

\footnote{In Fostel and Geanakoplos (2015), a \textit{financial asset} is defined as an asset that “gives no direct utility to investors, and pays the same dividends no matter who owns it”. Note that the capital good $K$ used as collateral in Chapter 1 is not a financial asset under this definition, due to its differential productivity under different agents.}
the senior tranche never defaults, and the junior tranche only defaults in the worst state of the world.

The Multinomial Max-Min theorem nests the Binomial No-Default theorem when there are only two states of the nature in every period, and it shows that the analytical framework in Geanakoplos (1997) can be generalized to models where debt contracts are not explicitly collateralized, but are instead implicitly backed by the residual value on the assets of the debtor.

3.2 Set-up

Consider a simple model with two periods $t = \{1, 2\}$: with a certain initial period $S_1 = \{0\}$ and three possible states of nature in the second period $S_2 = \{U, M, D\}$. Let there be a single perishable consumption good $c$ and a risky asset $Y$ that pays dividends $d^Y = (d_U, d_M, d_D)$ in units of the consumption numeraire in the second period. Suppose, without loss of generality, $d_U > d_M > d_D > 0$. Agents $h \in H$ derive utility from their holding of the consumption numeraire $c^h$ in every period.

In the initial period, the agents in the model can trade in the risky asset $Y$, and also in simple non-state-contingent debt contracts with a creditor hierarchy that determines the priority of claims in the event of default or insolvency. At the top of the creditor hierarchy is senior secured debt that promises to repay $j$ units of the consumption numeraire in all three states in period 2. Every unit of senior secured debt sold must be backed by 1 unit of the risky asset $Y$ as collateral. The agents can trade an entire spectrum of such senior secured debt contracts $J = \left\{ \begin{pmatrix} j \\ j \\ j \end{pmatrix} , \begin{pmatrix} Y = 1 \\ Y_{-1} = 0 \\ Y_{-2} = 0 \end{pmatrix} \right\}$ that differ only in the promised repayment amount $j$ (the first vector), which we will use to index the debt instrument being traded. The second component of $J$ represents the collateral used to back the contract, which is one unit
of \( Y \) for all contracts \( j \in J \). The terms \( Y_{-1} \) and \( Y_{-2} \) denote any residual claims on \( Y \) and will be defined shortly.

If the promised repayment \( j \) ever exceed the value of the collateral \( Y \) in the second period, the issuer of the debt will default. Therefore, the **actual delivery on the senior secured debt** \( j \) is given by:

\[
\delta (j) = \begin{pmatrix} \min \{ j, d_U \} \\ \min \{ j, d_M \} \\ \min \{ j, d_D \} \end{pmatrix} = \min \{ j \tilde{1}, d^Y \} 
\]

(3.1)

The key observation here is that whenever a senior secured debt \( j \) is sold, it creates a “new” financial asset that is given by the residual claim on \( Y \). Let’s denote this **first-order residual claim** by \( Y_{-1} (j) \), which comes with a period 2 payoff vector of 

\[
d^Y_{-1} (j) = d^Y - \delta (j).
\]

For given senior secured debt \( j \) issued against \( Y \), a **junior subordinated debt** instrument can also be written that is either contractually or implicitly backed by the residual claim \( Y_{-1} (j) \). Let the promised repayment on junior subordinated debt be denoted by \( k \). The spectrum of possible junior subordinated debt instruments is given by: 

\[
K (j) = \begin{cases} \begin{pmatrix} k \\ k \\ k \end{pmatrix} & \begin{cases} Y = 0 \\ Y_{-1} = e_j \\ Y_{-2} = 0 \end{cases} \end{cases}, \text{ where } e_j \text{ denotes a } 1 \times (J + 1) \text{ vector with } 1 \text{ on the } (j + 1)'\text{th element and zeros elsewhere.}^{23} \text{ The actual delivery on junior subordinated debt } (j, k) \text{ is given by:}
\]

\[
\delta_{-1} (j, k) = \min \{ k \tilde{1}, d^Y_{-1} (j) \}
\]

(3.2)

---

23Note that \( Y = Y_{-1} (0) \), so the dimension of \( e_j \) is \( J + 1 \) to accommodate the case where there are no senior secured debt issued, and any junior subordinated debt effectively becomes senior secured debt.
This junior subordinated debt in turn creates a **second-order residual claim** on \( Y \), denoted by \( Y_{-2} (j, k) \), with a period 2 payoff vector of \( d_{-2}^Y (j, k) = d^Y - \delta (j) - \delta_{-1} (j, k) \). We can also think of this second-order residual claim as simply the **equity** stake in \( Y \). An interesting observation is that a state contingent equity contract can therefore also be expressed as a collateralized debt contract with non-state-contingent promise \( l \) backed by the second-order residual claim on \( Y \): 
\[
L (j, k) = \begin{cases} 
  \left( \begin{array}{c}
  l \\
  Y = 0
\end{array} \right), & \text{if } Y_{-1} = 0 \\
  \left( \begin{array}{c}
  l \\
  Y_{-2} = e_{j,k}
\end{array} \right), & \text{otherwise}
\end{cases}
\]

, where \( e_{j,k} \) denotes a \((J + 1) \times (K + 1)\) matrix with 1 on the \((j + 1, k + 1)\) element, and zeros elsewhere. The **actual delivery on equity** \((l, j, k)\) is given by:
\[
\delta_{-2} (j, k, l) = \min \{l \tilde{1}, d_{-2}^Y (j, k)\}
\]

(3.3)

The key point here is that even though the promised repayment amount on equity contract \((l, j, k)\) is not state-contingent, the actual delivery on the contract is very much state-contingent due to the changing value of the collateral across states.

In summary, the **set of feasible contracts** are composed of senior secured debt \( J \subseteq \mathbb{R}_+ \), junior unsecured debt \( K (j) \subseteq \mathbb{R}_+^J \) and equity contracts \( L (j, k) \subseteq \mathbb{R}_+^K \). All contracts are essentially backed by the same unit of the risky asset \( Y \), but the claims are ordered in a creditors’ hierarchy. The contract space is clearly very large. However, although all contracts will be priced in equilibrium, I will show that we can restrict attention to a much smaller subset of actively traded contracts in equilibrium without any loss of generality, a result that I will refer to as the Multinomial Max-Min Theorem.

Before moving on to the statement of the theorem, it is helpful to first define the **max-min contract** for each creditor tier (which will be the only actively traded contracts in the equilibria of interest). First, I will refer to \( j^* = d_D \) as the max-min senior secured debt contract. This is because the promise to repay \( j^* = d_D \) will always
be honored when it is backed by 1 unit of $Y$ as collateral. So $j^* = d_D$ is the maximum credible promise that ensures minimum credit risk to lenders (it is essentially risk free). In a similar sense, I will refer to $k^* (j^*) = d_M - d_D$ as the max-min junior subordinated debt contract. This is because given that a senior secured claim $j^*$ has already been issued on the underlying collateral $Y$, the best junior creditors can hope for is to be made whole in states other than the worst state $D$. So $k^* (j^*) = d_M - d_D$ is the maximum promise that can be honored in all states other than $D$. Lastly, the max-min equity contract $l^* (j^*, k^*) = d_U - d_M$ is worth zero in states $M$ and $D$, but pays off $d_U - d_M$ in the best state $U$. This max-min equity contract $l^* (j^*, k^*)$ can be equivalently thought of as purchasing the risky asset $Y$ on leverage whilst simultaneously issuing tiered debt contracts $j^*$ and $k^*$ against $Y$.

As a final point on notation, let $\varphi^h (j) > 0$, $\varphi^h_{-1} (j, k) > 0$ and $\varphi^h_{-2} (j, k, l) > 0$ denote respectively the number of senior secured contracts $j$, junior subordinated contracts $(j, k)$ and equity contracts $(j, k, l)$ sold by agent $h$. Let $\pi (j)$, $\pi (j, k)$, $\pi (j, k, l)$ be the price of these respective contracts; and let $p$ denote the price of the risky asset $Y$ at period 1. We are now ready to state the Multinomial Max-Min theorem and its relation to the Binomial No-Default Theorem.

### 3.3 The Multinomial Max-Min Theorem

The Binomial No-Default Theorem of Fostel and Geanakoplos (2015) states that “in binomial economies with financial assets serving as collateral, any equilibrium is equivalent in real allocations and prices to another equilibrium in which there is no default”. The Multinomial Max-Min Theorem generalizes this powerful result to the three states set-up outlined above with tiered collateralized debt contracts.

**Theorem 1.** ![Multinomial Max-Min Theorem]: Suppose that $S = \{0, U, M, D\}$, that $Y$ (and its corresponding residuals $Y_{-1}$, $Y_{-2}$) are financial assets, and that the max-min debt contracts $j^* = d_D \in J$, $k^* = d_M - d_D \in K^J$ are available for trade in the
Then given any equilibrium \( \left( p, \{ \pi_j \}_{j \in J}, \{ \pi_k(j) \}_{k \in k^j} \right), \left( c^h, y^h, \varphi^h, \varphi^{-1}_h \right)_{h \in H} \), we can construct another equilibrium \( \left( p, \{ \pi_j \}_{j \in J}, \{ \pi_k(j) \}_{k \in k^j} \right), \left( c^h, \bar{y}^h, \bar{\varphi}^h, \bar{\varphi}^{-1}_h \right)_{h \in H} \) with the same asset and contract prices and the same consumption, in which \( j^* \) and \( k^* \) are the only debt contract traded: \( \varphi^h_k = \varphi^h_j = 0 \) if \( j \neq j^* \) and \( k \neq k^* \).

Like the Binomial No-Default Theorem, the power of the Multinomial Max-Min Theorem lies in the observation that even though the agents can trade a very wide spectrum of debt contracts \( J \) and \( K^J \), we can instead restrict attention to the equilibrium where the only contracts traded are the max-min debt contracts \( j^* \) and \( k^* (j^*) \), without any loss of generality. However, unlike the Binomial No-Default Theorem where there is no default in equilibrium, with the three states set-up above and when junior subordinated debt are issued (i.e. when \( \varphi^h_k > 0 \) for some \( h \)), defaults can occur in equilibrium. Such defaults are kept to the “minimal” extent, in the sense that the buyers of junior subordinated debt \( k^* \) never expected to be paid in state \( D \). When trading the max-min debt contracts in equilibrium, the senior tranche never defaults, and the junior tranche only defaults in the worst state of the world.

The proof of the Multinomial Max-Min Theorem mirrors that of the Binomial No-Default Theorem and proceeds in three steps. First, the Payoff Cone Lemma shows that the portfolio of assets and contracts that any agent \( h \) holds in equilibrium that delivers payoff vector \( (w^h_U, w^h_M, w^h_D) \) must lie in the cone positively spanned by the three max-min contracts \( j^*, k^* (j^*) \) and \( l^* (j^*, k^*) \). Second, the State Pricing Lemma proves the existence of unique state prices \((a, b, c)\) such that if any agent \( h \) holds a portfolio delivering \( (w^h_U, w^h_M, w^h_D) \), the portfolio must cost \( aw^h_U + bw^h_M + cw^h_D \). Lastly, I show that for any given equilibrium, it is possible to construct a new equilibrium that is equivalent in consumption and asset prices, but where the agents trades only the max-min debt contracts. Details of the formal proof is left to the appendix.
3.4 Concluding Remarks

The goal of the Multinomial Max-Min Theorem is to show that the insights of Fostel and Geanakoplos (2015) are even more general than previously envisaged. The theorem improves the tractability of the endogenous leverage framework in models with a continuum of agents and more than two states. The confines of simple non-state-contingent collateralized debt contracts are also not as restrictive as it might first appear. With appropriately defined residual claims on the underlying asset, we can encompass both senior secured and junior unsecured debt using the same analytical toolkit. One can even characterize the payoffs of a state-contingent equity instrument as equivalent to that of a suitably collateralized and subordinated debt contract. I hope the analysis presented here can form an important step in incorporating the insights of the endogenous leverage framework of Geanakoplos (1997) into larger DSGE-style macroeconomic models.


3.5 Appendix - Proof

The proof of the Multinomial Max-Min Theorem proceeds in three steps.

Payoff Cone Lemma

First, I show that any agent \( h \) can replicate any payoff vector \( (w^h_U, w^h_M, w^h_D) \) across the three states using only the three max-min contracts \( j^*, k^* (j^*) \) and \( l^* (j^*, k^*) \).

Lemma 11. [Payoff Cone Lemma]: The portfolio of assets and contracts that any agent \( h \) holds in equilibrium delivers payoff vector \( (w^h_U, w^h_M, w^h_D) \) which lies in the cone positively spanned by the three max-min contracts: (1) the \( j^* = d_D \) senior secured debt contract with actual delivery \( \delta (j^*) = (j^*, j^*, j^*) \); (2) the \( k^* (j^*) = d_M - d_D \) junior subordinated contract with actual delivery \( \delta_{-1} (j^*, k^*) = (k^*, k^*, 0) \); and (3) the \( l^* (j^*, k^*) = d_U - d_M \) equity contract with actual delivery \( \delta_{-2} (j^*, k^*, l^*) = (l^*, 0, 0) \).

Note that the \( l^* (j^*, k^*) \) equity contract is equivalent to an Arrow-U security, with actual delivery \( \delta_{-2} (j^*, k^*, l^*) = d_Y - \delta (j^*) - \delta_{-1} (j^*, k^*) \) that can be obtained by buying the asset \( Y \) while simultaneously selling the max-min debt contracts \( j^* \) and \( k^* \) (i.e. purchasing \( Y \) on leverage by issuing senior secured and junior subordinated debt against it). Since there are no Arrow-M nor Arrow-D securities available for purchase, we must have \( w^h_U \geq w^h_M \geq w^h_D \) in equilibrium, and it is easy to see that all feasible \( (w^h_U, w^h_M, w^h_D) \) satisfying this condition can be replicated using only the three max-min contracts.

State Pricing Lemma

The next step of the proof is to demonstrate the existence of unique state prices at \( t = 1 \) for the delivery of the consumption numeraire in each state at \( t = 2 \):
Lemma 12. [State Pricing Lemma]: There exist unique state prices $a, b, c$ such that if any agent $h$ holds a portfolio delivering $(w^h_M, w^h_M, w^h_D)$, the portfolio costs $aw^h_U + bw^h_U + cw^h_D$.

Proof. In three broad stages: (1) Establish that the max-min contracts can be priced using a unique vector of state prices. (2) Use the payoff cone lemma to establish these state prices as the upper bound on contract and asset prices. This is because buyer won’t pay more than the state prices because otherwise they can replicate the delivery using max-min contracts instead at lower costs. (3) Argue that the state prices are also the lower bound on contract and asset prices, because otherwise for sellers the cost of obtaining the collateral required to sell the max-min contracts would exceed the value received.

1. [Unique State Prices] Find state prices for the max-min debt contracts $j^* = d_D$, $k^* = d_M - d_D$, $l^* = d_U - d_M$. That is, solving:

$$
\begin{pmatrix}
\pi_{j^*} \\
\pi_{k^*} \\
\pi_{l^*}
\end{pmatrix} = \begin{pmatrix}
d_D & d_D & d_D \\
 d_M - d_D & d_M - d_D & 0 \\
 d_U - d_M & 0 & 0
\end{pmatrix} \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
$$

which gives: $a = \frac{\pi_{l^*}}{d_U - d_M}$; $b = \frac{\pi_{k^*}}{d_M - d_D} - a$; and $c = \frac{\pi_{j^*}}{d_D} - a - b$.

• Note that $\pi_{l^*} = a(d_U - d_M)$ is the price of buying the $(d_U - d_M, 0, 0)$ Arrow $U$ securities, which equals to buying the asset $Y$ and selling the max-min contracts $j^*$ and $k^*(j^*)$, so $p = \pi_{j^*} + \pi_{k^*} + \pi_{l^*}$, and thus the asset $Y$ itself is also priced using the state prices.

• Note also that the promised interest rate on the junior subordinated debt is higher than that on the senior secured debt:

$$
\frac{k^*}{\pi_{k^*}} = \frac{1}{a + b} > \frac{1}{a + b + c} = \frac{j^*}{\pi_{j^*}}
$$
2. [Upper bound on residual assets prices $p_{-1}^Y (j)$ and $p_{-2}^Y (j,k)$] We want to show that the dividends on residual assets, $d_{-1}^Y (j) = d^Y - \delta (j)$ and $d_{-2}^Y (j,k) = d_{-1}^Y - \delta_{-1} (j,k)$, for $Y_{-1} (j)$ and $Y_{-2} (j,k)$ respectively, cannot be priced more than their valuation under the state prices:

$$p_{-1}^Y (j) \leq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta (j)$$

$$p_{-2}^Y (j,k) \leq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1} (j,k)$$

By the payoff-cone lemma, the dividends on the residual assets $Y_{-1} (j), Y_{-2} (j,k)$ are spanned by the max-min debt contracts, so their price cannot exceed $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_0 (j)$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1} (j,k)$ respectively. Otherwise buyers can replicate the same payoff for cheaper using the max-min contracts instead.

3. [Upper bound on $\pi_j$ and $\pi_k$] Suppose that the senior secured debt contract $j$ with $j \neq j^* = d_D$ is positively traded in equilibrium, then $\pi_j \leq a \min \{j, d_U\} + b \min \{j, d_M\} + c \min \{j, d_D\} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta (j)$. Similarly for $k (j)$ with $j \neq j^*$ or $k (j^*) \neq k^* (j^*) = d_M - d_D$, we must have $\pi_{k(j)} \leq a \min \{k, d_U - \min \{j, d_U\}\} + b \min \{k, d_M\} + c \min \{k, d_D\} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta (j)$.
\[ b \min \{k, d_M - \min \{j, d_M\}\} + c \min \{k, d_D - \min \{j, d_D\}\} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1}(k, j). \]

This is because by the Payoff Cone Lemma, the delivery of contracts \( j \) and \( k \) are both positively spanned by the Arrow \( U \) security and the max min contracts \( j^*, k^* \), all of which are priced by \( a \) and \( b \) (by the previous step). Thus if \( \pi_j \), \( \pi_k \) were priced higher than the value of its delivery under \((a, b, c)\), the buyer can achieve the same deliveries by buying a linear combination of the max-min contracts \( j^*, k^*, l^* \) (where \( l^* = d_U - d_M \) is equivalent to the Arrow \( U \) contract).

We have therefore established an upper bound for \( \pi_j, \pi_k \).

4. **[Lower bound on \( \pi_j \)]** Suppose that a senior secured debt contract \( j \) with \( j \neq j^* = d_D \) is positively traded in equilibrium, then we want to show that

\[ \pi_j \geq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta(j). \]

(a) Consider first the case \( j \leq j^* = d_D \). This contract \( j \) fully delivers in all states and is proportional to the delivery under contract \( j^* \). So if its price were less than \( \frac{j}{j^*} \pi_j^* = a_j + b_j + c_j \), then the seller could have sold \( \frac{j}{j^*} \) units of contract \( j^* \) instead (which is feasible because it requires less collateral).

(b) Consider next the case \( j \in (d_D, d_U] \). In order to sell contract \( j \), the seller need to purchase the asset \( Y \) on leverage, incurring net costs \( p - \pi_j \), and receiving the residual delivery vector \( d_{j,1}^Y(j) \). By a previous step, we have established that the price of \( d_{j,1}^Y(j) \) (denoted \( p_{j,1}^Y(j) \)) cannot be higher than \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{j,1}^Y(j) \), so for the seller to be willing to sell contract \( j \), we
must have:

\[
p - \pi_j \leq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{Y_1}^Y(j)
\]

\[
\Leftrightarrow \pi_j \geq p - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{Y_1}^Y(j) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d^Y(j) - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{Y_1}^Y(j)
\]

\[
= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot (d^Y(j) - d_{Y_1}^Y(j)) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta(j)
\]
Similarly, since $\pi_{k(j)}$ is bounded above by $\left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot \delta_{-1}(j, k)$, we have:

$$p - \pi_j - \pi_{k(j)} \leq \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot d^Y_{-2}(j, k)$$

$$\Leftrightarrow \pi_j \geq p - \pi_{k(j)} - \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot d^Y_{-2}(j, k)$$

$$\geq \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot d^Y(j) - \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot \delta_{-1}(j, k) - \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot d^Y_{-2}(j, k)$$

$$= \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot \left( d^Y(j) - \delta_{-1}(j, k) - d^Y_{-2}(j, k) \right)$$

$$= \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot \delta(j)$$

as required. Taken the two previous steps together, we have now established $\pi_j = \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \cdot \delta(j)$.
5. **[Lower bound on \( \pi_k \):** For \( k(j) \) with \( j \neq j^* \) or \( k(j) \neq k^*(j^*) = d_M - d_D \), we want to show that \( \pi_{k(j)} \geq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1}(k,j) \).

Following a similar argument to above, but now with \( \pi_j = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta(j) \), we must have:

\[
p - \pi_j - \pi_{k(j)} \leq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{-2}^{Y}(j,k)
\]

\[
\Leftrightarrow \pi_{k(j)} \geq p - \pi_j - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{-2}^{Y}(j,k)
\]

\[
= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d^{Y}(j) - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta(j) - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot d_{-2}^{Y}(j,k)
\]

\[
= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1}(k,j)
\]

As required. We have now established that \( \pi_{k(j)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1}(k,j) \).
6. **Lower bound on residual asset prices** $p_{Y-1}$ and $p_{Y-2}$. Given $\pi_j = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$\delta(j)$ and $\pi_{k(j)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \delta_{-1}(j, k)$, we must have $p_{Y-n} \geq d_{Y-n}$ for $n = 1, 2$ (otherwise there would be infinite demand for the asset $Y$ which violates market clearing).

**Construction of the new max-min equilibrium**

All that remains is to show that any equilibrium $\left( (p, \{\pi_j\}_{j \in J}, \{\pi_{k(j)}\}_{k(j) \in k^J}) , (c^h, y^h, \varphi^h, \varphi_{-1}^h)_{h \in H} \right)$ with period 2 portfolio payoffs $(w^h_U, w^h_M, w^h_D)$ can be replicated in a new max-min equilibrium $\left( (p, \{\pi_j\}_{j \in J}, \{\pi_{k(j)}\}_{k(j) \in k^J}) , (c^h, \bar{y}^h, \bar{\varphi}^h, \bar{\varphi}_{-1}^h)_{h \in H} \right)$ where only the max-min contracts are traded.

1. Define:

$$(w^h_U, w^h_M, w^h_D) = d^Y y^h - \sum_j \delta(j) \varphi^h(j) - \sum_k \delta_{-1}(k, j) \varphi_{-1}^h(k(j))$$

$$\bar{y}^h = \frac{w^h_U - w^h_M}{(d_U - d_M)}$$

$$\varphi^h(j^*) = \bar{y}^h - \frac{w^h_D}{d_D}$$

$$\varphi_{-1}^h(k^*) = \bar{y}^h - \frac{(w^h_M - w^h_D)}{(d_M - d_D)}$$

2. Verify that all agents are optimizing under the new max-min equilibrium:

- It is easy to check that the proposed portfolio $\left( \bar{y}^h, \varphi^h(j^*), \varphi_{-1}^h(k^*) \right)$ also yields the payoff $(w^h_U, w^h_M, w^h_D)$.
• By the payoff cone lemma and the state pricing lemma, the portfolio then must have the same cost.

• Note that since $\varphi^h(j^*) \leq \bar{y}^h$ and $\varphi^h_{-1}(k^*) \leq \bar{y}^h$, this portfolio choice satisfies the collateral constraints.

• Lastly, since $Y$ is a financial asset that yields the same payoff to every agent, every agent is optimizing given the new proposed portfolio $(\bar{y}^h, \varphi^h_0(j^*), \varphi^h_{-1}(k^*))$.

3. Verify that markets clears:

• Sum over individuals to get:

$$
\sum_h \bar{y}^h \begin{pmatrix}
d_U \\
d_M \\
d_D
\end{pmatrix} - \sum_h \varphi^h(j^*) \begin{pmatrix}
d_D \\
d_D \\
d_D
\end{pmatrix} - \sum_h \varphi^h_{-1}(k^*) \begin{pmatrix}
d_M - d_D \\
d_M - d_D \\
0
\end{pmatrix}
$$

$$
= \sum_h \begin{pmatrix}
w^h_U \\
w^h_M \\
w^h_D
\end{pmatrix}
$$

$$
= \sum_h y^h \begin{pmatrix}
d_U \\
d_M \\
d_D
\end{pmatrix} - \sum_h \sum_j \varphi^h(j) \delta(j) - \sum_h \sum_j \sum_k \varphi^h_{-1}(k(j)) \delta_{-1}(k,j)
$$

$$
= \sum_h y^h \begin{pmatrix}
d_U \\
d_M \\
d_D
\end{pmatrix} \quad \text{(since $\sum_h \varphi^h_0(j) = \sum_h \varphi^h_{-1}(k(j)) = 0$ in the original equilibrium)}
$$
Therefore we must have:

\[
\sum_h (\bar{y}^h - y^h) \begin{pmatrix} d_U \\ d_M \\ d_D \end{pmatrix} - \sum_h \varphi^h_0 (j^*) \begin{pmatrix} d_D \\ d_D \\ 0 \end{pmatrix} - \sum_h \varphi^h_{-1} (k^*) \begin{pmatrix} d_M - d_D \\ d_M - d_D \\ 0 \end{pmatrix} = 0
\]

Since the vectors \((d_U, d_M, d_D), (d_M - d_D, d_M - d_D, 0), (d_D, d_D, d_D)\) are linearly independent, we must have:

\[
\sum_h (\bar{y}^h - y^h) = 0
\]
\[
\sum_h \varphi^h_0 (j^*) = 0
\]
\[
\sum_h \varphi^h_{-1} (k^*) = 0
\]

so all markets clear.

This completes the proof to the Multinomial Max-Min Theorem.
4 Capital requirements, the safe real interest rate and the fundamental problem of bank risk taking

4.1 Introduction

The great financial crisis that started in 2007-08 revealed that many banks in many countries had taken excessive risks. Asymmetry of information between banks and those that provide them with debt, high leverage and limited liability makes a tendency for banks to take excessive risks more than just an occasional phenomenon linked to episodes of excessive exuberance, but rather something more fundamental and persistent. How best to counter this excessive risk taking is a major policy issue facing governments and their central banks. It has prompted some questioning of whether it is wise to use monetary policy simply to hit inflation and real activity targets, without attention to incentives that low interest rates might give to excessive risk taking in the banking sector. It has raised a pressing policy issue about the right level of bank capital requirements. It also raised questions about how the level of interest rates and limits on bank funding structure (capital requirements) interact and about which has the greater effect on excessive risk taking. These are the issues we address in this paper.

We develop a model where banks take excessive risks because of asymmetric information and limited liability. By excessive risks we mean investment risks that would be rejected by a well-informed, risk-neutral agent who is entitled to the entirety of the investment payoff. Excessive risk taking by banks arises because banks know more about the probability of the success of the risky loans extended than do investors (including households) who provide the debt funding to banks. Acquiring assets with excessive risks is optimal for banks because they are protected in the down-state by limited liability. Recognizing this, investors charge a higher interest rate on loans

24Joint work with David Miles.
made to banks with lower internal equity (or 'skin-in-the-game'), disincentivising them from carrying out socially valuable, but privately costly, screening of potential investments. Thus in the absence of intervention, we have sub-optimally low levels of screening by banks; and excessive risk taking by those banks that do screen.

There are only imperfect tools available to bank regulators, central banks and governments to counter this tendency for banks to take too many risks. One tool is to use capital requirements to force banks to use more equity funding than they would choose themselves. Another tool is to raise the safe real interest rate to push up the real cost of debt to banks. Limits on bank leverage offset incentives to gamble at the expense of those supplying debt; a higher safe interest rate raises the required return on bank liabilities, incentivising them to be more selective on the assets they acquire. We show that capital requirements and the safe real interest rates are (imperfect) substitutes when used to combat excessive risk taking by banks. Neither policy instruments are perfect remedies and both incur costs. We illustrate under what conditions each tool is more effective and when both might be used.

There are a number of ways through which a national authority might be able to influence the local safe real interest rate. A government could choose to offer a savings vehicle paying a guaranteed real interest rate (and that has been the original rationale for many governments to issue real, ie inflation-proof, debt). Central banks set a nominal interest rate that feeds through to the safe real interest rate given nominal rigidities. For the remainder of the paper, we refer to attempts by a fiscal or monetary authority to affect the safe real interest rate as 'interest rate policy'. The scope to use such policies may be limited however. With a perfectly integrated global capital market the safe real rate may be exogenous to any government. In this case one can interpret the results in this paper in another way, which is about what effect changes in a global real interest rate should have on the appropriate setting of bank capital requirements. We find such effects are substantial. Given the large decline in
the average real yields on inflation-proof government bonds in recent decades this set of results is of significant practical relevance.

There is a growing literature on the impact of capital requirements on bank lending; there is also a much larger (and older) literature on the effects of interest rates on banks. But there is less research on the interaction of bank capital requirements with variations in the safe real interest rate in offsetting inefficient lending - which is the focus of this paper.

The broad consensus in the existing literature is that stricter capital requirements can curb excessive risk-taking and lending (see for instance Furlong and Keeley (1989); and Gensbach and Rochet (2017)); and a tightening of monetary policy could achieve similar effects, albeit potentially at a cost to other macroeconomic considerations (Farhi and Tirole (2009) and Nicolò et al. (2010)). A separate body of work exists on the role of bank capital in the transmission of monetary policy (such as Kashyap and Stein (1994), Kashyap and Stein (2000) and Bolton and Freixas (2006))25.

More recently, Angeloni and Faia (2013) compared the performance of different Taylor Rules under four banking regimes. They concluded that the optimal monetary policy rule should incorporate some response to financial conditions. Angelini et al. (2014) analyzed the impact of monetary policy and capital requirements on macroeconomic performance and stability, and discussed the need for cooperative arrangements between the authorities responsible for each instrument.

Woodford (2012) asks to what extent one might tilt monetary policy away from a 'pure' inflation targeting stance because of its potential impact on financial stability. He says:

"The real issue, I would argue, should not be one of controlling the possible mispricing of assets in the marketplace ... but rather one of seeking to deter extreme levels of leverage and of maturity transformation in the

25See Borio and Zhu (2012) for a survey of this literature
financial sector. ... Even modest changes in short-term rates can have a significant effect on firms’ incentives to seek high degrees of leverage or excessively short-term sources of funding. ... Acceptance that monetary policy deliberations should take account of the consequences of the policy decision for financial stability will require a sustained research effort.”

Much of the literature which explores whether monetary policy should play a role in reducing problems in the financial sector assumes that interest rates can be used to control leverage and that - for reasons not modeled - leverage is sometimes too high. An exception is Stein (2012) who is precise about the source of market failure and we follow that route. Not being clear about exactly why financial outcomes are problematic means that policy conclusions are not robust. The fundamental source of market failure in our model is precisely that which provides the most compelling rationale for the existence of banks - namely their specialization in collecting and interpreting information on types of assets whose risks are hard to assess and which make their financing through the issuance of securities in capital markets problematic.

Our paper develops a simple theoretical framework to illustrate the trade-offs involved in using interest rate policy and capital requirements to influence banks’ risk taking decisions. We analyze what an optimal combination of capital requirements and interest rate policy might look like for a policy-maker maximizing the aggregate expected return on investments undertaken. The model we use is an extension of the framework developed in Bernanke and Gertler (1990), which focused on financial fragility and economic performance. Their model described an economy with two types of risk-neutral agents: entrepreneurs that have access to risky investment projects; and households from whom the entrepreneurs must borrow in order to fund their investment projects. Entrepreneurs must undertake costly screening before they can find out the probability of success of their project, so some with low endowments (and thus high funding requirements) may choose to forego the lending opportunity
even before the screening stage. Those that do undertake screening are incentivized to take excessive risks. This is because entrepreneurs enjoy limited liability and cannot credibly reveal their probability of success to their creditors. The extent to which an individual entrepreneur is prone to excessive risk-taking is also exacerbated by lower initial endowments. Bernanke and Gertler concluded that the dependence of aggregate investment on the initial distribution of endowment introduces 'financial fragility'. A 'financially fragile' economy, defined as one with a sizeable proportion of its entrepreneurs operating around the threshold level of endowment between screening and not screening, may experience a dramatic collapse in investment if subjected to a negative shock to endowments.

We make a number of changes to the Bernanke and Gertler (1990) model:

1. **Banks**: We introduce bank intermediation by distinguishing between two types of Bernanke and Gertler 'entrepreneurs': those that can credibly communicate the quality of their assets to creditors and access funding directly (we call these 'firms'); and those that cannot and thus rely on banks. We assume that banks fund projects where it is hard for individual investors to assess their risks. Banks effectively come to own these projects and finance them from internal equity and from debt (and possibly equity) raised from outside investors. We therefore have two types of corporations that invest: (1) 'firms' which fund risky assets that are easy to assess and monitor by creditors; and (2) 'banks' which fund assets that are hard to value and where information is not shared with other investors.

2. **Debt and Equity**: We distinguish between two types of funding contracts: banks in our model can raise funds from households either in the form of debt or in the form of equity. Debt contracts promise to pay the creditor a fixed sum; equity contracts promise a share of the return on the risky project (net of any debt obligations). There may be a non-zero pay-off for the risky project in the
event it fails. Providers of debt have priority claim over this liquidation value of the risky project.

3. **Policy tools**: The policy maker has two levers - interest rate policy and regulatory capital requirements - with which to ameliorate the effects of the asymmetry of information in the banking sector that combine with limited liability to make market outcomes inefficient. Interest rate policy sets the safe rate paid on the outside option (i.e. a risk-free deposit facility) that is available to all agents. One could think of this safe rate as paid by the central bank on deposits it takes, and recouped through taxes set by the fiscal authority. Regulatory capital requirements prohibit banks with internal equity less than the regulatory minimum from proceeding with the risky lending project, unless they raise the additional equity from households. Banks for which the capital regulation is binding will be financed by a mixture of debt and external equity.

We find that interest rate policy and prudential capital requirements operate as imperfect substitutes. A tightening of either instrument can improve 'prudence' (by disincentivising the banks against acquiring excessively risky assets with sub-optimally low probability of success); but only at the cost of decreased 'participation' (where decreased 'participation' means that more banks will choose to forego the risky asset even before they discover its probability of success through costly screening). The substitutability between the policy instruments, and this trade-off between 'prudence' and 'participation', implies that the optimal level of prudential capital requirements rises as the safe real interest rate falls. We find that, across a broad range of calibrations, using capital requirements is more efficient as a way to get the level of risk-taking right in the economy than changing the safe real interest rate. But there are situations where using both tools is optimal and some where using only the interest rate is optimal. However in a truly global market the government might be a price taker, in which case the safe real rate is not a policy variable but an exogenous
factor. Our results then illustrate how exogenous variations in that rate should affect the design of capital limits on banks.

The rest of the paper is organized as follows. Section 2 lays out the basic theoretical model for analyzing bank risk taking decisions. Section 3 calculates the outcome under the first-best scenario. Section 4 characterizes the equilibrium under asymmetric information. The two policy instruments are introduced in Section 5 where we also describe the main propositions of the model. Section 6 provides quantitative results from numerical simulations of the model. Section 7 concludes with a discussion of policy implications.

4.2 The Model

We develop a two-sector, two-period, model at the center of which is the decision of banks to acquire risky assets. We show conditions under which banks take excessive risks and how capital requirements and variations in the safe real interest rate can offset the extent of such mis-allocation. Banks are intermediaries that screen risky projects, and then finance these risky assets by raising funds from households. By undertaking costly screening, banks receive private information on the value of the assets they acquire; and by borrowing from households, banks can maximize their returns through leverage. When a bank’s assets drop in value, the bank benefits from limited liability. The information asymmetry in the model give rise to moral hazard and the Modigliani-Miller theorem does not hold.

Agents: The banking sector and the non-financial corporate sector

Banks On the asset side, the banks in our model have access to a risky asset and a risk-free deposit facility at the central bank.

The risk-free deposit facility remunerates any amount deposited at the risk-free rate \((1 + r)\). This is a purely real model so we could think of \(r\) as a safe real rate
offered by the central bank and potentially requiring fiscal backing to make good on the promise. One could equally think of the policy rate as being directly chosen by the fiscal authorities who offer real government debt to households and finance the real interest costs by levying taxes.

The risk-free asset is common to all banks while each bank has access to a risky asset that is unique to them.\(^{26}\) (We will use the terms "risky asset" and "project" interchangeably). The risky project requires 1 unit of input in the first period. In the second period, it pays out \(y_h > 1 + r\) when it succeeds (with probability \(p\)), and \(y_l < 1 + r\) when it fails (with probability \((1 - p)\)). The probability of success for each project, \(p\), is independently and identically distributed between \([0, 1]\) with probability density function \(h(p)\) and cumulative density function \(H(p)\).

A key feature of banks is that they can undertake costly screening (fixed cost \(C \ll 1\)) to discover the realization of \(p\) for their own projects in period 1, before having to commit a full unit of input. We assume that the risky project is not worthwhile in the absence of screening, so \(y_l + E[p](y_h - y_l) < 1 + r\).

We allow for a continuum of banks in our model that are heterogeneous in two dimensions. First, they differ in the realization of their probability of success \((p_i)\) for their risky projects. Second each bank makes a draw of initial endowment \(\omega_i\) (i.e. initial bank equity). The p.d.f. and c.d.f. of \(\omega\) are denoted by \(f(\omega)\) and \(F(\omega)\) respectively, with support \([\omega_{lb}, \omega_{ub}]\), where \(0 \leq \omega_{lb} < \omega_{ub} \ll 1\).

Since \(\omega_{ub} \ll 1\), all banks must seek funding from households in order to proceed with their project. This external funding can take the form of additional equity \((\tilde{\omega})\) or debt (i.e. deposits from the general public).

Both the distribution of \(\omega\) and each bank’s realization of \(\omega_i\) are publicly observed. But whilst the ex ante distribution of \(p\) is also common knowledge, each bank’s

\(^{26}\)We do not think of the risky asset as literally being one single asset. Rather we interpret it as a portfolio of assets which pays out \(y_h\) in a good state, and \(y_l\) in a bad state. Risky assets differ only in their probability of success \(p\).
individual realization of $p_i$ is assumed to be private information. We assume $p$ and $\omega$ to be independently distributed.

The assumption that the probability of success of a bank’s assets $p_i$ is the private information of the bank is crucial. This captures the idea that banks have assets (projects) where the information gathered in screening is hard to share and where the probability of success cannot be credibly announced. Households depositing in banks know that they don’t know much about the quality of the assets.

**Firms** The second sector in our model is composed of a continuum of firms. Firms also have access to both the risk-free deposit facility and the risky asset/project. But critically, firms are assumed to be sufficiently transparent that investors know as much about the chances of success ($p_i$) as the firms themselves know after screening. So firms directly issue securities to investors with no intermediation by banks. The fraction of “firms” and “banks” in the population are given by $\mu$ and $(1 - \mu)$ respectively.

The existence of asymmetry of information in the banking sector, and the lack of it in the non-financial corporate sector, lies at the core of our model and results. The model captures the idea that banks specialize in acquiring information on the probabilities of good and bad outcomes for the value of the assets they acquire - assets where the assessment of outcomes is time consuming and tricky and where continued monitoring may be required. In many ways that is the fundamental reason why banks, who largely hold non-marketable assets, exist (see Diamond (1984); and Gale and Hellwig (1985)).
Households Households provide funding to both banks and firms. For simplicity, we do not model the households explicitly and assume they, like all other agents, are risk-neutral and so demand an expected return of at least $(1 + r)$.

Timing

The model has two periods. The timing in the first period is sub-divided into three stages:

1. In the first stage of the first period, each firm and bank draws a realization of their initial endowments, $\omega_i$, from the common distribution for endowments. Given the realization of $\omega_i$, each then decides whether or not to undertake costly screening to discover the success rate of the risky assets that they can acquire, $p_i$. Those that do not screen put the entirety of their endowments in the risk-free facility.

2. In the second stage, those firms or banks that did screen discover their probability of success, $p_i$, and decide whether to proceed with the acquisition of risky assets. When firms/banks do proceed, they commit the entirety of their initial endowment and begin to seek external funding.

3. In the third stage, those firms/banks that are proceeding with their projects decide whether to fund the shortfall through debt, additional equity, or a combination of the two. Any external equity injections take place before the firm/bank seeks debt. By assumption, households can observe the capital of the firm/bank ($\omega + \tilde{\omega}$, composed of the initial endowment plus any subsequent

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27Our framework can be extended to a general equilibrium setting where households maximize utility across two periods, and the risk-free interest rate on the deposit facility $r$ adjust endogenously to clear the market for household funds. Such an extension does not affect our key results regarding the behavior of banks.

28Firms and banks would only proceed with the project if its expected return exceeds the certain return from the deposit facility (recall that all agents are assumed to be risk neutral). Therefore, when a project is deemed worthwhile, a firm or bank would prefer to commit the entirety of their endowment rather than to place a portion in the safe deposit facility.
equity injections). Firms can also credibly communicate their probability of success \( p_i \), whereas banks cannot.

- An equity contract takes the form \( \{ \tilde{\omega}, \frac{\tilde{\omega}}{\omega+\tilde{\omega}} \} \), where \( \tilde{\omega} \) is the size of the equity injection, \( \omega \) is the firm/bank’s own endowment, and \( \frac{\tilde{\omega}}{\omega+\tilde{\omega}} \) is the share of net return promised to the external equity providers.

- A debt contract between a household and a bank takes the form \( \{ R(\omega + \tilde{\omega}, r), (1 - \omega - \tilde{\omega}) \} \), where \( \omega + \tilde{\omega} \) is the total capital of the bank and \( (1 - \omega - \tilde{\omega}) \) is the amount of debt sought. \( R(\omega + \tilde{\omega}, r) \) denotes the gross amount (principal plus interest) that is promised on debt. A bank that funds its shortfall only through debt sets \( \tilde{\omega} = 0 \); whereas a bank that uses only equity sets \( \tilde{\omega} = 1 - \omega \). All intermediate cases are permitted.

- A debt contract between a household and a firm takes a similar form, except the terms of the contract is also conditioned on \( p \).

- All firms and banks have limited liability, so when the project fails the return to debtors is \( \min \{ y_l, R \} \).

In the second period, the returns from the risky projects are realized and deposits in the safe facility mature, contracts are fulfilled (or defaulted).

**Mechanics**

Since there is no asymmetry of information between households and firms, their analysis is relatively straight-forward. Inefficiencies arise in the relationship between households and banks.

**Two key thresholds** drive the mechanics of the model for banks:

1. The threshold level of a banks’ initial endowment, \( \tilde{\omega} \), at which point it is indifferent between screening and simply depositing its endowments in stage 1 to earn the safe rate; and
2. The threshold level of the success probability of projects, $\hat{p}$, at which point a bank in stage 2 is indifferent between proceeding after screening and depositing at the safe-rate.

The level of bank endowment matters for the screening decision because a lower initial endowment implies a larger funding gap and higher cost of funding. A bank with initial endowment (i.e. internal equity) less than $\hat{\omega}$ would find that the funding cost, plus the fixed cost of screening, exceed the expected return from acquiring risky assets. The endowment threshold $\hat{\omega}$ is defined formally in Section 4.3.

The realization of a bank’s success probability $p_i$ determines whether it proceeds with the project after screening. A bank with $p_i < \hat{p}$ finds that it can achieve a higher expected return by using the risk-free deposit facility, so will forego the lending opportunity. We define $\hat{p}$ more formally in Section 4.3.

For the rest of the paper, we will refer to $\hat{\omega}$ as the 'participation threshold' of a bank and $\hat{p}$ as the 'prudence threshold' of a bank. $\hat{\omega}$ is described as the 'participation threshold' because banks with initial endowment (or internal equity) below this level do not 'participate' in acquiring risky assets. $\hat{p}$ is described as the 'prudence threshold' because the success rate banks are willing to accept is an indicator of how prudent they are in handling funds from households. Imprudent banks will tend to take excessive risks to take advantage of their private information and limited liability – that is they acquire assets with a probability of success lower than a fully informed, risk-neutral agent would find acceptable.

4.3 The First-Best - the model under perfect information

Under the first-best, households can observe a bank’s probability of success, $p_i$, as well as its level of capitalization $(\omega_i + \tilde{\omega}_i)$. Therefore there is no information asymmetry, and no distinction between firms and banks. Thus for the rest of this section, we will
use the “firms” to encapsulate the behavior of both firms and banks under perfect information.

The Prudence Threshold - First-Best Contracts

Working backwards, we start by examining the prudence threshold (at Stage 2 of the game) for those firms that have discovered their individual realization of $p_i$ through screening. The prudence threshold, $\hat{p}$, is defined as the success rate required to make a firm indifferent between undertaking the risky project and simply depositing its endowments in the risk-free facility.

Specifically, for a firm with initial endowment $\omega_i$, and which is looking for $\bar{\omega}_i$ in additional equity and $1 - (\omega_i + \bar{\omega}_i)$ in debt from households, its prudence threshold $\hat{p}_i = \hat{p}(\omega_i + \bar{\omega}_i)$ and the terms of its funding contract are jointly determined by the following three equations\(^{29}\):

1. The condition where the firm is indifferent between the risky asset and the safe deposit facility:

$$\frac{\omega}{\omega + \bar{\omega}} \left( \hat{p} (y_h - R) + (1 - \hat{p}) \max [0, y_l - R] \right) = \omega (1 + r); \text{ or}$$

$$\hat{p} (y_h - R) + (1 - \hat{p}) \max [0, y_l - R] = (\omega + \bar{\omega}) (1 + r)$$

(4.1)

where $\bar{\omega}$ is the size of the equity injection from households; $\omega + \bar{\omega}$ is the total capital of the firm; and $\frac{\omega}{\omega + \bar{\omega}}$ is the share of the return retained by the firm.

A firm with capital of $\omega + \bar{\omega}$ needs to finance the remainder of the project $(1 - \omega - \bar{\omega})$ through debt. $R(\omega + \bar{\omega}, p)$ denotes the gross amount (principal plus interest) that is promised on that debt.

\(^{29}\)For brevity, we suppress the $i$ subscript henceforth where possible.
2. The condition where the households are receiving, in expectation, their required return from providing debt funding to a firm:

\[ pR + (1 - p) \min [y_l, R] = (1 - (\omega + \tilde{\omega})) (1 + r) \]  

(4.2)

3. The condition where for the households providing additional equity to the firm their expected return from the capital injection is at least equal to the expected return on debt/deposits:

\[
\frac{\tilde{\omega}}{\omega + \tilde{\omega}} (p (y_h - R) + (1 - p) \max [0, y_l - R]) \geq \tilde{\omega} (1 + r)
\]

or \[ p (y_h - R) + (1 - p) \max [0, y_l - R] \geq (\omega + \tilde{\omega}) (1 + r) \]  

(4.3)

This last condition is always satisfied given equation 4.1, and the fact that corporations will only proceed with \( p \geq \hat{p}_{fb} \), where \( \hat{p}_{fb} \) is the first-best level of prudence that reflects risk neutrality of all agents. Specifically, \( \hat{p}_{fb} \) is the lowest level of \( p \) that gives an expected return of at least \((1 + r)\).

**Proposition 11.** When \( p \) is common knowledge between all agents, firms will not take excessive risks, regardless of the level of their capitalization.

\[
\hat{p}(\omega_i + \tilde{\omega}_i) = \frac{(1 + r) - y_l}{(y_h - y_l)} \equiv \hat{p}_{fb} \text{ for } \forall (\omega_i + \tilde{\omega}_i)
\]  

(4.4)

**Proof.** See Annex. The intuition of the proof is simple. Creditors to the firm can perfectly observe the risks the firm is taking, so they can structure their contracts such that there are no incentives for the firm to engage in excessive risk-taking, where excessive risk taking is defined as \( p_i < \hat{p}_{fb} \).

Let \( \omega^*(p) \) denote the level of capital \((\omega + \tilde{\omega})\) such that \( y_l = R(p, \omega^*) \), so for a firm with \{\( p, \omega^*(p) \)\}, we can re-arrange equation 4.2 and 4.1 to give:
\[ \omega^* (p) = \omega^* = 1 - \frac{y}{(1+r)} \text{[so } \omega^* \text{ is independent of } p]; \]

\[ R (p, \omega^*) = R (\omega^*) = (1 - \omega^*) (1 + r) \text{[so } R (\omega^*) \text{ is independent of } p]; \text{ and} \]

\[ \hat{p} (\omega^*) = \frac{(1+r)-y}{(yh-y)} = \hat{p}_{fb} \]

We will refer to any firm with \( \omega + \tilde{\omega} \geq \omega^* \) as a 'fully capitalized' firm, because such a firm can pay its debt in full - even in the bad state\(^{30}\).

**Proposition 12. After screening in the first-best:**

1. firms invest in the risky asset if and only if \( p \geq \hat{p}(\omega + \tilde{\omega}, r) \); and

2. 'fully-capitalised' firms weakly prefer debt funding to equity funding. 'Not-fully-capitalised' firms (with \( \omega_i + \tilde{\omega}_i < \omega^* \)) are indifferent between using all debt, or using a mixture of debt and additional equity (provided that the additional equity injection does not make the firm 'fully-capitalised').

For simplicity, we assume that when firms are indifferent between debt and equity, they will proceed with the project using debt finance only.

**Proof.** See Annex. Part (1) follows from the definition of \( \hat{p}(\omega + \tilde{\omega}) \): firms with success probability \( p < \hat{p}(\omega + \tilde{\omega}, r) \) would receive a higher expected return from the risk-free facility than the risky asset. Part (2) holds because for 'fully-capitalised' firms \( (\omega + \tilde{\omega} > \omega^*) \), debt can be obtained at the risk-free rate [see the proof for proposition 11], but equity comes at the cost of sharing a proportion of the expected return - and that expected return exceeds the expected return on debt. We know that a firm will proceed with the project if and only if the expected return is weakly greater than the safe-rate of return (from part (1) and equation 4.1). Thus equity is more expensive than debt for a 'fully-capitalised' firm. For 'not-fully capitalized' firms, households with perfect information set debt contracts as a function of both \( p \) and \( \omega \). Additional

\(^{30}\)These firms can still 'fail' in the sense that their equity holders may be wiped out in the bad state.
equity injections lowers the cost on the bank’s entire stock of debt, so equity becomes just as attractive as debt [the Modigliani-Miller Theorem holds]. We offer a more formal proof of this proposition in the Annex.

**Corollary 2.** *In the absence of capital regulations, firms will not seek external equity injections when they undertake risky lending.*

**The Participation Threshold - First Best Case**

Having established the 'prudence threshold' of firms, we continue to work backwards to find firms' 'participation threshold' - the threshold amount of internal capital below which screening is not worthwhile. Upon discovering its realization of $\omega_i$ at Stage 1 of the game, a firm decides whether to engage in costly screening. The 'participation threshold' of firms is defined as the level of initial endowment that equates the expected value of screening ($V$) to the fixed cost of screening ($C$). The costly screening process can be interpreted as buying an option on the risky asset. In general, the value of the option will depend on the funding structure of the firm, in particular how much additional equity it is trying to attract from households.

Formally, $V$ is given by:

\[
V(\omega, \tilde{\omega}, r) \equiv E_p \left[ \max \left\{ 0, \frac{\omega}{\omega + \tilde{\omega}} \left( \frac{p (y_h - R)}{(1 - p)} \max [0, y_l - R] \right) - \omega (1 + r) \right\} \right]
\]

\[
= \frac{\omega}{\omega + \tilde{\omega}} \int_{p_1}^{p} [p (y_h - y_l) + y_l - (1 + r)] h(p) dp \quad (4.5)
\]

where the last line follows because the debt is provided by households at an expected return of $1 + r$ (more detail can be found in the annex).
Definition. The participation threshold for firms, $\hat{\omega}$, is implicitly defined by:

$$V(\hat{\omega}, \tilde{\omega}, r) = \frac{\hat{\omega}}{\tilde{\omega} + \omega} \int_{\hat{p}}^{1} \left[ p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp = C$$  \hspace{1cm} (4.6)

where $C$ is the fixed cost of screening.

We know that in the absence of any capital requirements ($\omega_{\text{reg}} = 0$), $\tilde{\omega} = 0$; and in the first-best $\hat{p}(\omega + \tilde{\omega}) = \hat{p}_{fb} = \frac{1+r-y_l}{(y_h-y_l)}$, so

$$V_{fb}(\omega_{\text{reg}} = 0, r) = \int_{\hat{p}_{fb}}^{1} \left[ p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp$$  \hspace{1cm} (4.7)

We impose by assumption that screening is socially optimal in the first-best:

$$\int_{\hat{p}_{fb}}^{1} \left[ p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp > C$$

so that in the first-best, every firm screens in the absence of any policy intervention and the participation threshold is zero:

$$\hat{\omega}_{fb} (\omega_{\text{reg}} = 0, r) = 0$$  \hspace{1cm} (4.8)

The aggregate value-added of firms (and banks) is given by:

$$V_{fb} = \int_{\hat{p}_{fb}}^{1} \left[ y_l + p(y_h - y_l) - (1 + r) \right] h(p) dp - C$$  \hspace{1cm} (4.9)

Summary

In the first-best households can observe the probability of success ($p_i$). This is sufficient to ensure that firms (and banks) operate in the most appropriate and prudent manner, regardless of the level of their capitalization/leverage. Specifically, every firm sets the prudence threshold at $\hat{p}_{fb} = \frac{(1+r)-y_l}{(y_h-y_l)}$, the level which ensures
that every risky project undertaken after screening is expected to deliver at least the risk-free return. In addition, with the participation threshold ($\hat{\omega}$) equal to zero, all firms screen so no profitable lending opportunities are wasted. Leverage and limited-liability, in the absence of asymmetric information, do not lead to inefficiencies and do not necessitate any policy intervention.

4.4 The Model with Asymmetry of Information

Frictions in the model arise because of the interaction between asymmetric information, leverage and limited liability. When households cannot observe banks’ probability of success ($p_i$), moral hazard becomes prevalent in the banking sector. This problem manifests itself in a sub-optimal level of screening for the risky asset; and for those banks that do screen, a lower level of prudence than ideal in deciding whether to invest in the risky asset.

Analytically, the structure of the model remains unchanged under asymmetric information (the majority of the equations are identical). The main distinction is that for banks $\hat{p}$ becomes a function of $\omega$, and therefore can no longer be evaluated outside of the integral over $\omega$. The analysis for firms is identical to the first-best case.

The Prudence Threshold

As with the first-best, we start by examining the prudence threshold (at Stage 2 of the game) for those banks that have discovered their individual realization of $p_i$ through screening. Under asymmetric information, the analogues of equations 4.1-4.4 are:
1. The condition that a bank with initial capital $\omega$ and additional equity injection $\tilde{\omega}$ is indifferent between the risky asset and the risk-free asset:

$$\frac{\omega}{\omega + \tilde{\omega}} \{\hat{p}(y_h - R) + (1 - \hat{p}) \max[0, y_l - R]\} = \omega(1 + r)$$

or:

$$\hat{p}(y_h - R) + (1 - \hat{p}) \max[0, y_l - R] = (\omega + \tilde{\omega})(1 + r) \quad (4.10)$$

this is identical to equation 4.1.

2. The condition that the households are indifferent between providing debt and the risk-free facility:

$$A(\hat{p}) R + (1 - A(\hat{p})) (\min[y_l, R]) = (1 - \omega - \tilde{\omega})(1 + r) \quad (4.11)$$

where $A(\hat{p}) \equiv E[p \mid p > \hat{p}] \equiv \frac{1}{1-H(\hat{p})} \int_{\hat{p}}^{1} ph(p) \, dp$ denotes the conditional expectation of $p$ given that $p \geq \hat{p}$.

Note that the key difference here (relative to equation 4.2) is that instead of observing the $p$ directly, households now need to form a conditional expectation of $p$.

3. The condition that for households providing additional equity to the bank their expected return from the capital injection is at least equal to the safe rate of return:

$$\frac{\tilde{\omega}}{\omega + \tilde{\omega}} \{A(\hat{p})(y_h - R) + (1 - A(\hat{p})) \max[0, y_l - R]\} \geq \tilde{\omega}(1 + r)$$

or:

$$A(\hat{p})(y_h - R) + (1 - A(\hat{p})) \max[0, y_l - R] \geq (\omega + \tilde{\omega})(1 + r) \quad (4.12)$$
This last condition is always satisfied given equation 4.10 and the fact that \( A(\hat{p}) \geq \hat{p} \).

**Proposition 13.** With asymmetric information, banks that are not 'fully capitalized' will take excessive risks (\( \hat{p} < \hat{p}_{fb} \)):

1. As before, let \( \omega^* = \omega + \tilde{\omega} \) denote the level of capital such that \( R(\omega^*) = y_l \), so \( \omega^* = 1 - \frac{y_l}{(1+r)} \). We refer to any bank with \( \omega_i + \tilde{\omega}_i \geq \omega^* \) as a "fully capitalized" bank.

2. Banks become more prudent as they increase their capital, up to a cap when they become fully capitalized.

   (a) For \( \omega + \tilde{\omega} \geq \omega^* \), we have \( \hat{p} = \frac{(1+r)-y_l}{(y_h-y_l)} \); \( R = (1 - (\omega + \tilde{\omega})) (1 + r) \);

   (b) For \( \omega + \tilde{\omega} < \omega^* \), we have \( \frac{\partial \hat{p}}{\partial \omega} = \frac{(A(\hat{p})-\hat{p})(1+r)}{(y_h-R)A(\hat{p})+\hat{p}A'(\hat{p})(R-y_l)} \); and \( \hat{p} \leq \frac{(1+r)-y_l}{(y_h-y_l)} = \hat{p}_{fb} \).

The function for \( \hat{p}(\omega + \tilde{\omega}) \) kinks at \( \omega^* \), with right-hand derivative given by \( \partial_+ \hat{p}(\omega^*)/\partial \omega = 0 \) and left-hand derivative given by \( \partial_- \hat{p}(\omega^*)/\partial \omega = \frac{(A(\hat{p})-\hat{p})(1+r)}{(y_h-R)A(\hat{p})+\hat{p}A'(\hat{p})(R-y_l)} > 0 \).

An increase in internal equity has the same marginal impact on prudence as an increase in external equity. Banks behavior is influenced by the quantity of their equity rather than its source.

**Proof.** A full proof is in the annex, here we give the intuition. In the first part of the proposition, \( \omega^* \) is defined as the level of internal capital such that \( y_l = R(\omega^*, r) \), so \( \omega^* = 1 - \frac{y_l}{(1+r)} \) follows directly from re-arranging equation 4.11. Banks with capital above this level can borrow at the risk free rate because the liquidation value of their risky assets is sufficient to cover the bank’s debt obligations (again from re-arranging equation 4.11 after setting \( R(\omega^*) = y_l \)). The second part of the proposition says
that banks that are not fully capitalized will take excessive risks. Limited liability encourages banks to take excessive risks when the liquidation value of the assets is less than the bank’s debt obligations. Households can anticipate this, so will increase the cost of debt \( R(\omega + \tilde{\omega}, r) \) for banks with low capital (and thus high funding needs). The fact that banks cannot credibly communicate their success rate \( (p_i) \) to households exacerbates this issue as the debt contract can only be formulated as a function of the bank’s observable capital \( (\omega + \tilde{\omega}) \). This means two banks with the same level of capital will face the same borrowing costs regardless of the success probabilities on their assets - the bank with the better project is effectively subsidizing the borrowing costs of the one with the worse asset. Taken together, limited liability and this cross-subsidization towards less attractive assets give rise to moral hazard. The fact that prudence is an increasing function of capital reflects the observation that the moral hazard issue becomes less pronounced when banks have more ’skin in the game’.

\[ \text{Proposition 14. Preference for Debt} \]

1. After screening banks invest in the risky asset if and only if \( p \geq \hat{p}(\omega + \tilde{\omega}, r) \);

   and

2. when banks acquire risky assets they prefer to use as much debt finance as possible (i.e. banks will choose to set \( \tilde{\omega} = 0 \)).

\[ \text{Proof.} \] See Annex. Part (1) follows from the definition of \( \hat{p}(\omega + \tilde{\omega}) \); banks with success probability \( p < \hat{p}(\omega + \tilde{\omega}) \) would receive a higher expected return from investing in the safe asset. Part (2) holds because banks expect to pay out a higher rate of return to external equity providers than to debt providers. We show this formally in the annex. Qualitatively, equity is more costly in this model because debt is available at a rate which makes households indifferent (in expectation) between providing the debt and using the safe deposit facility. Equity, on the other hand, allows the household to share the surplus from banks acquiring risky assets, and therefore delivers a rate of
return for households that is greater than the risk-free facility. The original owners of bank equity have the value of their claims diluted by raising new equity from outsiders. Consequently banks stick with debt in the absence of any policy intervention, because this is the cheapest way for them to raise funds at the margin.

Corollary 3. In the absence of capital regulations, banks will not seek external equity injections when they acquire risky assets.

The Participation Threshold

As in the first-best case, the value of screening for banks is still given by equation 4.5:

\[
V(\omega, \tilde{\omega}, r + \delta_r) = E_p \left[ \max \left\{ 0, \frac{\omega}{\omega + \tilde{\omega}} \left( \frac{p (y_h - R)}{(1 - p) \max [0, y_l - R]} - \omega (1 + r) \right) \right\} \right]
\]

\[
= \frac{\omega}{\omega + \tilde{\omega}} \int_{\hat{\hat{p}}(\omega + \tilde{\omega})}^{1} [p (y_h - y_l) + y_l - (1 + r)] h(p) dp \tag{4.13}
\]

And the participation threshold \(\hat{\omega}\) is still given implicitly by equation 4.6:

\[
\frac{\hat{\omega}}{\omega + \tilde{\omega}} \int_{\hat{\hat{p}}(\omega + \tilde{\omega})}^{1} [p (y_h - y_l) + y_l - (1 + r)] h(p) dp = C \tag{4.14}
\]

with the only difference that \(\hat{p}\) is now a function of \(\omega + \tilde{\omega}\). Since \(\hat{p}(\omega + \tilde{\omega}) \leq \hat{p}_{fb}\), this means that even when \(\tilde{\omega} = 0\), the participation threshold under asymmetric information will in general be a positive number.

Proposition 15. \(\frac{\partial V(\omega, \tilde{\omega} = 0, r)}{\partial \omega} \left\{ \begin{array}{l} > 0 \text{ for } \omega < \omega^* \\ = 0 \text{ for } \omega \geq \omega^* \end{array} \right. \). In the absence of capital regulations (and \(\tilde{\omega} = 0\), the expected value of screening is increasing in the bank’s initial
endowments (up to a cap when the bank becomes fully well capitalized). (see proof in the annex)

Since we have established through corollary 3 that banks will only use debt funding in the absence of capital regulations, for now we can restrict attention to the case where $\bar{\omega} = 0$.

Summary and Implications

The presence of moral hazard in the model give rise to sub-optimal outcomes compared to the first-best scenario. Banks have no incentives to top-up their capital. Consequently, the hurdle for participation, $\hat{\omega} > 0$, is too high; and the level of prudence, $\hat{p}(\omega, r) \leq \frac{(1+r) - y}{(y_h - y_l)} = \hat{p}_{fb}$ is too low. As a result aggregate value added of the banking sector is lower than it is in the first-best (where we assumed that $p$ is common knowledge). Banks do not do enough screening (and so will be missing opportunities that investors would value), but those that do screen will accept some investments that well-informed, risk-neutral investors would reject.

4.5 Policy Tools and Their Transmission

Interest rate policy and capital regulations can be used to drive outcomes closer to the first-best level in the banking sector. Both policy tools function through their effects on the participation and prudence thresholds.

Interest Rate Policy / Variations in the safe real interest rate

Let us assume that the policy maker (either through its central bank or directly by offering assets to the private sector) can set the risk-free rate of return on the deposit facility ($r$) and this affects the attractiveness of the risky asset relative to the risk-free deposit. (We return to the issue of whether the national authority have control over $r$ below). The central bank remunerates all resources placed in its deposit facilities at
the rate $r$, the servicing cost of which is covered by lump sum taxation of households. This interest rate ($r$) is revealed to all agents at the very beginning of the game.

We normalize $r = 0$ as the neutral interest rate. In the first-best scenario, there is no need for interest rate policy intervention, so $r = 0$ and the first-best level of prudence becomes:

$$
\hat{p}_{fb} = \frac{1 - y_t}{y_h - y_t}
$$

(4.15)

The first-best level of the participation threshold $\hat{\omega}_{fb}$ ($\omega_{reg} = 0$) = 0 remains the same as before (see equation 4.8)

Note that the possibility of interest rate policy intervention introduces a wedge between the socially optimal outcome and what is privately optimal in the absence of asymmetric information. In particular, the privately optimal level of prudence is given by

$$
\hat{p}^* (r) = \frac{1 + r - y_t}{y_h - y_t}
$$

(4.16)

which will be different to $\hat{p}_{fb}$ (the socially optimal level of prudence) for non-zero $r$. So using interest rate policy to combat the inefficiencies of the banking sector will lead to distortions in the non-bank sector (i.e. the risk-taking behavior of “firms”). This makes the relative size of the banking sector to that of the non-financial sector ($\mu$) a factor in assessing the extent to which the authorities should use the influence they may have over $r$ to curb excess risk taking.

The effect of interest rate policy on banks’ behavior is summarized in the proposition below.

**Proposition 16.** Interest rate tightening improves 'prudence' at the cost of decreasing 'participation'. Policy loosening achieves the converse:

1. $\frac{\partial \hat{p}}{\partial r} > 0$; and

2. $\frac{\partial \hat{\omega}}{\partial r} > 0$
Proof. We describe the intuition for the proposition here; formal proof is in the annex. Interest rate policy tightening improves prudence for all firms and banks by increasing the attractiveness of the outside option. A quick way to illustrate the result $\frac{\partial \hat{p}}{\partial r} > 0$ is by appealing to proposition 13: $\hat{p}(\omega + \tilde{\omega}) \leq \hat{p}_{fb}$. Recall from equation 4.15: $\hat{p}_{fb} = \frac{1 - y_l}{(y_h - y_l)}$; and from Proposition 13: $\hat{p}(\omega + \tilde{\omega}, r) \leq \frac{1 + r - y_l}{(y_h - y_l)} = \hat{p}^*(r)$. So by increasing $r$, a central bank can increase $\hat{p}^*(r)$ and thus push $\hat{p}(\omega + \tilde{\omega}, r)$ closer to $\hat{p}_{fb}$ for all $\omega$.

Interest rate policy tightening decreases the level of participation because it increases the relative attractiveness of the outside option. Recall that $\hat{\omega}(\tilde{\omega}, r)$ is implicitly defined by

$$V(\hat{\omega}, \tilde{\omega}, r) = \hat{\omega} \int_{\hat{p}(\omega + \tilde{\omega})}^{1} \left[ p \left( y_h - y_l \right) + y_l - (1 + r) \right] h(p) dp = C$$

Increasing $r$ reduces the value of screening, so for a given fixed cost of screening $C$, banks need a higher amount of internal equity to make the screening process worthwhile.

Prudential Policy

We model capital regulations in the simple form of a leverage ratio restriction. Regulators impose a minimum capital requirement $\omega_{reg}$ such that banks with endowments $\omega_i < \omega_{reg}$ must seek outside equity of at least $(\omega_{reg} - \omega_i)$ or are barred from investing in the risky asset. We rule out the possibility that the regulator can set a capital requirement that depends on the riskiness of bank assets by assuming the asset specific chance of success $p_i$ remains private bank information.

The capital requirement $\omega_{reg}$ is announced as soon as banks find out their individual realizations of $\omega_i$. A strengthening of capital requirements achieves the same qualitative effect as a tightening of interest rate policy on banks’ decisions, albeit through different means. Critically, capital regulations can be imposed on banks
alone, without affecting the behavior of firms. This is an important difference between the blunt instrument of interest rate policy (one "that gets in all the cracks") and the targeted instrument of bank capital requirements.

**Proposition 17.** *Capital regulations improve prudence by forcing some banks to seek additional equity funding, but do so at the cost of decreased participation:*

1. \( \tilde{\omega} = \max [0, \omega_{\text{reg}} - \omega] \). Only banks that are constrained by the capital regulations will seek equity injections from households. And when they do, they will only top-up to the minimum capital standard.

2. \( \frac{\partial \hat{p}(\omega + \tilde{\omega}, r)}{\partial \omega_{\text{reg}}} \geq 0 \) with strict inequality as long as \( \tilde{\omega}(\tilde{\omega} = 0, r) < \omega_{\text{reg}} < \omega^* \). Capital requirement will improve the prudence level of banks for which it binds, so long as they are not already 'fully capitalized'.

3. \( \hat{\omega}(\omega_{\text{reg}} - \omega, r) > \hat{\omega}(\tilde{\omega} = 0, r) \) for all \( \omega_{\text{reg}} \) such that \( \omega_{\text{reg}} > \hat{\omega}(\tilde{\omega} = 0, r) \). The participation threshold is higher for banks that require external equity injections (i.e. banks that falls short of the capital requirement), than for banks that are not bound by the capital requirement.

4. \( \frac{\partial \omega(\omega_{\text{reg}} - \omega, r)}{\partial \omega_{\text{reg}}} \geq 0 \). A tightening of capital standards increases the participation threshold.

**Proof.** We give the intuition of the result here, details are in the annex.

1. \( \tilde{\omega} = \max [0, \omega_{\text{reg}} - \omega] \) follows directly from the observation that banks weakly prefer debt to equity (proposition 14). Consequently, banks for which the regulations are binding will only top-up their capital to the regulatory minimum, and seek the remaining funds in the form of debt.

2. \( \frac{\partial \hat{p}(\omega + \tilde{\omega}, r)}{\partial \omega_{\text{reg}}} \geq 0 \) is a corollary of proposition 13 (\( \frac{\partial \hat{p}(\omega + \tilde{\omega}, r)}{\partial \omega} \geq 0 \)) and the previous observation that \( \tilde{\omega} = \max [0, \omega_{\text{reg}} - \omega] \). In particular, \( \frac{\partial \hat{p}(\omega + \tilde{\omega}, r)}{\partial \omega_{\text{reg}}} = 0 \) for banks
with \( \omega_i > \omega_{\text{reg}} \) or \( \omega_i \geq \omega^* \). Banks with \( \omega_i > \omega_{\text{reg}} \) obviously will remain unaffected by a marginal increase in \( \omega_{\text{reg}} \). Banks with \( \omega_i \geq \omega^* \) already operate at the optimal level of prudence, so pushing \( \omega_{\text{reg}} \) above \( \omega^* \) does not bring any further gains in prudence.

3. \( \dot{\omega} (\omega_{\text{reg}} - \omega, r) > \dot{\omega} (0, r) \) holds because equity funding is more costly than debt funding due to the dilution effect (proposition 14). Therefore, the value of screening is lower for banks that are constrained by the capital requirement. (\( \omega_{\text{reg}} \) is announced in Stage 1, so banks can anticipate capital needs in advance of screening.) Consequently, the presence of binding capital regulations increases the participation threshold for a given fixed cost of screening.

4. \( \frac{\partial \dot{\omega}}{\partial \omega_{\text{reg}}} \geq 0 \) follows from part 1 and 3. Higher capital requirements increase the size of equity injection required, and thus reduce the value of screening and increase the threshold for participation.

\[\Box\]

Summary of Policy Implications

An increase in the risk-free interest rate increases the opportunity cost of funding risky assets. This pushes up prudence for all banks at all levels of initial endowments / internal equity. So poorly endowed (or highly leveraged) banks become more prudent, but firms which do not suffer asymmetric information will become too cautious\(^{31}\). A strengthening of bank capital standards forces more banks to seek out more external equity, inducing them to behave as if they are a better endowed bank.

The two policy instruments will also affect participation. Because external equity is more costly than debt, tougher capital requirements make the break-even level of initial endowment for banks higher for a given fixed cost of screening. So more banks

\(^{31}\)Very well capitalized banks will also become too cautious. But this side effect falls away if we restrict the upper support of \( \omega \) to a value significantly below 1.
will drop out from screening due to a low level of initial endowments. A tightening of interest rate policy shares the same drawback.

So the trade-off between 'prudence' and 'participation' is common across both interest rate policy and capital regulations. But there are two important differences. First, interest rate policy distorts incentives for all banks and firms; whereas capital regulations only affect those banks for which it is binding (i.e. poorly capitalized banks). Second, a fall in the safe real interest rate below the neutral level (i.e. an negative \( r \)) can increase participation of banks beyond the level possible in the absence of any intervention (and thus gets closer to the state of universal participation under the first-best). In contrast, capital regulations can be at best non-binding. This means capital requirements alone can never correct the participation distortions relative to the first best.

**Aggregate Outcomes**

**Expected bank investment** In the presence of a binding regulatory regime (where \( \omega_{\text{reg}} > \hat{\omega} (\omega_{\text{reg}} r) > \hat{\omega} (0, r) \)), the expected amount of investment by banks in the risky asset is given by:

\[
I (r, \omega_{\text{reg}}) = (1 - \mu) \left[ \int_{\omega_{\text{reg}}}^{\omega_{ub}} (1 - H (\hat{\omega} (\omega_{\text{reg}}, r))) dF (\omega) + \int_{\omega_{\text{reg}}}^{\omega_{ub}} (1 - H (\hat{\omega} (\omega, r))) dF (\omega) \right]
\]  

(4.17)

where:

- \( \omega_{ub} \) is the upper bound for the distribution of \( \omega \)
- \( \int_{\omega_{\text{reg}}}^{\omega_{ub}} (1 - H (\hat{\omega} (\omega_{\text{reg}}, r))) dF (\omega) \) is the amount of investment expected from banks whose initial endowment fell short of the capital regulations (and hence need to raise additional equity); and
\[
\int_{\omega_{\text{ub}}}^{\omega_{\text{ub}}} (1 - H (\hat{p} (\omega, r))) dF (\omega)
\]

is the amount of investment expected from banks that are not affected by the capital regulations (and hence can finance themselves with debt and internal equity).

In the absence of prudential regulation (or if the regulation is completely non-binding: \(\omega_{\text{reg}} \leq \hat{\omega} (0, r)\)), the expected level of bank investment is given by:

\[
I (r) = (1 - \mu) \int_{\omega(0,r)}^{\hat{\omega}_{\text{ub}}} (1 - H (\hat{p} (\omega, r))) dF (\omega)
\]

(4.18)

The expected level of bank investment in the first-best is:

\[
I_{fb} = (1 - \mu) (1 - H (\hat{p}_{fb})) \quad \text{where} \quad \hat{p}_{fb} = \frac{1 - y_l}{y_h - y_l}
\]

(4.19)

A comparison of these equations shows that whilst both interest rate policy and capital requirements can be used to push up the prudence threshold of banks to a level that is closer to the first best, this can only be achieved with a decline in the funding of risky assets. Part of the fall in the investment in risky assets is socially desirable - banks with low \(p_i\) should rightly give up their risky asset after screening. What is not desirable is the decline in bank acquisition of risky assets due to falling participation. When capital requirements and the stance of interest rate policy are strict, more banks may choose to pass up the chance to fund risky investments even before the screening stage. This fall in participation is sub-optimal because \(\omega\) and \(p\) are assumed to be independently distributed. Banks with low initial endowments (low \(\omega_i\)) will be disincentivised from screening by the regulatory environment, even though they might discover very good assets (with high \(p_i\)) if they screen.
Expected value added  Under a binding regulatory regime, where $\omega_{\text{reg}} > \hat{\omega} \left( \omega_{\text{reg}} r \right) > \hat{\omega} \left( 0, r \right)$, the expected amount of value added from firms and banks is given by:

$$V \left( \omega_{\text{reg}}, r \right) = \mu \int_{\hat{\omega} \left( 0, r \right)}^{\omega_{\text{ub}}} \left[ \int_{p^* \left( r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) - C \right] dF \left( \omega \right)$$

$$+ \left( 1 - \mu \right) \int_{\hat{\omega} \left( \omega_{\text{reg}}, r \right)}^{\min \left[ \omega_{\text{reg}}, \omega_{\text{ub}} \right]} \left[ \int_{\hat{p} \left( \omega, r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) \right] dF \left( \omega \right)$$

$$+ \left( 1 - \mu \right) \int_{\omega_{\text{ub}}}^{\omega_{\text{ub}}} \left[ \int_{\hat{p} \left( \omega, r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) \right] dF \left( \omega \right)$$

$$- \left( 1 - \mu \right) C \left[ \int_{\hat{\omega} \left( \omega_{\text{reg}}, r \right)}^{\omega_{\text{ub}}} dF \left( \omega \right) \right]$$

(4.20)

where $V \left( \omega_{\text{reg}}, r \right)$ is composed of:

- The value-added of firms $\mu \int_{\hat{\omega} \left( 0, r \right)}^{\omega_{\text{ub}}} \left[ \int_{p^* \left( r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) - C \right] dF \left( \omega \right)$, where $\hat{p}^* \left( r \right) = \frac{1 + r - y_t}{y_h - y_t}$.

- The value-added of banks that are constrained by the capital requirement:

$$\left( 1 - \mu \right) \int_{\hat{\omega} \left( \omega_{\text{reg}}, r \right)}^{\min \left[ \omega_{\text{reg}}, \omega_{\text{ub}} \right]} \left[ \int_{\hat{p} \left( \omega, r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) \right] dF \left( \omega \right)$$

- The value-added of banks that are not constrained by the capital requirement:

$$\left( 1 - \mu \right) \int_{\omega_{\text{ub}}}^{\omega_{\text{ub}} \min \left[ \omega_{\text{reg}}, \omega_{\text{ub}} \right]} \left[ \int_{\hat{p} \left( \omega, r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) \right] dF \left( \omega \right)$$

- Minus the cost of screening by all banks: $\left( 1 - \mu \right) C \left[ \int_{\hat{\omega} \left( \omega_{\text{reg}}, r \right)}^{\omega_{\text{ub}}} dF \left( \omega \right) \right]$

In the absence of capital regulations (or when it is entirely non-binding), the expected aggregate value-added is given by:

$$V \left( r \right) = \mu \int_{\hat{\omega} \left( 0, r \right)}^{\omega_{\text{ub}}} \left[ \int_{p^* \left( r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) - C \right] dF \left( \omega \right)$$

$$+ \left( 1 - \mu \right) \int_{\hat{\omega} \left( 0, r \right)}^{\omega_{\text{ub}}} \left[ \int_{\hat{p} \left( \omega, r \right)}^{1} \left[ y_t + p \left( y_h - y_t \right) - 1 \right] dH \left( p \right) - C \right] dF \left( \omega \right)$$
Lastly, recall that the aggregate value added in the first-best is given by:

\[ V_{fb} = \int_{\hat{p}_{fb}}^{1} [y_l + p(y_h - y_l) - 1] dH(p) - C \]  \hspace{1cm} (4.21)

where \( \hat{p}_{fb} = \frac{1-y_l}{y_h-y_l} \).

In the following section, we treat the expected level of aggregate value added \( V(\omega_{reg}, r) \), which is also the weighted average rate of return on all savings invested in firms and banks, as the objective function for policymakers\(^{32}\). This allows us to employ numerical techniques to calculate the optimal setting of interest rate policy and capital requirements under illustrative parameterizations of the model.

### 4.6 Numerical Simulations

We use numerical simulations to illustrate the trade-off between interest rate policy and capital requirements. The model we have outlined above is necessarily a significant simplification of the financial intermediation process. A precise calibration of the model is not straightforward because there are no direct empirical counterparts to some of the key parameters of the model, specifically the pair of binary outcomes \((y_h, y_l)\) for the risky asset. We chose a wide range of parameter combinations for \((y_h, y_l)\) which cover high and low risk investment environments. And we rely on robustness tests to demonstrate that our main qualitative results are sound, namely that: (1) interest rate policy and capital regulations function as imperfect substitutes; (2) these policy interventions can trade-off between 'prudence' and 'participation' of banks; (3) the level of the optimal capital requirement is sensitive to variations in the safe real interest rates; and (4) across a wide, plausible, range of calibrations, the optimum setting of policy is one where the capital requirement plays the main role in addressing the asymmetric information distortions in the banking sector.

\(^{32}\)Using the expected aggregate value added as the objective function makes sense given the assumption of risk neutrality.
We allow the binary outcomes for the risky asset \((y_h, y_l)\) to vary between permissible values in the model\(^{33}\), whilst making some assumptions on the remaining model parameters. Specifically, we assume that the probability of success, \(p\), is uniformly distributed between \([0, 1]\); initial endowments \(\omega\) are uniformly distributed between \([0, 0.1]\)\(^{34}\); and the cost of screening \(C\) is 1%. In our model, \(\mu\) represents the proportion of initial endowments in the economy that are available to firms rather than banks. We set \(\mu\) to 0.85, to reflect the fact that financial companies account for roughly 15% of the capitalization of major stock indices in the US and the UK.

Table 4.1 presents the optimal policy combinations when we allow \((y_h, y_l)\) to vary. The range of combinations of \(y_h\) and \(y_l\) that we consider runs from very low to very high risk worlds. A world in which \(y_h\) is as low as 1.1 and \(y_l\) as high as 0.8 is one in which at the first best losses on those investments which fail are only 3% of total investment. But the expected return on investments are also quite low at 5%. At the other extreme when \(y_h\) is 1.9 and \(y_l = 0\) optimal investment implies that losses from projects failing are around 24% of total investment, but the expected return is over 45%.

\(^{33}\)As part of the theoretical model, we impose two conditions on the permissible levels for \((y_h, y_l)\):

1. The risky asset must not be so attractive that banks will always proceed without screening:

\[
y_l + E[p](y_h - y_l) < 1
\]

2. The risky asset must be attractive enough such that all banks will screen in the first-best:

\[
\int_{\hat{\rho}_{fb}}^{1} [p(y_h - y_l) + y_l - 1] h(p) dp > C
\]

where \(\hat{\rho}_{fb} = \frac{1-y_l}{y_h-y_l}\).

In the numerical simulation, we also required that the risky asset must be attractive enough such that at least some banks will screen under asymmetric information when there is no policy intervention: \(\hat{\omega}(r = 0, \omega_{reg} = 0) < \omega^{ub}\). This additional restriction ruled out the combination \((y_h = 1.1, y_l = 0.6)\) in Table 4.1.

\(^{34}\)Note that because the prudence threshold of firms is independent of \(\omega\), and that firms are essentially indifferent between debt and equity funding, the results of the model is invariant to the distribution of firm’s initial endowments. So for simplicity, we assume that banks and firms share the same distribution for \(\omega\). In the Appendix, we show that when we extend the model to allow for taxes (that have debt deductions), the distribution of endowments for firms does matter.
In Table 4.1 the figures in curved brackets for each combination of \((y_h; y_l)\) are the optimal interest rate (deviation from the neutral policy setting) and the capital requirement. The figure in square brackets is the associated maximal value added (as per equation 4.20, which is also the weighted average rate of return on all savings invested in firms and banks). The results here shows that, across these wide-ranging parameterizations, capital regulations are generally the main tool policy makers should use in addressing banking inefficiencies. Only in the safest investment environment \((y_h = 1.1; y_l = 0.8)\) should interest rate policy be used on its own, and then the safe real interest rate should be set to 0.5 percentage points above a neutral level. In most cases capital requirements are substantial - ranging from around 10% to up to 30%. In the most risky environment \((y_h = 1.9; y_l = 0.9)\), where incentives for banks to raise too much debt and invest too much are at their greatest, both interest rate policy and capital requirements are very tight; the safe real interest rate is 2.5% above neutral and capital should be 30% of total bank assets.
Table 4.1: Optimal Policy Setting across \((y_h, y_l)\) pairs

<table>
<thead>
<tr>
<th>Optimal Policy ((r%, \omega_{reg}%); [VA%])</th>
<th>Good state payoff ((y_h))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1</td>
</tr>
<tr>
<td>0.0 (-)</td>
<td>(0.0, 0.0); [0.57]</td>
</tr>
<tr>
<td>0.1 (-)</td>
<td>(0.5, 0.0); [0.70]</td>
</tr>
<tr>
<td>0.2 (-)</td>
<td>(0.0, 0.0); [0.85]</td>
</tr>
<tr>
<td>0.3 (-)</td>
<td>(0.0, 10.0); [1.04]</td>
</tr>
<tr>
<td>0.4 (-)</td>
<td>(0.0, 12.5); [1.30]</td>
</tr>
<tr>
<td>0.5 (-)</td>
<td>(0.0, 12.5); [1.64]</td>
</tr>
<tr>
<td>0.6 (-)</td>
<td>(0.0, 12.5); [2.12]</td>
</tr>
<tr>
<td>0.7 ((0.0, 0.0); [0.21])</td>
<td>(0.0, 12.5); [2.81]</td>
</tr>
<tr>
<td>0.8 ((0.5, 0.0); [0.60])</td>
<td>-</td>
</tr>
</tbody>
</table>

In order to examine the trade-offs between interest rate policy and capital regulations, and to demonstrate the more detailed implications of the model, we take a specific illustrative parameterization (see Table 4.2) and present the associated results. We have chosen the pair \((y_h, y_l) = (1.4, 0.5)\) for this illustrative example. This combination generates first-best outcomes where about 20% of those assets acquired (after screening) become non-performing (and in this low state, the loss of value is such that the residual worth of the asset is only one half of the investment). This scale of risk and of losses is not implausible for loans to finance corporate activity - particularly of small businesses that rely on bank intermediation. Given the losses on sub-prime mortgage lending that was behind the financial crisis of 2007-08, this scale of risk may not be excessive even for lending backed by residential property which used to be considered relatively safe. Our aim in choosing these values is not however
an attempt to closely match returns on past investment, but rather to provide a numerical illustration of outcomes that are broadly indicative of the levels of risk.

Table 4.2: Illustrative Baseline Parameterization

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Illus. Calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good state payoff on risky asset ($y_h$)</td>
<td>1.4</td>
</tr>
<tr>
<td>Bad state payoff on risky asset ($y_l$)</td>
<td>0.5</td>
</tr>
<tr>
<td>Success prob. of risky asset ($p$)</td>
<td>$p \sim U [0, 1]$</td>
</tr>
<tr>
<td>Cost of screening ($C$)</td>
<td>0.01</td>
</tr>
<tr>
<td>Proportion of firms relative to banks ($\mu$)</td>
<td>0.85</td>
</tr>
<tr>
<td>Initial endowments for banks and firms ($\omega$)</td>
<td>$\omega \sim U [0, 0.1]$</td>
</tr>
</tbody>
</table>

With the set of parameters in Table 4.2, we compute the aggregate value-added of firms and banks for different settings of the safe interest rate and regulatory capital requirements. Table 4.3 present the results for $\omega_{reg}$ between 0% and 30% (step size of 2.5 percentage points), and $r$ between −2% and 10% (step size of 50 basis points).

Table 4.3: Aggregate value added under selected policy combinations

<table>
<thead>
<tr>
<th>VA (%)</th>
<th>0.0</th>
<th>2.5</th>
<th>5.0</th>
<th>7.5</th>
<th>10.0</th>
<th>12.5</th>
<th>15.0</th>
<th>17.5</th>
<th>20.0</th>
<th>22.5</th>
<th>25.0</th>
<th>27.5</th>
<th>30.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0</td>
<td>7.13</td>
<td>7.16</td>
<td>7.23</td>
<td>7.29</td>
<td>7.36</td>
<td>7.41</td>
<td>7.44</td>
<td>7.46</td>
<td>7.47</td>
<td>7.48</td>
<td>7.47</td>
<td>7.46</td>
<td>7.45</td>
</tr>
<tr>
<td>-1.5</td>
<td>7.16</td>
<td>7.20</td>
<td>7.26</td>
<td>7.32</td>
<td>7.38</td>
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<td>6.95</td>
<td>6.91</td>
<td>6.86</td>
<td>6.82</td>
<td>6.77</td>
<td>6.72</td>
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Each row in Table 4.3 illustrates how expected aggregate value added changes as we vary the level of capital requirements, for a given safe real interest rate. This allows us to trace a path for the optimal capital requirement conditioned on the safe interest rate (italic, underlined, blue fill). Table 4.3 thus shows that the optimal regulatory capital standards strengthens as the safe real interest rate falls.

In particular, under this illustrative calibration, the optimal capital requirement is 20% when the safe real interest rate is at its neutral stance \(r = 0\). Even when the safe real interest rate rises to 5.5% above its neutral level, the optimal capital requirement is still as high as 12.5%. Given that \(\omega\) is assumed to be distributed between 0 and 0.1, this means a leverage ratio that is binding for all banks in the population. For no capital requirements to be (locally) optimal, the safe real rate of interest need to rise to 6 percentage points above the neutral rate and beyond.

We could alternatively read Table 4.3 on a column-by-column basis. This would allow us to trace out a path for the optimal the real safe interest rates for given level of bank capital requirements (bold, green fill). We find that the optimal stance of interest rate policy loosens as regulatory capital standard strengthens. Under this specific set of parameters, the globally optimal outcome is achieved when capital requirement is set to 17.5% and the real safe interest rate is 0.5 percentage points above the neutral level.

The model also allows us to describe the banks’ behavior under each policy setting (see Table 4.4). Under the globally optimal combination of \(r = 0.5\%\) and \(\omega_{\text{reg}} = 17.5\%\), the capital regulation is binding for all banks in the population. The participation threshold equals 0.023, meaning that 23% of banks would choose not to screen the risky asset. For banks that do screen, they will invest if and only if the probability of success exceeds the prudence threshold of 0.41 (the first-best level is 0.56). The conditional expectation of success is therefore \(E[p|p \geq \hat{p}(\omega_{\text{reg}}) = 0.41] \approx\)
0.70, which implies that around 30% of the risky investments made are expected to underperform. In aggregate about 46% of banks invest in risky assets.

Table 4.4 also shows that across a broad range of measures - including aggregate value added, investment in the risky asset, and probability of failure - the outcomes under optimal policy are between first-best levels (where there is no asymmetry of information, no capital requirements and interest rate policy is neutral) and levels under asymmetry of information but no policy intervention. The table shows that investment in the risky asset is significantly lower in the first-best case and under optimal policy than compared to the case of no-intervention. Free market outcomes with asymmetric information generate much too much bank funding of risky assets.

Note that although the banks in our model are heterogeneous at the start (they take independent draws from the distribution of $\omega$), having a minimum capital requirement that exceeds the upper bound of the $\omega$ distribution (i.e. $\omega_{\text{reg}} \geq \omega_{\text{ub}}$) imposes homogeneity of leverage amongst those banks that do proceed with investing in the risky asset. All banks that proceed will top-up their equity to the minimum level required ($\omega = 17.5\%$ in the optimal case) and fund the remainder through debt. These banks therefore share a common prudence threshold: $\hat{p}(\omega_{\text{reg}}) = 0.41$. In contrast, if $\omega_{\text{reg}}$ was lower than $\omega_{\text{ub}}$ (say $\omega_{\text{reg}} = 5\%$), then we would observe a cluster of banks with capital equal to the regulatory minimum, as well as some banks with a surplus of capital for regulatory purposes (having started with a better endowment). Since the prudence threshold $\hat{p}$ is a function of capital $\omega$, banks would no longer behave in a homogeneous fashion.

An alternative interpretation - the safe rate is set in global capital markets

We have assumed so far that the authorities (possibly through the central bank) can offer a safe asset with an offered real rate of return that is a choice variable – though it must accept the fiscal implications of that choice. Suppose instead that the safe
rate is a price set in a global capital market. We can still use our results to draw implications for the setting of bank capital requirements. We have shown that the lower is the safe real interest rate the worse is the problem of excessive risk taking by banks. An exogenous fall in the safe real rate should be countered by tight capital requirements.

Table 4.3 illustrates the scale of the effects. In recent decades global safe real rates – proxied by the average yield on inflation-proof debt issued by developed country governments – have fallen dramatically, from nearly 4% 30 years ago to close to 0% today. At a 4% safe real rate Table 4.3 shows the optimal capital requirement is 12.5%; at a 0% safe real rate the optimal capital requirement is 20%. Movements in real rates appear to have a sizeable effect on appropriate capital requirements.

4.7 Concluding Remarks

We find that prudential capital requirements and variations in the safe real interest rate (implemented as part of an interest rate policy) operate as imperfect substitutes when used to counter excessive risk taking by banks. A tightening of either instruments can improve 'prudence' (by disincentivising banks against investing in risky assets with low probability of success); but only at the cost of decreased 'participation' (as more banks will choose to forego the opportunity to acquire a risky asset even before they discover its probability of success through costly screening).

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Participation Threshold</th>
<th>Prudence Threshold</th>
<th>Avg. prob. of failure on risky investments made</th>
<th>Investment in risky assets</th>
<th>Value Added (%)</th>
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</thead>
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<tr>
<td>First-Best</td>
<td>0</td>
<td>0.56</td>
<td>0.22</td>
<td>0.44</td>
<td>7.89</td>
</tr>
<tr>
<td>Optimal Policy Combination</td>
<td>0.0230</td>
<td>0.41</td>
<td>0.30</td>
<td>0.46</td>
<td>7.50</td>
</tr>
<tr>
<td>No Policy Intervention</td>
<td>0.0037</td>
<td>Varies with ω, avg.=0.27</td>
<td>0.37</td>
<td>0.71</td>
<td>7.25</td>
</tr>
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</table>
Numerical simulations of our model help to illustrate, and to broadly quantify, the magnitude of this interaction between the safe real interest rate and capital regulation. We find that the trade-off between 'prudence' and 'participation' is such that the optimal level of prudential capital requirements rises as the safe real interest rate falls. In general the global optimum is one where interest rates should be slightly higher than a neutral setting and capital requirements should be substantial and binding. Investment in risky assets is much lower when policy is set optimally. This suggests how very weak is the argument that regulations on banks should not be so tight as to materially reduce bank lending. Our findings also suggest that the significant and simultaneous decline in global real interest rates on inflation proof debt of many governments in recent years should have prompted a rise in bank capital requirements.

There are several interesting ways one could extend the baseline analytical framework presented here. One of these is to introduce corporate taxation and explore whether using the tax rate can help offset a tendency to over-invest in risky assets. We present one version of this extension in the appendix and discuss preliminary results.

We assumed here that banking regulators have no more information on the riskiness of bank assets than outside investors. If they have some - but less than complete - information on risks \( p_i \) then risk based capital requirements become feasible. That could mean that optimal risk weighted bank capital requirements are on average lower than the leverage ratios we have calculated. It is not clear however that regulators having very imperfect information on bank risk \( p_i \) should mean that average leverage requirements would be very substantially lower. If they are not - so that capital requirements of 10% or more of bank assets are indeed optimal - then this suggests that current rules on bank leverage that allow capital to be as low as 3% or so of total assets may well be far too lax.
References


4.8 Appendices

A.1 Proof of Proposition 11- No excessive risk-taking under perfect information

Consider the case where \( \min [y_t, R(p, \omega + \tilde{\omega})] = R(p, \omega + \tilde{\omega}) \) (the debt obligation is less than the liquidation value), re-arranging equations 4.1 and 4.2 gives:

\[
R(p, \omega + \tilde{\omega} \mid y_t > R) = (1 - (\omega + \tilde{\omega}))(1 + r)
\]

\[
\hat{p}(\omega_i + \tilde{\omega}_i \mid y_t > R) = \frac{1 + r - y_t}{y_{th} - y_t}
\]

Consider the case where \( \min [y_t, R(p, \omega + \tilde{\omega})] = y_t \) (the debt obligation is greater than the liquidation value): So from equation 4.2 we have \( pR + (1 - p)y_t = (1 - (\omega + \tilde{\omega}))(1 + r) \), or

\[
R(p, \omega + \tilde{\omega} \mid y_t < R) = \frac{(1 - (\omega + \tilde{\omega}))(1 + r) - (1 - p)y_t}{p}
\]

Substitute \( R(\hat{p}, \omega + \tilde{\omega} \mid y_t < R) \) into equation 4.1 to give \( \hat{p}\left(y_{th} - \frac{(1-(\omega + \tilde{\omega}))(1+r)-(1-\hat{p})y_t}{p}\right) = (\omega + \tilde{\omega})(1 + r) \), which simplifies to

\[
\hat{p}(\omega + \tilde{\omega} \mid y_t < R) = \frac{1 + r - y_t}{y_{th} - y_t}
\]

Since it is easy to see that \( R \) is a decreasing function of \((\omega + \tilde{\omega})\), from the definition of \( \omega^* \) we can conclude that \( y_t < R \) whenever \( \omega + \tilde{\omega} < \omega^* \), and \( y_t > R \) whenever \( \omega + \tilde{\omega} > \omega^* \). So \( \hat{p}((\omega + \tilde{\omega}) > \omega^*) = \hat{p}((\omega + \tilde{\omega}) < \omega^*) = \hat{p}(\omega^*) = \frac{(1+r) - y_t}{y_{th} - y_t} \). We define this first-best level of prudence as \( \hat{p}_{fb} \equiv \frac{(1+r) - y_t}{y_{th} - y_t} \).

A.2 Proof of Proposition 12 - Weak preference for debt in the First-Best

1. Case A: For a firm with \( \omega \geq \omega^* \), and thus \( y_t > R(\omega) \), we show that the firm has at least a weak preference for external finance using debt, through proof by contradiction:
(a) Statement A: suppose that the firm with \( \omega \geq \omega^* \) strictly prefers using some amount of external equity as opposed to only using debt:

\[
p (y_h - R (\omega)) + (1 - p) (y_l - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - R (\omega + \tilde{\omega})) + (1 - p) (y_l - R (\omega + \tilde{\omega}))]
\]

for some \( \tilde{\omega} \in [0, 1 - \omega] \).

Then this implies:

\[
p (y_h - y_l) + (y_l - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - y_l) (y_l - R (\omega + \tilde{\omega}))]
\]

From proposition 11: \( R (\omega + \tilde{\omega}) = (1 - (\omega + \tilde{\omega})) (1 + r) \), so:

\[
p (y_h - y_l) + (y_l - (1 - \omega) (1 + r)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - y_l) + (y_l - (1 - (\omega + \tilde{\omega})) (1 + r))]
\]

\[
\tilde{\omega} [p (y_h - y_l) + y_l] < -\omega (1 - (\omega + \tilde{\omega})) (1 + r) + (\omega + \tilde{\omega}) (1 - \omega) (1 + r)
\]

\[
\tilde{\omega} [p (y_h - y_l) + y_l] < [(\omega + \tilde{\omega}) - \omega (\omega + \tilde{\omega}) - \omega + \omega (\omega + \tilde{\omega})] (1 + r)
\]

\[
p (y_h - y_l) + y_l < (1 + r)
\]

\[
p < \frac{(1+r) - y_l}{y_h - y_l} = \hat{p}_{fb}. \text{ Contradiction - since we know from the definition of the prudence threshold } \hat{p}_{fb}, \text{ that firms will only proceed with lending if } p \geq \hat{p}_{fb}.
\]

2. Case B: Consider firms with \( \omega < \omega^* \), and thus \( y_l < R (\omega) \). As before, we show weak preference for debt through proof by contradiction.

(a) Statement B1: firms strictly prefer to use some external equity when the equity injection means they are still not 'fully capitalized':

\[
p (y_h - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - R (\omega + \tilde{\omega}))] \text{ for some } \tilde{\omega} \in [0, \omega^* - \omega]
\]

(b) Statement B2: firms strictly prefer to use some external equity when the equity injection means they become 'fully capitalized':

\[
p (y_h - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - R (\omega + \tilde{\omega})) + (1 - p) (y_l - R (\omega + \tilde{\omega}))] \text{ for some } \tilde{\omega} \in [\omega^* - \omega, 1 - \omega]
\]

Consider statement B1, which simplifies to

\[
y_h - R (\omega) < \frac{\omega}{\omega + \tilde{\omega}} [(y_h - R (\omega + \tilde{\omega}))]
\]
From equation 4.1 and proposition 11, we have:
\[
\hat{p}_{fb} (y_h - R (\omega)) = \omega (1 + r) = \frac{\omega}{\omega + \tilde{\omega}} [\hat{p}_{fb} (y_h - R (\omega + \tilde{\omega})]]
\]

So \((y_h - R (\omega)) = \frac{\omega}{\omega + \tilde{\omega}} (y_h - R (\omega + \tilde{\omega}))\). Contradiction. In fact, the same proof shows that a 'not-fully-capitalised firm' would be indifferent between using all debt, or using a mixture of debt and additional equity (provided that the additional equity injection does not make the firm 'fully-capitalised').

Consider statement B2, which simplifies to:
\[
p (y_h - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - y_l) + y_l - R (\omega + \tilde{\omega})],
\]
where \(R (\omega < \omega^*) = \frac{(1-\omega)(1+r)-(1-p)\omega}{p}\), and
\[
R (\omega + \tilde{\omega} > \omega^*) = (1 - (\omega + \tilde{\omega})) (1 + r).
\]

From equation 4.1 and proposition 11, we have:
\[
\hat{p}_{fb} (y_h - R (\omega)) = \omega (1 + r) = \frac{\omega}{\omega + \tilde{\omega}} [\hat{p}_{fb} (y_h - R (\omega + \tilde{\omega})) + (1 - \hat{p}_{fb}) (y_l - R (\omega + \tilde{\omega}))],
\]
so when \(p = \hat{p}_{fb}\), the statement is false.

For the remainder of the proof we just need to show that the statement is also false for \(p > \hat{p}_{fb}\).

The derivative of the LHS of statement B2 w.r.t. \(p\) is:
\[
(y_h - R (\omega)) - p \frac{\partial R(\omega)}{\partial p} = (y_h - R (\omega)) - p \left(- \frac{R(\omega)}{p} + \frac{\omega}{p} \right) = (y_h - y_l)
\]

The derivative of the RHS of statement B2 w.r.t. \(p\) is \(\frac{\omega}{\omega + \tilde{\omega}} (y_h - y_l) < (y_h - y_l)\). The LHS increases faster than the RHS when \(p\) increases from \(\hat{p}_{fb}\).

Therefore \(p (y_h - R (\omega)) \geq \frac{\omega}{\omega + \tilde{\omega}} [p (y_h - y_l) + y_l - R (\omega + \tilde{\omega})]\) for any \(p \geq \hat{p}_{fb}\). Contradiction reached.
A.3 Proof of Proposition 13 - Prudence Threshold under Asymmetric Information

Proposition 13.1

The proof for \( \omega^* = 1 - \frac{y_l}{(1+r+\delta r)} \) follows directly from re-arranging equation 4.11, which also shows that banks with \( \omega_i + \tilde{\omega}_i = \omega^* \) can borrow at the risk-free rate \( 1 + r \). For banks with \( \omega_i + \tilde{\omega}_i > \omega^* \), their funding shortfall is smaller and thus their gross debt obligation is lower: \( R(\omega_i + \tilde{\omega}_i) < R(\omega^*) = y_l \) (since \( \frac{\partial R(\omega_i + \tilde{\omega}_i,r)}{\partial \omega} < 0 \), a result we will show in the proof to part 2 of the proposition below). The rate at which the debt is provided is still floored at \( 1 + r \), again from re-arranging equation 4.11.

Proposition 13.2

1. Consider case A where \( y_l \geq R \).

   Re-arrange the banks’ and the households’ indifference equations to give: \( R = (1 - (\omega + \tilde{\omega}))(1 + r) \) and \( \hat{p} = \frac{(1+r)-y_l}{(y_h - y_l)} \).

   Note that under this case: \( \frac{\partial R}{\partial \omega} < 0 \) so we can deduce that \( y_l > R \) holds whenever we have \( (\omega + \tilde{\omega}) > \omega^* \).

2. Consider case B where \( y_l \leq R \).

   Re-arranging the indifference equations gives:

   \( \hat{p}[y_h - R] = (\omega + \tilde{\omega})(1+r) \); and \( A(\hat{p}) R + (1 - A(\hat{p})) y_l = (1 - (\omega + \tilde{\omega}))(1 + r) \).

   Partially differentiate both w.r.t. \( \omega \), [or \( \tilde{\omega} \)], and solve the system of linear equations to give:

   \[
   \begin{bmatrix}
   \frac{\partial \hat{p}}{\partial \omega} \\
   \frac{\partial R}{\partial \omega}
   \end{bmatrix}
   = \begin{bmatrix}
   \frac{(A(\hat{p}) - \hat{p})(1+r)}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)} \\
   \frac{[A'(\hat{p})(R - y_l) + (y_h - R)(1+r)]}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)}
   \end{bmatrix}
   \]

   Since \( A'(\hat{p}) > 0 \) and \( A'(\hat{p}) > 0 \) unless \( \hat{p} = 1 \) (i.e. the degenerate case where no banks lend - which we will rule out by parameterization), and \( y_h \geq R \geq y_l \),
we can deduce that $\frac{\partial \hat{p}}{\partial \omega} > 0$ and $\frac{\partial R}{\partial \omega} < 0$. Also, $y_l < R$ whenever we have $(\omega + \tilde{\omega}) < \omega^*$. 

3. To summarize: whenever $(\omega + \tilde{\omega}) \geq \omega^*$, we have $\hat{p}(\omega + \tilde{\omega}) = \frac{(1+r)-y_l}{(y_h-y_l)}$, and whenever $(\omega + \tilde{\omega}) < \omega^*$, we have $\frac{\partial \hat{p}}{\partial \omega} > 0$. Therefore $\hat{p} \leq \frac{(1+r)-y_l}{(y_h-y_l)}$ for all $(\omega + \tilde{\omega})$.

A.4 Proof of Proposition 14 - Preference for Debt under Asymmetric Information.

Part 1 of the proposition follows directly from the definition of $\hat{p}(\omega + \tilde{\omega}, r)$ as the 'prudence threshold' for banks (see discussion in main body).

For part 2 of the proposition, we show that for any given level of initial endowment banks weakly prefer to fund the remainder of the project through debt rather than through any other funding arrangements:

- Case A: consider a bank with $\omega \geq \omega^*$, and thus $y_l \geq R$, we show that the bank has a weak preference to top-up the remainder using debt through proof by contradiction:

Statement A: suppose that the bank with $\omega \geq \omega^*$ strictly prefers using some amount of external equity as opposed to only using debt:

$$p (y_h - R(\omega)) + (1-p) (y_l - R(\omega)) < \frac{\omega}{\omega + \tilde{\omega}} (p (y_h - R(\omega + \tilde{\omega})) + (1-p) (y_l - R(\omega + \tilde{\omega})))$$

for some $0 < \tilde{\omega} \leq 1 - (\omega + \tilde{\omega})$.

Then this implies $y_l - R(\omega) + p (y_h - y_l) < \frac{\omega}{\omega + \tilde{\omega}} [y_l - R(\omega + \tilde{\omega}) + p (y_h - y_l)]$

Recall from Proposition 13, for $\omega \geq \omega^*$: $R(\omega) = (1 - \omega)(1 + r)$; $R(\omega + \tilde{\omega}) = (1 - (\omega + \tilde{\omega})) (1 + r)$

So $\left(1 - \frac{\omega}{\omega + \tilde{\omega}}\right) (y_l + p (y_h - y_l)) < R(\omega) - \frac{\omega}{\omega + \tilde{\omega}} R(\omega + \tilde{\omega})$

$$\frac{\omega}{\omega + \tilde{\omega}} (y_l + p (y_h - y_l)) < (1 - \omega)(1 + r) - \frac{\omega}{\omega + \tilde{\omega}} (1 - (\omega + \tilde{\omega})) (1 + r)$$
\[ \bar{\omega} (y_l + p (y_h - y_l)) < (\omega + \bar{\omega}) (1 - \omega) (1 + r) - \omega (1 - (\omega + \bar{\omega})) (1 + r) \]

\[ \bar{\omega} (y_l + p (y_h - y_l)) < [ (\omega + \bar{\omega}) (1 - \omega) - \omega (1 - (\omega + \bar{\omega})) ] (1 + r) \]

\[ \bar{\omega} (y_l + p (y_h - y_l)) < [ (\omega + \bar{\omega}) - \omega (\omega + \bar{\omega}) - \omega (\omega + \bar{\omega}) ] (1 + r) \]

\[ \bar{\omega} (y_l + p (y_h - y_l)) < \bar{\omega} (1 + r) \]

\[ y_l + p (y_h - y_l) < (1 + r) \]

\[ p < \frac{(1 + r) - y_l}{(y_h - y_l)} \]

But recall from Proposition 1 that for any bank with \( \omega \geq \omega^* \), \( \hat{p} = \frac{(1+r) - y_l}{(y_h - y_l)} \).

Therefore, given banks will proceed with the project if and only if \( p > \hat{p} = \frac{(1+r) - y_l}{(y_h - y_l)} \), statement A cannot be true, and the bank will weakly prefer to use debt for any \( \bar{\omega} \in [0, 1 - (\omega + \bar{\omega})] \). [For simplicity, we assume that when banks are indifferent between debt and equity, they will proceed with the project using debt finance].

- Case B: consider a bank with \( \omega < \omega^* \), and thus \( y_l < R(\omega) \), we show a weak preference for debt through proof by contradiction:

1. Statement B1: the bank strictly prefers to use some external equity:

\[ p (y_h - R(\omega)) < \frac{\omega}{\omega + \bar{\omega}} p (y_h - R(\omega + \bar{\omega})) \text{ for some } \bar{\omega} \in [0, \omega^* - \omega); \text{ and} \]

2. Statement B2: the bank strictly prefers to use some external equity:

\[ p (y_h - R(\omega)) < \frac{\omega}{\omega + \bar{\omega}_2} [ p (y_h - R(\omega + \bar{\omega}_2)) + (1 - p) (y_l - R(\omega + \bar{\omega}_2))] \text{ for some } \bar{\omega}_2 \in [\omega^* - \omega, 1 - \omega] \]

Consider Statement B1, which simplifies to \( y_h - R(\omega) < \frac{\omega}{\omega + \bar{\omega}} (y_h - R(\omega + \bar{\omega})) \).

Let \( w = \omega + \bar{\omega} \) denote total capital. Recall from the bank’s indifference condition \( \hat{p}(w) [y_h - R(w)] = w(1 + r) \); so \( y_h - R(w) = \frac{w(1+r)}{\hat{p}(w)} \)

So Statement B1 becomes \( \frac{\omega (1+r)}{\hat{p}(w)} < \frac{\omega}{\omega + \bar{\omega}} \frac{(\omega + \bar{\omega})(1+r)}{\hat{p}(\omega + \bar{\omega})} \); or \( \hat{p}(\omega + \bar{\omega}) < \hat{p}(\omega) \) when \( \omega + \bar{\omega} < \omega^* \).
i.e. a better capitalized bank is less prudent. Contradiction [see Proposition 13].

Consider Statement B2, which simplifies to:

\[ p (y_h - R (\omega)) < \frac{\omega}{\omega + \tilde{\omega}_2} [(y_l - R (\omega + \tilde{\omega}_2)) + p (y_h - y_l)] \]

for some \( \tilde{\omega}_2 \in [\omega^* - \omega, 1 - \omega] \), where \( R (\omega + \tilde{\omega}_2) = (\omega + \tilde{\omega}_2) (1 + r) \)

But from bank’s indifference condition we know:

\[ \hat{p} (\omega) [y_h - R (\omega)] = \omega (1 + r); \text{ and } \frac{\omega}{\omega + \tilde{\omega}_2} ((y_l - R (\omega + \tilde{\omega}_2)) + \hat{p} (\omega + \tilde{\omega}_2) (y_h - y_l)) = \omega (1 + r) \]

So \( \hat{p} (\omega) [y_h - R (\omega)] = \frac{\omega}{\omega + \tilde{\omega}_2} ((y_l - R (\omega + \tilde{\omega}_2)) + \hat{p} (\omega + \tilde{\omega}_2) (y_h - y_l)) \)

Since \( \omega < \omega^* \leq \omega + \tilde{\omega}_2 \) by assumption, we know from proposition 13 that \( \hat{p} (\omega) < \hat{p} (\omega + \tilde{\omega}_2) \)

Therefore we have \( p (y_h - R (\omega)) \geq \frac{\omega}{\omega + \tilde{\omega}_2} [(y_l - R (\omega + \tilde{\omega}_2)) + p (y_h - y_l)] \) for any \( p \in [\hat{p} (\omega), \hat{p} (\omega + \tilde{\omega}_2)] \), Contradiction.

For the last part of the proof by contradiction, just need to show that as \( p \)

increases, the return from debt financed investments increases faster than that from mixed financed investments:

\[ \text{i.e. } (y_h - R (\omega)) \geq \frac{\omega}{\omega + \tilde{\omega}_2} (y_h - y_l) \cdot \left[ \frac{\partial \hat{p}}{\partial \omega} \frac{\partial R}{\partial \omega} \right] = \left[ \frac{(A(\hat{p}) - \hat{p})(1+r)}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)} - \frac{[A'(\hat{p})(R - y_l) + (y_h - R)](1+r)}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)} \right] \]

Recall again \( y_h - R (\omega) = \frac{\omega (1+r)}{\hat{p}(\omega)} \), and \( \hat{p} (\omega) < \frac{(1+r) - y_l}{(y_h - y_l)} \), we have \( y_h - R (\omega) > \omega (1 + r) \frac{(y_h - y_l)}{(1+r) - y_l} = \omega (y_h - y_l) \frac{(1+r)}{(1+r) - y_l} \)

so to complete the contradiction to statement B2, just need to show \( \omega (y_h - y_l) \frac{(1+r)}{(1+r) - y_l} \geq \frac{\omega}{\omega + \tilde{\omega}_2} (y_h - y_l) \); or \( \frac{(1+r)}{(1+r) - y_l} \geq \frac{1}{\omega + \tilde{\omega}_2} \)

Recall \( \omega^* = 1 - \frac{y_l}{(1+r)} = \frac{(1+r) - y_l}{(1+r)} \), so \( \frac{1}{\omega^*} = \frac{(1+r)}{(1+r) - y_l} \). And given \( \omega + \tilde{\omega}_2 \geq \omega^* \), we have \( \frac{1}{\omega^*} = \frac{(1+r)}{(1+r) - y_l} \geq \frac{1}{\omega + \tilde{\omega}_2} \). QED.
A.5 Value of Screening (equation 4.5)

We present the proof for the last line of equation 4.5:

\[
V(\omega, \tilde{\omega}, r)
\equiv \mathbb{E}_p \left\{ \max \left\{ 0, \frac{\omega}{\omega + \tilde{\omega}} \left( \frac{p(y_h - R)}{(1 - p) \max [0, y_l - R]} \right) - \omega(1 + r) \right\} \right\}
\]

\[
= \frac{\omega}{\omega + \tilde{\omega}} \int_0^1 \left[ p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp
\]

The last equality follows because household provide debt at the risk-free rate:

1. For fully capitalized banks with \( \omega + \tilde{\omega} \geq \omega^* = 1 - \frac{y}{(1 + r)} \):

\[
V(\omega, \tilde{\omega}, r) = \mathbb{E}_p \left[ \max \left\{ 0, \frac{\omega}{\omega + \tilde{\omega}} \left\{ \frac{p(y_h - R)}{(1 - p) (y_l - R)} \right\} - \omega(1 + r) \right\} \right]
\]

So from equation 4.10 we have:

\[
V(\omega, \tilde{\omega}, r) = \int_0^1 \left[ \frac{\omega}{\omega + \tilde{\omega}} \left\{ \frac{p(y_h - R)}{(1 - p) (y_l - R)} \right\} - \omega(1 + r) \right] h(p) dp
\]

Which can be simplified to:

\[
V(\omega, \tilde{\omega}, r) = \frac{\omega}{\omega + \tilde{\omega}} \int_0^1 \left[ p(y_h - y_l) + y_l - R(\omega + \tilde{\omega}) - (\omega + \tilde{\omega})(1 + r) \right] h(p) dp
\]

But when \( \omega + \tilde{\omega} \geq \omega \), we know from proposition 13 that \( R(\omega + \tilde{\omega}) = (1 - (\omega + \tilde{\omega}))(1 + r) \)

So \( V(\omega, \tilde{\omega}, r) = \frac{\omega}{\omega + \tilde{\omega}} \int_0^1 \left[ p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp \).

2. For banks with \( \omega + \tilde{\omega} < \omega^* \):
\[ V(\omega, \tilde{\omega}, r) = E_p \left[ \max \left( 0, \frac{\omega}{\omega + \tilde{\omega}} p (y_h - R(\omega + \tilde{\omega})) - \omega(1 + r) \right) \right] \]

Which can be simplified again to:

\[ V(\omega, \tilde{\omega}, r) = \frac{\omega}{\omega + \tilde{\omega}} \int \left[ p(y_h - R(\omega + \tilde{\omega}, r)) - (\omega + \tilde{\omega})(1 + r) \right] h(p) dp \]

Recall from equation 4.11 \( A(\hat{p}) R(\omega + \tilde{\omega}) + (1 - A(\hat{p})) y_l = (1 - \omega - \tilde{\omega})(1 + r) \), so \( A(\hat{p}) (R(\omega + \tilde{\omega}, r) - y_l) + y_l = (1 - \omega - \tilde{\omega})(1 + r) \).

Substituting in the definition of \( A(.) \) as the conditional expectation:

\[ \left[ \frac{1}{1-H(\hat{p})} \int \left( p(y_h - y_l) + y_l - (1 + r) \right) h(p) dp \right] (R(\omega + \tilde{\omega}) - y_l) + y_l = (1 - (\omega + \tilde{\omega})) (1 + r) \]

Re-arrange to give:

\[ \int p(R(\omega + \tilde{\omega}) - y_l) h(p) dp = [(1 - (\omega + \tilde{\omega})) (1 + r) - y_l] \left[ \int \left( p(y_h - y_l) + y_l - (1 + r) \right) h(p) dp \right] \]

... because \( R(.) \) is not a function of \( p \), and \( \omega \) and \( p \) are independently distributed.

So \( \int p R(\omega + \tilde{\omega}) h(p) dp = \int [p(y_h - y_l) + (1 - \omega - \tilde{\omega})(1 + r)] h(p) dp \)

And \( V(\omega, \tilde{\omega}, r) = \frac{\omega}{\omega + \tilde{\omega}} \int p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp \) Q.E.D.

3. Note that when \( \tilde{\omega} = 0 \) (i.e. in the absence of capital regulation):

\[ V(\omega, 0, r) = \int [p(y_h - y_l) + y_l - (1 + r)] h(p) dp. \]

Note further that when \( \tilde{\omega} = \omega_{\text{reg}} - \omega \) (i.e. when a bank tops-up to the regulatory minimum - see proposition 17):

\[ V(\omega, \tilde{\omega} = \omega_{\text{reg}} - \omega, r) = \frac{\omega}{\omega_{\text{reg}}} \int p(y_h - y_l) + y_l - (1 + r) \right] h(p) dp = \frac{\omega}{\omega_{\text{reg}}} V(\omega = \omega_{\text{reg}}, \tilde{\omega} = 0, r) \]

So the value of screening for a bank which falls short of the regulatory capital requirement \( (\omega_i < \omega_{\text{reg}}) \) is a \( \frac{\omega_i}{\omega_{\text{reg}}} \) fraction of the value of screening for a bank that is just meeting the regulatory requirement:

\[ V(\omega, \tilde{\omega} = \omega_{\text{reg}} - \omega) = \frac{\omega}{\omega_{\text{reg}}} V(\omega = \omega_{\text{reg}}, \tilde{\omega} = 0) \quad (4.22) \]
and the participation threshold for a capital constrained bank is given by:

$$\hat{\omega} (\bar{\omega} = \omega_{\text{reg}} - \omega) = \frac{\omega_{\text{reg}} C}{V(\omega = \omega_{\text{reg}}, \bar{\omega} = 0)}$$ (4.23)

(see definition 4.6).

A.6 Proof of Proposition 15 - Participation Threshold

Recall: \( V(\omega; \bar{\omega}, r) = \frac{\omega}{\omega + \bar{\omega}} \int_{\hat{p}}^{1} [p (y_{h} - y_{l}) + y_{l} - (1 + r)] h(p) dp \)

Using the Product Rule and the Leibniz Integral Rule\(^{35}\) we get:

$$\frac{\partial V(\omega; \bar{\omega})}{\partial \omega} = \frac{\bar{\omega}}{(\omega + \bar{\omega})^2} \int_{\hat{p}}^{1} [p (y_{h} - y_{l}) + y_{l} - (1 + r)] h(p) dp - \frac{\omega}{\omega + \bar{\omega}} [\hat{p} (y_{h} - y_{l}) + y_{l} - (1 + r)] h(\hat{p}) \frac{\partial \hat{p}}{\partial \omega}$$

or

$$\frac{\partial V(\omega; \bar{\omega})}{\partial \omega} = \frac{1}{(\omega + \bar{\omega})} \omega V - \frac{\omega}{\omega + \bar{\omega}} [\hat{p} (y_{h} - y_{l}) + y_{l} - (1 + r)] h(\hat{p}) \frac{\partial \hat{p}}{\partial \omega}.$$

Recall \( \bar{\omega} = \max [0, \omega_{\text{reg}} - \omega] \) (see proposition 17):

- So when \( \bar{\omega} = 0 \), \( \frac{\partial V(\omega; \bar{\omega} = 0, r)}{\partial \omega} = -[\hat{p} (y_{h} - y_{l}) + y_{l} - (1 + r)] h(\hat{p}) \frac{\partial \hat{p}}{\partial \omega} \geq 0 $\)

where the inequality holds because \( \frac{\partial \hat{p}}{\partial \omega} \geq 0 \) with equality when \( (\omega + \bar{\omega}) \geq \omega^{*} \); and \( \hat{p} (y_{h} - y_{l}) + y_{l} - (1 + r) \leq 0 \) with equality when \( (\omega + \bar{\omega}) \geq \omega^{*} \).

- When \( \bar{\omega} = \omega_{\text{reg}} - \omega \), equation 4.22 shows \( V(\omega; \bar{\omega} = \omega_{\text{reg}} - \omega) = \frac{\omega}{\omega_{\text{reg}}} V(\omega = \omega_{\text{reg}}, \bar{\omega} = 0) \), so \( \frac{\partial V(\omega; \bar{\omega} = \omega_{\text{reg}} - \omega)}{\partial \omega} = \frac{1}{\omega_{\text{reg}}} V(\omega = \omega_{\text{reg}}, \bar{\omega} = 0) \geq 0. \)

A.7 Proof of Proposition 16 - Transmission of Interest Rate Policy

Proposition 16.1: \( \frac{\partial \hat{p}}{\partial r} > 0 \)

1. For banks with \( \omega + \bar{\omega} \geq \omega^{*} \), recall that \( \hat{p} (\omega + \bar{\omega}) = \frac{(1+r) - y_{l}}{y_{h} - y_{l}} \), so \( \frac{\partial \hat{p}}{\partial r} > 0. \)

\(^{35}\)Leibniz Integral Rule:

$$\frac{d}{dx} \int_{y_{0}}^{y_{1}} f(x, y) dy = \int_{y_{0}}^{y_{1}} f_{x}(x, y) dy + \frac{dy_{y}}{dy} \frac{dy}{dx} - f(x, y_{0}) \frac{dy_{y}}{dx} \text{ provided } f \text{ and } f_{x} \text{ are both continuous over a region in the form } [x_{0}, x_{1}] \times [y_{0}, y_{1}]$$
2. For banks with $\omega + \bar{\omega} < \omega^*$ (and thus $y_l < R(\omega + \bar{\omega})$) we have from equation 4.10 and equation 4.11:

$$\hat{p} (y_h - R(\omega + \bar{\omega})) = (\omega + \bar{\omega}) (1 + r); \text{ and}$$

$$A (\hat{p}) R (\omega + \bar{\omega}) + (1 - A (\hat{p})) y_l = (1 - \omega - \bar{\omega})(1 + r).$$

Partially differentiate both equations with respect to $r$, and solving the resulting system of linear equations to give:

$$\frac{\partial \hat{p}(\omega + \bar{\omega}, r)}{\partial r} = A (\hat{p})(\omega + \bar{\omega}) + \hat{p}(1 - \omega - \bar{\omega}) (y_h - R)A (\hat{p}) + \hat{p}A'(\hat{p})(R - y_l) > 0. \text{ Q.E.D.}$$

**Proposition 16.2:** $\frac{\partial \hat{\omega}}{\partial r} > 0$

Recall that $\hat{\omega}(\bar{\omega}, r)$ is implicitly defined by:

$$V(\hat{\omega}, \bar{\omega}, r) = \frac{\hat{\omega}}{\omega + \bar{\omega}} \int_{\hat{p}}^{1} [p(y_h - y_l) + y_l - (1 + r)] h(p) dp = C$$

We know also from proposition 15 that $\frac{\partial V(\hat{\omega}, \bar{\omega}, r)}{\partial \omega} \begin{cases} > 0 \text{ for } \omega + \bar{\omega} < \omega^* \\ = 0 \text{ for } \omega + \bar{\omega} \geq \omega^* \end{cases}$. So to show $\frac{\partial \hat{\omega}}{\partial r} > 0$ for $\hat{\omega} < \omega^*$, it would be sufficient to show that $\frac{\partial V(\hat{\omega}, \bar{\omega}, r)}{\partial r} < 0$ for $\omega + \bar{\omega} < \omega^*$ (by the implicit function theorem).

Using the Leibniz Integral Rule we get:

$$\frac{\partial V(\omega, \bar{\omega}, r)}{\partial r} = - \frac{\omega}{\omega + \bar{\omega}} [1 - H(\hat{p})] + \frac{\omega}{\omega + \bar{\omega}} [(1 + r) - y_l - \hat{p}(y_h - y_l)] h(\hat{p}) \frac{\partial \hat{p}}{\partial r}$$

So $\frac{\partial V}{\partial r} < 0$ for $\omega + \bar{\omega} < \omega^*$ if $\frac{\partial \hat{p}}{\partial r} (\omega + \bar{\omega} < \omega^*) < \frac{1 - H(\hat{p})}{(1+r) - y_l - \hat{p}(y_h - y_l)} h(\hat{p})$.

For $\omega + \bar{\omega} < \omega^*$, recall from the first part of this proposition that:

$$\frac{\partial \hat{p}}{\partial r} (\omega + \bar{\omega} < \omega^*) = \frac{A(\hat{p})(\omega + \bar{\omega}) + \hat{p}(1 - \omega - \bar{\omega})}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)}$$

\[36 \frac{\partial \hat{\omega}}{\partial r} \text{ for } \omega + \bar{\omega} \geq \omega^* \text{ is not well defined.} \]

\[37 \text{We are looking at cases where } \omega + \bar{\omega} < \omega^*, \text{ so } [(1 + r + \delta_r) - y_l - \hat{p}(y_h - y_l)] > 0. \]
Substituting this into the inequality above means we need to show:
\[
\frac{A(\hat{p})(\omega + \hat{\omega}) + \hat{p}(1 - \omega - \hat{\omega})}{(y_h - R)A(\hat{p}) + \hat{p}A'(\hat{p})(R - y_l)} < \frac{1 - H(\hat{p})}{[(1 + r) - y_l - p(y_h - y_l)]h(p)}
\]

Re-arranging equation 4.10 and 4.11 (when \(\omega + \hat{\omega} < \omega^*\)) gives:
\[
y_h - R = \left(\frac{\omega + \hat{\omega}}{p}\right)(1 + r); \quad R = \frac{(1 - (\omega + \hat{\omega}))(1 + r) - y_l}{A(\hat{p})} + y_l; \quad \text{and} \quad (y_h - y_l) = \left(\frac{\omega + \hat{\omega}}{p}\right)(1 + r) + \frac{(1 - (\omega + \hat{\omega}))(1 + r) - y_l}{A(\hat{p})}.
\]

It can also be shown generically that \(\frac{1 - H(\hat{p})}{h(\hat{p})} = \frac{A(\hat{p}) - \hat{p}}{A'(\hat{p})}\) (to be shown in Annex # below).

So substituting these results into the inequality above we get:
\[
\frac{A(\hat{p})(\omega + \hat{\omega}) + \hat{p}(1 - \omega - \hat{\omega})}{(\frac{\omega + \hat{\omega}}{p})(1 + r)A(\hat{p}) + \hat{p}A'(\hat{p})} \times \frac{(1 - (\omega + \hat{\omega}))(1 + r) - y_l}{A(\hat{p})} < \frac{1}{[(1 + r) - y_l - p(y_h - y_l)]A'(\hat{p})}
\]

Re-arrange and simplify the RHS to give:
\[
\frac{A(\hat{p})(\omega + \hat{\omega}) + \hat{p}(1 - \omega - \hat{\omega})}{(\omega + \hat{\omega})(1 + r)A(\hat{p}) + \hat{p}A'(\hat{p})} < \frac{1}{[(1 + r) - y_l - p(y_h - y_l)]A'(\hat{p})}
\]

So:
\[
[(A(\hat{p})(\omega + \hat{\omega}) + \hat{p}(1 - (\omega + \hat{\omega}))][(1 - (\omega + \hat{\omega})](1 + r) - y_l] < \left\{ \frac{(\omega + \hat{\omega})(1 + r)}{p} A(\hat{p}) \frac{A(\hat{p})}{A'(\hat{p})} + \hat{p} \left(1 - (\omega + \hat{\omega})\right)(1 + r) - y_l \right\}
\]

... which can be further simplified to:
\[
A(\hat{p})(1 - (\omega + \hat{\omega})) (1 + r) - A(\hat{p}) y_l - \hat{p} \left[(1 - (\omega + \hat{\omega})) (1 + r) - y_l\right] < \frac{A(\hat{p})}{p} A'(\hat{p}) (1 + r).
\]

Recall from equation 4.11: \((1 - (\omega + \hat{\omega})) (1 + r) - y_l = A(\hat{p})(R - y_l) > 0\), so for the above inequality to hold it is sufficient to show that:
\[
A(\hat{p})(1 - (\omega + \hat{\omega})) (1 + r) < \frac{A(\hat{p})}{p} A'(\hat{p}) (1 + r); \quad \text{or} \quad (1 - (\omega + \hat{\omega})) < \frac{1}{p} A'(\hat{p}).
\]

For \(p \sim U[0, 1]\), \(\frac{A(\hat{p})}{p} A'(\hat{p}) = \frac{(\hat{p} + 1)}{p} > 1 > (1 - (\omega + \hat{\omega}))\). Therefore, \(\frac{\partial V}{\partial r} < 0\) for \(\omega + \hat{\omega} < \omega^*\), and \(\frac{\partial \omega}{\partial r} > 0\) for \(\hat{\omega} < \omega^*\). Q.E.D.
A.8 Relationship between conditional expectation and probability density function

**Proposition 18.** \( \frac{1-H(\hat{p})}{h(\hat{p})} = \frac{A(\hat{p})-\hat{p}}{A'(\hat{p})} \), where \( A(\hat{p}) \equiv \frac{1}{1-H(\hat{p})} \int_{\hat{p}}^{p_{ub}} ph(p) \, dp \) and \( p_{ub} \) is the upper bound of the \( p \) distribution with pdf \( h(.) \) and cdf \( H(.) \).

**Proof.** Let \( g(p) = \int ph(p) \, dp \), then \( A(\hat{p}) = \frac{1}{1-H(\hat{p})} [g(p_{ub}) - g(\hat{p})] \), and

\[
A'(\hat{p}) = \frac{h(\hat{p})}{[1-H(\hat{p})]^2} [g(p_{ub}) - g(\hat{p})] + \frac{1}{1-H(\hat{p})} g'(\hat{p}) = A(\hat{p}) \frac{h(\hat{p})}{1-H(\hat{p})} + \frac{1}{1-H(\hat{p})} \hat{p}h(\hat{p})
\]

So \( A'(\hat{p}) = (A(\hat{p}) - \hat{p}) \frac{h(\hat{p})}{1-H(\hat{p})} \), and \( \frac{1-H(\hat{p})}{h(\hat{p})} = \frac{A(\hat{p})-\hat{p}}{A'(\hat{p})} \) \( \square \)

A.9 Proof of Proposition 17

Part 1 and 2 of the proposition are proved in the main text (these are corollaries of previous propositions). We start with the proof for part 4 of the proposition here, and use the result \( \frac{\partial \omega(\omega_{reg}-\omega, r)}{\partial \omega_{reg}} > 0 \) to prove part 3 of the proposition.

**Proof of Proposition 17.4:** \( \frac{\partial \omega(\omega_{reg}-\omega, r)}{\partial \omega_{reg}} \geq 0 \)

Given equation 4.23 in Annex 4.8, we have \( \omega(\omega = \omega_{reg} - \omega) = \omega_{reg}C \), so

\[
\frac{\partial \omega(\omega = \omega_{reg} - \omega, r)}{\partial \omega_{reg}} = \frac{C}{V(\omega = \omega_{reg}, \omega = 0)} \frac{\omega_{reg}C}{[V(\omega = \omega_{reg}, \omega = 0)]^2} \frac{\partial V(\omega = \omega_{reg}, \omega = 0)}{\partial \omega_{reg}} = C \left\{ \frac{V(\omega = \omega_{reg}, \omega = 0) - \omega_{reg}}{V(\omega = \omega_{reg}, \omega = 0)} \right\} \]

implying that for non-zero \( C \) and \( V(\omega = \omega_{reg}, \omega = 0) \), \( \frac{\partial \omega(\omega = \omega_{reg} - \omega, r)}{\partial \omega_{reg}} \geq 0 \) iff

\( V(\omega = \omega_{reg}, \omega = 0) - \omega_{reg} \frac{\partial V(\omega = \omega_{reg}, \omega = 0)}{\partial \omega_{reg}} \geq 0 \).

Recall that \( V(\omega, \bar{\omega}) \geq 0 \) by definition (\( V \) is the value of an option in risky bank lending), so when \( \omega_{reg} = 0 \): \( V(\omega = \omega_{reg}, \bar{\omega} = 0) = \omega_{reg} \frac{\partial V(\omega = \omega_{reg}, \bar{\omega} = 0)}{\partial \omega_{reg}} = V(0, 0) \geq 0 \).

All that remains is to show \( V(\omega = \omega_{reg}, \bar{\omega} = 0) - \omega_{reg} \frac{\partial V(\omega = \omega_{reg}, \bar{\omega} = 0)}{\partial \omega_{reg}} \) is non-decreasing for \( \omega_{reg} \in [0, 1] \).

Differentiating with respect to \( \omega_{reg} \) gives:
\[
\frac{\partial}{\partial \omega_{\text{reg}}} \left\{ V(\omega = \omega_{\text{reg}}, \tilde{\omega} = 0) - \omega_{\text{reg}} \frac{\partial V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}} \right\} = \left\{ \frac{\partial V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}} + \ldots \right. \\
\left. - \omega_{\text{reg}} \frac{\partial^2 V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}^2} \right\} = -\omega_{\text{reg}} \frac{\partial^2 V(\omega = \omega_{\text{reg}}, \tilde{\omega} = 0)}{\partial \omega_{\text{reg}}^2}
\]

So we need to show that \(\frac{\partial^2 V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}^2} \leq 0\).

Recall from the proof to proposition 15 that:

\[
\frac{\partial V(\omega, \tilde{\omega})}{\partial \omega} = -\left[ \tilde{p}(y_h - y_l) + y_l - (1 + r) \right] h(\tilde{p}) \frac{\partial \tilde{p}}{\partial \omega}
\]

So we have:

\[
\frac{\partial^2 V(\omega = \omega_{\text{reg}}, \tilde{\omega} = 0)}{\partial \omega_{\text{reg}}^2} = \frac{\partial^2 V(\omega, 0)}{\partial \omega^2} (\omega = \omega_{\text{reg}})
\]

\[
= - \left\{ \frac{\partial \tilde{p}(\omega_{\text{reg}})}{\partial \omega} (y_h - y_l) \right\} h(\tilde{p}(\omega_{\text{reg}})) \frac{\partial \tilde{p}(\omega_{\text{reg}})}{\partial \omega} + \\
\left\{ h'(\tilde{p}(\omega_{\text{reg}})) \tilde{p}(\omega_{\text{reg}}) (y_h - y_l) + y_l - (1 + r) \right\} \frac{\partial \tilde{p}(\omega_{\text{reg}})}{\partial \omega} + \\
\left\{ \tilde{p}(\omega_{\text{reg}}) (y_h - y_l) + y_l - (1 + r) \right\} h(\tilde{p}(\omega_{\text{reg}})) \frac{\partial^2 \tilde{p}(\omega_{\text{reg}})}{\partial \omega^2}
\]

For uniformly distributed \(p\), \(h'(\cdot) = 0\). So \(\frac{\partial^2 \tilde{p}(\omega_{\text{reg}})}{\partial \omega^2} \leq 0\) is sufficient for \(\frac{\partial^2 V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}^2} \leq 0\). This in turn implies that \(\frac{\partial V(\omega=\omega_{\text{reg}},\tilde{\omega}=0)}{\partial \omega_{\text{reg}}} \geq 0\) and finally \(\frac{\partial \omega_{\text{reg}}}{\partial \omega_{\text{reg}}} \geq 0\).

\(^{38}\)Recall for \(\omega + \tilde{\omega} < \omega^*\): \(\frac{\partial \tilde{p}}{\partial \omega} = \frac{(A(p) - \tilde{p})(1+r)}{(p_{\text{ub}} - p_{\text{lb}})} > 0\). For uniformly distributed \(p\), we have \(h(p) = \frac{1}{p_{\text{ub}} - p_{\text{lb}}} \), \(h'(p) = 0\) and \(A(\tilde{p}) = \frac{1}{2} (\tilde{p} + p_{\text{ub}})\).

So \(\frac{\partial \tilde{p}}{\partial \omega} = \frac{(\frac{1}{2} (\tilde{p} + p_{\text{ub}}) - p)(1+r)}{(p_{\text{ub}} - p_{\text{lb}})} = \frac{(p_{\text{ub}} - \tilde{p})(1+r)}{(p_{\text{ub}} - p_{\text{lb}})} > 0\).

And \(\frac{\partial^2 \tilde{p}}{\partial \omega^2} = \frac{\partial \tilde{p}}{\partial \omega} \leq 0\) since \(\frac{\partial \tilde{p}}{\partial \omega} > 0\) and \(\frac{\partial R}{\partial \omega} \leq 0\).
Proof of Proposition 17.3: \( \hat{\omega}(\hat{\omega} = \omega_{reg} - \omega, r) > \hat{\omega}(\hat{\omega} = 0, r) \) for all \( \omega_{reg} > \hat{\omega}(0, r) \)

From the (implicit) definition of \( \hat{\omega} \), we have:

\[
V(\hat{\omega}(0, r), \hat{\omega} = 0) = C; \quad \text{and} \quad V(\hat{\omega}(\omega_{reg} - \omega, r), \hat{\omega} = \omega_{reg} - \omega) = C.
\]

We also know from equation 4.22: \( V(\omega, \hat{\omega} = \omega_{reg} - \omega) = \frac{\omega}{\omega_{reg}} V(\omega = \omega_{reg}, \hat{\omega} = 0) \) [i.e. the value of screening for a bank which falls short of the regulatory capital requirement \( (\omega_i < \omega_{reg}) \) is a \( \frac{\omega_i}{\omega_{reg}} \) fraction of the value of screening for a bank that is just meeting the regulatory requirement].

Taken together, the above equations imply

\[
V(\hat{\omega}(\omega_{reg} - \omega, r), \hat{\omega} = \omega_{reg} - \omega) = \frac{\hat{\omega}(\omega_{reg} - \omega, r)}{\omega_{reg}} V(\omega = \omega_{reg}, \hat{\omega} = 0) = C
\]

and therefore \( \hat{\omega}(\omega_{reg} - \omega, r) = \omega_{reg} \) iff \( \omega_{reg} = \hat{\omega}(0, r) \). In other words, when the minimum capital requirement is set at the level of participation threshold which prevails in the absence of the requirement, then the capital requirement has no impact on participation. We can define a 'binding regulatory regime' as one where \( \omega_{reg} > \min[\hat{\omega}(\omega_{reg} - \omega, r), \hat{\omega}(0, r)] \), because if the capital requirement is less than or equal to the lower of the two participation thresholds then no banks would be bound by it should their decide to engage in lending.

The rest of the proof is shown by contradiction:

Suppose \( \hat{\omega}(\hat{\omega} = \omega_{reg} - \omega, r) \leq \hat{\omega}(\hat{\omega} = 0, r) \) and the regulatory regime is binding \( \omega_{reg} > \min[\hat{\omega}(\omega_{reg} - \omega, r), \hat{\omega}(0, r)] \). We can then examine the participation thresholds for banks with \( \omega_i < \omega_{reg} \) (banks for which the regulatory regime is binding), and show that a contradiction must arise:

1. First consider regulatory regime (i): \( \hat{\omega}(\hat{\omega} = 0, r) \geq \omega_{reg} > \hat{\omega}(\hat{\omega} = \omega_{reg} - \omega, r) \).
Under this regime $\omega_{reg} > \omega (\tilde{\omega} = \omega_{reg} - \omega, r)$. So for the condition shown at the start of the proof

\[
V (\hat{\omega} (0, r) , \tilde{\omega} = 0) = V (\hat{\omega} (\omega_{reg} - \omega, r) , \tilde{\omega} = \omega_{reg} - \omega)
\]

\[
= \frac{\hat{\omega} (\omega_{reg} - \omega, r)}{\omega_{reg}} V (\omega = \omega_{reg}, \tilde{\omega} = 0)
\]

to hold we need $V (\omega = \omega_{reg}, \tilde{\omega} = 0) > V (\hat{\omega} (0, r) , \tilde{\omega} = 0)$, which given $\frac{\partial V (\omega, \tilde{\omega})}{\partial \omega} \geq 0$ (see proof to proposition 15) means $\omega_{reg} > \hat{\omega} (0, r)$. Contradiction.

2. Now consider regulatory regime (ii): $\omega_{reg} > \hat{\omega} (\tilde{\omega} = 0, r) \geq \hat{\omega} (\tilde{\omega} = \omega_{reg} - \omega, r)$

We have already shown that $\hat{\omega} (\omega_{reg} - \omega, r) = \omega_{reg}$ iff $\omega_{reg} = \hat{\omega} (0, r)$; which implies $\hat{\omega} (\omega_{reg} - \omega, r) = \hat{\omega} (0, r)$ when $\omega_{reg} = \hat{\omega} (0, r)$. So given $\frac{\partial V (\omega_{reg}, \omega, r)}{\partial \omega_{reg}} > 0$ (Proposition 17.4), when $\omega_{reg} > \hat{\omega} (0, r)$, $\hat{\omega} (\omega_{reg} - \omega, r) > \hat{\omega} (0, r)$. Contradiction.

Therefore $\hat{\omega} (\tilde{\omega} = \omega_{reg} - \omega, r) > \hat{\omega} (\tilde{\omega} = 0, r)$ for all $\omega_{reg} > \hat{\omega} (0, r)$. Q.E.D.

A.10 The Model with Corporate Taxation

Suppose the government imposes a proportional tax on corporate profits $t$.

The key equations for firms (and banks when there is no asymmetric information) become:

\[
\hat{p}_{firm} (y_h - R) + (1 - \hat{p}_{firm}) \max [0, y_l - R] = (\omega + \tilde{\omega}) (1 + r) + t\hat{p}_{firm} (y_h - R - (\omega + \tilde{\omega}))
\]

(4.24)

\[
pR + (1 - p) \min [y_l, R] = (1 - \omega - \tilde{\omega}) (1 + r)
\]

(4.25)

\[
p (y_h - R) + (1 - p) \max [0, y_l - R] \geq (\omega + \tilde{\omega}) (1 + r) + tp (y_h - R - (\omega + \tilde{\omega}))
\]

(4.26)
So for firms that are not 'fully capitalized', we can show that:

\[
\hat{p}_{\text{firm}}(r, t) = \frac{(1 + r) - y_t - t \left( (1 - \omega - \bar{\omega}) (1 + r) - y_t \right)}{y_h - y_t - t \left( y_h - y_t - (\omega + \bar{\omega}) \right)}
\]  

(4.27)

The participation threshold for firms is given by:

\[
V(\bar{\omega}, \bar{\omega}, r) = \int_{\hat{p}_{\text{firm}}}^{1} \left[ p (y_h - y_t) + y_t - (1 + r) - tp (y_h - R_{\text{firm}} - (\omega + \bar{\omega})) \right] h(p) dp = C
\]

Note that since debt payments are tax deductible in this set-up, the distribution of firms’ initial endowments now matters for the equilibrium outcome. Modigliani-Miller no longer holds.

The key equations for banks under asymmetric information (eqns 4.10, 4.11 and 4.12) become:

\[
\hat{p} (y_h - R) + (1 - \hat{p}) \max [0, y_t - R] = (\omega + \bar{\omega}) (1 + r) + t\hat{p} (y_h - R - (\omega + \bar{\omega}))
\]

(4.28)

\[
A (\hat{p}) R + (1 - A (\hat{p})) \min [y_t, R] = (1 - \omega - \bar{\omega}) (1 + r)
\]

(4.29)

\[
A (\hat{p}) (y_h - R) + (1 - A (\hat{p})) \max [0, y_t - R] \geq (\omega + \bar{\omega}) (1 + r) + tA (\hat{p}) (y_h - R - (\omega + \bar{\omega}))
\]

(4.30)

For banks that are not 'fully-capitalised', we can show that a sufficient condition for the prudence threshold to be increasing in \( t \) is:

\[
A' (\hat{p}) < \frac{A(\hat{p})}{\hat{p}}
\]

which is satisfied when \( p \) follows a uniform distribution. It is not difficult to show that banks that are not 'fully-capitalised' retains a preference for debt funding.
Next, the value of screening for banks is given by:

\[
V(\omega, \tilde{\omega}) = \mathcal{E}_p \left[ \max \left\{ 0, \frac{\omega}{\omega + \tilde{\omega}} \left( \frac{p(y_h - R) - tp(y_h - R - (\omega + \tilde{\omega})) + \ldots}{(1 - p) \max[0, y_l - R]} \right) - \omega(1 + r) \right\} \right]
\]

\[
= \int_{\tilde{p}(\omega + \tilde{\omega})}^{1} \left[ \frac{\omega}{\omega + \tilde{\omega}} \left( \frac{p(y_h - R) - tp(y_h - R - (\omega + \tilde{\omega})) + \ldots}{(1 - p) \max[0, y_l - R]} \right) - \omega(1 + r) \right] h(p) \, dp
\]

\[
= \frac{\omega}{\omega + \tilde{\omega}} \left[ \int_{\tilde{p}(\omega + \tilde{\omega})}^{1} (y_l + p(y_h - y_l) - (1 + r) - tp(y_h - R - (\omega + \tilde{\omega}))) \, h(p) \, dp \right] = C
\]

where the last equality can be shown by considering the two cases \( \omega \geq \omega^* \) and \( \omega < \omega^* \) separately.

The participation threshold \( \hat{\omega} \) is implicitly defined by the equation:

\[
\frac{\hat{\omega}}{\omega + \tilde{\omega}} \left[ \int_{\tilde{p}(\omega + \tilde{\omega})}^{1} (p(y_h - y_l) + y_l - (1 + r) - tp(y_h - R - (\omega + \tilde{\omega}))) \, h(p) \, dp \right] = C
\]

where \( \tilde{\omega} = \max \{0, \omega_{\text{reg}} - \omega\} \).

Accounting for corporate taxation as an additional dimension of policy does not affect the main results in our baseline model. All three tools: interest rate policy, capital regulation, and corporation tax operate as imperfect substitutes which trade-off prudence and participation. Similar to the interest rate policy, the corporation tax affects both the transparent 'firms' sector and the banking sector indiscriminately and therefore can lead to inefficiencies when used to tackle excessive risk taking in the banking sector. Numerical simulations based on parameterizations in table 4.2 shows that the optimal policy combination now depend on the initial distribution of endowments for firms: \( \omega_{\text{firm}} \). When leverage is just as high in the transparent firms sector as the banking sector: \( \omega_{\text{firm}} \sim U[0, 0.1] \), the optimal policy response is to rely on an (prohibitively) high rate of corporate tax (\( > 0.75 \)) to boost prudence, with a lower than neutral (i.e. negative) interest rate to offset the costs in participation. But
the level of leverage in the non-financial sector of developed economies is generally very much lower than it is for banks. When we allow for even slightly lower levels of leverage in the firm sector, say $\omega_{firm} \sim U[0,0.2]$, the optimal policy setting becomes $(r = 0, \omega_{reg} = 17.5\%, t = 5\%)$. Under more realistic assumptions, $\omega_{firm} \sim U[0,\omega_{firm}^{ub} \geq 0.3]$ , we revert back to the solution in the main body of the text: $(r = 0.5\%, \omega_{reg} = 17.5, t = 0)$. Thus blunt instruments, such as the interest rate, and particularly the corporate tax rate, are not the ideal tools when dealing with high leverage and excessive risk taking in the banking sector.