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Finite-Horizon Planning and Optimizing the Rate of Savings

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This paper follows up the recent debate between S. Chakravarty and A. Maneschi [2, 3, 4] in the *International Economic Review* on "Optimal Savings with Finite Planning Horizon" with the following specific purposes: (1) To argue that, in spite of the arbitrariness involved in choosing the terminal capital stock, *finite-horizon planning* makes more sense than planning with an infinite horizon. (2) To enquire into the nature of the time-path of optimum *rate of savings* in the model in question, particularly as regards the bearing of the *elasticity of marginal utility from consumption*, 'v', on the question whether optimum rate of saving falls, remains constant, or rises over time. While doing so, the non-disinvestment constraint suggested by Maneschi in [3] is not interpreted, as Maneschi rather artificially does, as imposing a lower bound on the choice of the terminal capital stock (as identified by the parameter 'g'); instead, the more usual and straight forward interpretation of the constraint as one delimiting the set of *feasible* programmes given *any* choice of 'g' is adopted. (3) To initiate, following the results obtained from (2) above, a discussion on *social choice of the parameter 'v'*, without a meaningful rationalization of which optimum growth analysis may very well be regarded as a game in abstract theory only.

1. The Case for Finite-Horizon Planning

Neither Chakravarty nor anyone else has ever seriously presented the case for finite-horizon planning, and the general presumption in the theoretical literature has, it appears, remained against it. Now that Maneschi in particular launches a determined attack on it, it has become necessary to present the case for finite-horizon planning at its strongest.

Notwithstanding the presumption that a nation lives for ever,[1] planning with a finite-horizon makes more sense than infinite-horizon planning because neither the production function nor the social welfare function can reasonably be anticipated for all times to come.[2] Estimation of the production function within a reasonable level of significance is possible only for some finite plan-period. However comprehensive (e.g., with technical change built in) the specification of the production function may be, it is a natural presumption to suggest that the margin of error from any estimation of the function would approach infinity as the time-argument approaches infinity. To put it differently, no estimation procedure exists that will give us the production function at "infinity" with any finite level of significance. Hence, any specification of the production function beyond some finite horizon would inevitably have to be arbitrary, and defensible only for abstract theoretical investigations but not for seeking guidance for planning actually to be put in operation.

Even assuming that the production function could have been estimated with reasonable accuracy for all times to come, or -- to think of a more credible 'if' -- even if one could agree to some particular specification as a working "null-hypothesis", social planners are neither in a position to know, nor to dictate, the other major argument in the issue, i.e., the utility function of society for all times to come. The validity of any specification of the social utility function for any particular period of time must rest on the support, through the political process of the day (not excluding a dictatorial choice by the political authority of the day), of the generation living in that particular time. If this fundamental principle of "temporal democracy" is accepted, it follows trivially that the specification of the social utility function for any period beyond the life-time of the generation to which the decision-maker actually belongs is beyond the latter's jurisdiction. Nor is there any means available at present for anticipating the utility function of societies yet unborn: there is no guarantee of any continuity in social decision-making as generations, governments, and key personalities in the decision-making offices change, amidst changes in the complex of national aspirations and international relations.

Social planners with a holier-than-thou attitude might consider themselves nevertheless qualified to dictate the nature of social preferences for all times to come and thus set sside the principle of temporal democracy mentioned above. Even this would fail to salvage the case for infinite-horizon planning in the absence of any means of enforcing one's dictates on future societies: notwithstanding the dictates of today's social planners, future societies would behave, and dispose of their resources, exactly as they wish, constrained only by the initial endowment of resources they are left to start with.

Here is really the most crucial argument. For the type of aggregative planning being considered, the only way social planners of today can effectively constrain economic action of societies tomorrow is through the size of the terminal capital stock to be left for the latter. Thus it is pointless, if not inefficient, to assume responsibility for the economic disposition of future societies beyond this one single variable.

Thus infinite-horizon planning, although fascinating as an abstract exercize, does not make practical sense, and the lack of a general interest in optimization over a finite planning-horizon in the theoretical literature is therefore to be deplored. It is heartening on the other hand to note that practical planners on the whole have refused to be dragged to infinity and have stuck to finite-horizon planning without being the less wiser for this.

This does not mean that finite-horizon planning is free from its own difficulties. As Maneschi has already brought out, arbitrariness of a major nature must be involved in choice of the terminal capital stock subject to which optimization is to be attained.[3] That it must be so should indeed be obvious, particularly when we note that choice of the terminal capital stock vitally affects two parties, so to say, 'current' and 'future' generations, of which only one is truly represented at the decision-making table with a choice before it which is therefore admittedly awkward. But certainly the situation is not going to be helped by assuming that the party present is in a position not only to decide on the amount of resources to be left for disposal by the other party, <u>but also to dictate how the latter should dispose of the resources to be thus left for them</u>!

2. Time-Path of Optimal Rate of Savings and the Role of the Elasticity of Marginal Utility

With the attack on the finite-horizon approach out of the way, we ought to take up more enthusiastically than ever the question of optimizing over a finite planning period. Neither Chakravarty nor Maneschi did really discuss this problem to a rounded completion, presumably because of their pre-occupation with the "sensitivity question". Specifically, the question of the optimum rate of saving, the title-issue of the whole discussion, remains untouched. In this section we shall be particularly concerned with this specific question.

We shall use the non-disinvestment constraint that has been suggested by Maneschi in [3]. We however reject the Maneschi approach of implying from this that the choice of the parameter 'g' has got to be restricted in an area where this constraint is not binding: such a procedure is not logically convincing; rather it merely confuses what constitutes a feasible solution to the problem in question and what does not. If for lower values of 'g' than Maneschi's g* the solution offered by Chakravarty in [1] violates the non-disinvestment constraint, all we need to do is simply to reject this as not constituting a feasible solution to the problem at all, and search for the best among solutions that are feasible; it is not necessary to disturb the pre-determined character of the choice of 'g' and require it to be governed by the irrelevant consideration that the non-disinvestment constraint should not be binding on the problem postulated.

Thus we keep the choice of 'g' entirely open in its whole feasible range, namely, anywhere between 0 and b, to be determined only by (admittedly arbitrary) value judgment of the social planners subject to whatever political forces are present to constraint this choice, a consideration totally exogenous to our model. The problem thus presented lends itself to a direct application of the "Maximum Principle" of Pontryagin and co-workers [5, Theorem 6]; the solution is identical, as it must be, to that obtained by Chakravarty [1] following the "classical" method in the range of 'g' where the non-disinvestment constraint is not binding, i.e., when 'g' exceeds or equals Maneschi's g*, but is different when 'g' is less than g*, the "classical" method failing in this case to provide the solution.

We shall not elaborate the detailed derivation of the solution here, but merely present the final result for 'g' $<$ g* in order only to complete the picture.[4] In this case, the optimum time-path of capital stock consists of an initial "growth phase" when capital stock would be growing, and a subsequent "stationary phase" when capital stock would remain constant. Both phases are non-empty. During the "growth phase" consumption remains smaller than income but is growing, at the rate of b/v; during the stationary phase consumption equals income and remains constant.

When $g \geqslant g^*$, the solution is already known from Chakravarty,[5] and it may simply be noted that there is no "stationary phase" in this case.

We shall now take a closer look at the focus of our enquiry, viz., the optimum rate of savings. It can be shown that the optimum rate of saving, which we shall call s*, (defined as the ratio of savings [= investment] to output), during the "growth phase" has the following general expression covering the whole range of admissible 'g' (during the "stationary phase" where it is relevant s* is of course zero):

$$s^*(t) = 1 - \frac{1}{A \cdot e^{b(1-\frac{1}{v})t} + \frac{v}{v-1}}$$

where **A** is given, when $g < g^*$, by

$$\text{(a).} \quad A = e^{\frac{b}{v}\tau - g\tau} - \frac{v}{v-1}$$

with τ equalling the length of the "growth phase", and, when $g \geqslant g^*$, by

$$\text{(b).} \quad A = \frac{v}{v-1}\left(\frac{e^{bT} - e^{\frac{b}{v}T}}{e^{bT} - e^{gT}} - 1\right)$$

It is easy to see that s*(t) thus obtained must either monotonically fall, remain constant, or monotonically rise as t rises. From this it follows easily that if the "growth phase" falls short of the total length of the plan-period (i.e., when $g < g^*$), s*(t) must be monotonically falling as t rises; for either a (non-zero) constant or a rising s*(t) means a growing capital stock, and hence output, and with this saving (investment) must rise further to give a rising rate of saving, so that the "growth phase" will be perpetuated and cannot come to a halt. The converse is not necessarily true, however, and what remains to be considered is how $s^*(t)$ behaves over time in the case when there is no "stationary phase" (i.e., when $g > g^*$). Here, the following interesting cases can be distinguished:

<u>$v < 1$</u>: In this case $e^{(1-\frac{1}{v})t}$ falls as t rises, and the relevant expression for A, given by (b), is positive; hence $s^*(t)$ falls as t advances.

<u>$v > 1$</u>: In this case $e^{(1-\frac{1}{v})t}$ rises as t rises; on the other hand $A \lesseqgtr 0$ according as $g \lesseqgtr \frac{b}{v}$. Thus $s^*(t)$ falls, remains constant, or rises with rising t according as $g \lesseqgtr \frac{b}{v}$.

It may be noted that the quantity b/v is always greater than the quantity g^* as given by $g^* = \frac{b}{v} + \frac{1}{T} \log_e \frac{1-v}{e^{b(\frac{1}{v}-1)} - v}$ where the second expression is negative.

To put the above findings together, optimum rate of savings over the course of the "growth phase". always falls monotonically if 'v' is less than unity irrespective of the choice of 'g'; on the other hand, if 'v' exceeds unity, then optimum rate of savings falls, remains constant, or rises monotonically over the course of the "growth phase" according as 'g' is chosen to be smaller than, equal to, or greater than, the quantity b/v. Yet another handy way of putting the whole result together is to say that, <u>given the choice of 'g', the optimum rate of savings monotonically falls, remains constant or monotonically rises according as 'v' is less than, equal to, or greater than, the quantity $\frac{b}{g}$.</u>

3. Conclusion: Towards Rationalizing the Choice of 'v'

This, we believe, brings the whole discussion to a more positive conclusion than where the Chakravarty-Maneschi debate left us. But this is only as far as pure theory goes. For relevance to actual quantitative application of such theory the pertinent question remains as to the choice of 'v', the elasticity of marginal utility from consumption, on which the character of optimum programmes depends very crucially. So far, no meaningful rationalization has been offered for choice of this elasticity, which therefore remains the weakest point in the whole body of optimum growth studies.[6] It is time that more serious attention be given to this most difficult nonetheless the most important and fundamental question, if an explicit utility function is to be used in actual quantitative planning in growing economies.

By way of initiating serious thinking on this question, we suggest that the results obtained in the previous section with respect to the bearing of 'v' on the temporal behaviour of the optimum rate of savings be considered as a starting point. Suppose we pose the question this way: Given society's initial capital stock, and its choice of the terminal capital stock, there is a given amount of savings that must be generated during the plan-period to satisfy the terminal requirement. In what way should the burden of this "sacrifice" be distributed among the various "generations" that make up the plan-period in question? Presented this way, the question bears a strikingly close analogy to the very familiar question in Public Finance: given the total size of required taxes, how should its burden be distributed between various sections of the community? Centuries of debates have brought us to an agreement at least about the general character of this distribution that is to be regarded as equitable, as embodied in the Principle of Progressive Taxation. Can we not apply this principle of inter-personal equity to the question of inter-temporal equity as well? If we can, then we really have quite an "objective" guidance at least about the general character of distribution of the required amount of savings among different "generations", for the Progressive Taxation is a *revealed behaviour* of societies as observed through tax-legislations of governments at least with respect to direct taxation, and is at least a *revealed faith* even if tax-systems, *taken as a whole*, of certain countries fail to bear the Progressive character.

If it is permitted to borrow principles of equity from the interpersonal to the inter-temporal sphere, and apply specifically the Principle of Progression to the question of optimum savings over time, then according to the results of the previous section an "equitable" choice of 'v' is clearly indicated as greater than the quantity b/g (note that this necessarily means that 'v' should exceed unity); for only then will a higher rate of saving would be imposed (a higher rate of "taxation", so to say) on later, higher-income "generations" than on earlier, lower-income "generations" as the Principle of Progression would require. This gives us a lower bound for the choice of 'v', given the choice of 'g', a lower bound that rises as a smaller 'g' is chosen, and vice versa. Since 'g' itself is a parameter in the choice of which considerations of inter-temporal equity (i.e., between plan-period and post-plan-period "generations") is involved, it is not surprising that an "equitable" choice of 'v' happens to have some relation with the choice of 'g': rather this is what one should expect for consistency. The inverse relation between the two has an easy interpretation: lowering the choice of 'g' implies in a sense asking less sacrifice from plan-period "generations" and more from post-plan-period "generations" for any overall growth rate covering any length of time greater than the plan-period actually conceived; translating the same principle (i.e., of higher progression) for distribution of the burden of growth among the different "generations" within the plan-period indicates a higher 'v' than otherwise.

The above gives, of course, only a lower bound for a rational (equitable) choice of 'v', and hence a consistency rule of a very broad nature. It does not actually pin-point a rational choice of this social parameter that is crucial for objectively determining optimum saving rates for a growing economy. But the considerations outlined above provides at least a basis around which debates about a more specific choice of 'v' can follow, and thus takes us one step closer to actual operational use of theoretical optimum growth analysis. It may particularly be worthwhile to try and see if specific choices of 'v' can be deduced, following the aforesaid analogy between inter-personal and inter-temporal equity, from specific rate-schedules of taxation that have found general acceptance in politically conscious societies through years of mature social debates. This, however, we reserve for a separate study.

Footnotes

[1]Maneschi [3], p. 109.

[2]Essentially this is an "uncertainty" argument, and one may feel that the use of an "uncertainty discount" as is often assumed in the theory of individual decision-making, takes care of the issue without making it necessary to have a finite planning horizon. This however, would be a confusion of issues, for two reasons: (1) uncertainty about preferences of future generations is analogous only to an indicision in one's own mind in the case of an individual decision-maker about what he may desire in the future. Such uncertainty has hardly been handled in individual decision-making theory, and the only sensible way to handle it would be to have the individual concerned leave an "adequate" provision for the future to "reasonably" accomodate what he may desire later on when he does make up his mind; this would be an exact parallel to leaving an "adequate" terminal capital stock for future societies in social decision-making. (2) An "uncertainty discount" in any case discriminates against the future. Such discrimination may be tenable for individual decision-making where discrimination is made both for and against the same individual (one's own self), only at different points of time; in social decision-making, however, such discrimination will, except for generations not very distant to each other through time, would favor one set of individuals in preference to another. As a matter of fact, in social decision-making, uncertainty about the future should give as much reason for having a negative time discount (cf., Sen [7]) as for a positive one, and this intriguing question would have to be settled before an uncertainty discount can be used in social decision-making involving the welfare of both current and future generations.

[3]What Maneschi has actually shown in Table 1 of [3] is that, in Chakravarty's model, with decumulation permitted, there exists only an initial area of "insensitivity" which must eventually give way to a later area of "sensitivity" while 'g' is being increased. It may be added, to complete enquiry of the "sensitivity question", that (a) the size of the "initial area of insensitivity", which apparently misled Chakravarty into his rather sweeping "insensitivity conclusion" in [1], is inversely related to the parameter 'b', and realistic values of 'b' can be conceived which makes the former give way to the "later area of sensitivity" well within the range of variation of 'g' that Chakravarty himself considered; (b) more significantly, even the "initial area of insensitivity" disappears if optimum growth paths are subjected to the non-disinvestment constraint, keeping the choice of 'g' open anywhere (unlike Maneschi). It can be shown, by solving the problem properly under this constraint as suggested in section 2 of this paper, that optimum consumption paths are sensitive to variation in 'g' significantly and in important respects (in fact cutting one another on their ways to the plan-terminal for choices of 'g' in areas where the non-disinvestment constraint becomes binding on the problem). It is this, (b) that really refutes Chakravarty's "insensitivity conclusion" completely. Nevertheless, as it has been argued above, the basic finite-horizon approach taken by Chakravarty seems to be the only sensible approach to take.

[4]The author has benefited from a discussion with Professor Ramesh Gangolly, now at the Institute of Advanced Studies, Princeton, over the analysis of the "Pontryagin equations" for solving the above problem. Responsibility for any inadequacy in the analysis is, however, the author's alone.

[5]An error in Chakravarty [1, p. 344], minor for his own purpose, by-passes the critical role of the quantity b/v in determining the time-course of the optimum rate of savings: for 'g' exceeding b/v, 'R' is positive and not negative as Chakravarty thinks it always is.

[6]An attempt was made by Tinbergen in [8] to estimate 'v' empirically from revealed behaviour of individual workers. Apparently it is the Tinbergen estimate of .6 that was taken over by Chakravarty and has featured through his debate with Maneschi (if so, it must have been missed by Chakravarty that the translation of the origin by putting subsistence-consumption equal to zero, actually requires a transformation of Tinbergen's estimate, according to Tinbergen's formula in p. 482; thus it is not correct to say that C is a "mere translation factor" and "does not affect the maximization problem" as Chakravarty says in [1, p. 342]: it does, through specification of 'v'). Apart from the questionable mixing up of issues pertaining to individual optimization with those pertaining to social optimization over time, the present author has found Tinbergen's estimation to be arbitrary, as it depended crucially on the two particular consumption levels that he actually used to derive his estimate (this has been discussed by the author in a separate paper [6], which shall be mailed to the reader on request).

References

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[5] Pontryagin and Others, The Mathematical Theory of Optimal Processes, Pergamon, 1964.

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[7] Sen, A.K., "On Optimizing the Rate of Saving", The Economic Journal, Vol. LXXI, No. 283, September, 1961.

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APPENDIX 1: MATHEMATICAL NOTE TO SECTION II[1]

$\theta = v$

1. The problem is to maximize $\int_o^T \frac{1}{1-\theta} (bK-\dot{K})^{1-\theta}\, dt$, i.e., to minimize

$\int_o^T \frac{1}{\theta-1} (bK-\dot{K})^{1-\theta}\, dt$, given

1.1 $\dot{K}(t) = u(t)$, the control variable, to be chosen for each point of time within bounds given by

1.2 $0 \leq u(t) \leq bK(t)$;

1.3 $K(o) = \bar{K} > o$, the initial condition;

1.4 $K(T) = K^T = \bar{K} \cdot e^{gT}$, the terminal condition, $o < g < b$;

1.5 $\theta \neq 1$, in order for optimization to be meaningful.

Without loss of generality, let $\bar{K}=1$, so that

1.3a $K(o) = 1$

1.4a $K(T) = e^{gT}$

2. By the Maximum Principle of Pouteryagin et al (Theorem 6), optimization requires u (t) to be so chosen as to maximize the following Hamiltonian function $H(K,u,t)$ at each point of time[2]:

2.1 $H(K,u,t) = \frac{1}{1-\theta} \{bK(t) - u(t)\}^{1-\theta} + \psi(t) \cdot u(t)$, where $u(t)$ is the shadow price of investment and is time-continuous, and its movement through time under optimization is governed by the relation

2.2 $\dot{\psi} = -\frac{\partial H}{\partial K} = -b\,(bK-u)^{-\theta} < 0.$

3. It can be shown that the Hamiltonian H is a concave function in u, given t and the time-path of K up to t and hence K (t). From this the following

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[1] See acknowledgment, page 4n.

Footnote 2 continued on next page.

can be said straightway about optimum u (t) {henceforth $u^*(t)$, denoting that value u (t) that maximizes the corresponding Hamiltoni function H (K,u,t)}:

3.1 $u^*(t) = o$ if and only if $H_u(K,u,t)\Big|_{u=0} \leq 0$ and

3.2 $u^*(t) > 0$ if and only if $H_u(K,u,t)\Big|_{u=0} > 0$

3.1 may be said to give a "corner solution", while 3.2 stands for the "regular" (classical) solution for $u^*(t)$, as given by

3.2a $H_u = \psi - (bK-u)^{-\theta} = 0.$

Note that $u^*(t)$ never reaches the upper bound u = bK(t), for $H_u\Big|_{u = bK} = \psi - \infty$, a negative quantity, so that H_u must vanish before u reaches bK.

4. We shall now prove the following:

Theorem I: If $u^*(t)$ is ever 0, it is 0 the rest of the plan-period.

Proof: Consider the time-path of the derivative $H_u(K,u,t)$ on the plane u = 0. Since $\dot{K} = u$ (by definition), K remains stationary on this plane. Hence, the time-derivative of $H_u(K,u,t)\Big|_{u=0}$ equals the time-derivative of ψ alone (see 3.2a), and thus (see 2.2) is a negative quantity.

Thus, on the plane u = o, the expression $H_u(K,u,t)$ monotonically falls as t increases. Hence, if for some t = t', we have $H_u(K,u,t)\Big/_{u=0} \leq 0$ then

$H_u(K,u,t)\Big/_{u=0} < 0$ for all $t > t'$.

[2] Formally, the Maximum Principle only gives a set of necessary conditions for optimum. In this particular problem, however, it should be easy to show that an optimum does exist (i.e., the solution space is compact); it should also be easy to see that the relevant differential equation, given values of the relevant parameters, has only one solution. The two together insure sufficiency.

This combined with 3.1 proves Theorem 1.

5. From Theorem 1, it follows that $u^*(t)$ cannot be zero for t=0, for then the terminal requirement 1.4 cannot be satisfied. Hence, by 3.2, we must have

$$H_u(K,u,t)\Big|_{\substack{u=0\\t=0}} > 0, \text{ i.e.,}$$

$$\psi(0) - b^{-\theta} > 0 \text{ (see 3.2a and 1.3a), or}$$

5.1 $$\psi(0) > b^{-\theta} > 0.$$

It is also easy to see that $H_u(K,u,t)\Big|_{u=0}$ is time-continuous, since $\psi(t)$ is time-continuous. Hence

<u>Theorem 2:</u> <u>There must at least be an initial phase that is non-empty, during which $u^*(t)$ is strictly positive, and hence given by 3.2, i.e., $H_u(K,u,t) = 0$.</u>

During this phase, capital stock would be growing with time, and accordingly we shall call this phase the "growth phase."

6. From Theorem 2, we have $u^*(t) = bK(t) - \frac{1}{\{\psi(t)\}^{\frac{1}{\theta}}}$ during the "growth phase." Combining this with 2.2, we have $\dot{\psi} = -b\psi$, whence $\psi(t) = \psi(0)e^{-bt}$. Hence, during the growth phase, we have

6.1 $$u^*(t) = bK(t) - B.e^{\frac{b}{\theta}t}, \quad \text{where} \quad B = \frac{1}{\{\psi(0)\}^{\frac{1}{\theta}}} > 0.$$

Equation 6.1 gives the optimum time-course of the rate of investment $\dot{K}$ during the "growth phase," yielding the general solution for optimum capital stock, denoted by $K^*(t)$, at each point of time during this phase as

6.2 $K^*(t) = C_1 e^{bt} + C_2 e^{\frac{b}{\theta} t}$,

with initial condition

6.3 $C_1 + C_2 = 1.$

Solving for C_1 and C_2 in terms of B, we obtain

6.4a $C_1 = 1 + \frac{\theta}{b(1-\theta)} . B = 1 + B^1$, and

6.4b $C_2 = - \frac{\theta}{b(1-\theta)} . B = -B^1$. Note that

6.5 B^1 (defined in 6.4b) $\gtrless 0$ for $\theta \lessgtr 1$.

Combining 6.2 with 6.4, we have

6.6 $K^*(t) = e^{bt} + B^1(e^{bt} - e^{\frac{b}{\theta} t})$ during the growth phase.

Note that 6.5 ensures that the expression $B^1(e^{bt} - e^{\frac{b}{\theta} t})$ is always negative, thus ensuring that the optimum rate of growth of capital is always less than maximum feasible rate given by the output-capital ratio β confirming a result we have already obtained in Section 3.

7. There are now two possibilities:

(a) The "growth phase" covers the entire plan period. In this case $u^*(t)$ is positive throughout, i.e., the non-disinvestment constraint is never binding. Thus, the optimum control is in the interior of the feasible region $0 \leq u\ (t) \leq bK(t)$ for all t at least in the range $0 \leq t < T$, and we have the so-called "classical solution" where the constant B^1 (hence B) is determined by the original terminal condition 1.4. This solution is already spelled out in Chakravarty (1), and we shall not repeat it here.[3]

(b) The "growth phase" stops short of the entire length of the plan-period, and is followed by what may be called a "stationary phase" which runs through the remaining period (by Theorem 1). This we shall call the "non-classical solution," and, as it has not been covered either by Chakaravarty or by Maneschi, we shall look at this solution more closely below.

Suppose we have a "non-classical solution," with the "stationary phase" starting at $t = \tau < T$. Then in order to satisfy the terminal requirement 1.4a, we must have $K^*(t) = K(T) = e^{gt}$ for all t in the range $\tau \leq t \leq T$. Specifically,

7.1 $K^*(\tau) = e^{gt}$, and this gives us the specific terminal condition to solve for B^1 and hence for the differential equation 6.1. Combining 6.6 with 7.1 we have

$$7.2 \quad K^*(\tau) = e^{b\tau} + B^1(e^{b\tau} - e^{\frac{b}{\theta}\tau}) = e^{gt}.$$

Furthermore, since the "stationary phase" <u>begins</u> at $t = \tau$, $u^*(t)$ must become zero for the first time at $t = \tau$, being determined <u>up to</u> this point of time by the differential equation 6.1. Setting $u^*(t=\tau) = 0$ in 6.1 we have $bK(\tau) = B.e^{\frac{b}{\theta}\tau}$, whence

$$7.3 \quad B = bK(\tau) \;.\; e^{-\frac{b}{\theta}\tau} = be^{gt-\frac{b}{\theta}\tau} .$$

Combining 7.2 and 7.3, we have the relation

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[3]See page 5n.

7.4 $\phi(\tau) = e^{b\tau} + \frac{\theta}{1-\theta} \cdot e^{gt-b(\frac{1}{\theta}-1)\tau} - \frac{1}{1-\theta} e^{gt} = 0$, which provides the solution for τ and hence for the whole system in the "non-classical" situation. Obviously, if we have the "non-classical" solution, then τ as given by 7.4 must be less than T. It can also be shown that there is only one positive value of τ that satisfies 7.4.[4]

8. We already have, from Maneschi in ( ), that the "classical" solution satisfies the non-disinvestment constraint only for $g \geq g^*$, where

$$g^* = \frac{b}{\theta} + \frac{1}{T} \log_e \frac{1-\theta}{e^{b(\frac{1}{\theta}-1)T} - \theta} .$$

Obviously, then, we have the "non-classical" solution for $g < g^*$. It can be shown easily that $0 < g^* < b$,[5] so that we have (Theorem 3): the ranges of 'g' for which the classical and non-classical solutions occur respectively are both non-empty.

9. Finally, the optimum rate of saving defined as $s^*(t) = \frac{u^*(t)}{bK^*(t)}$,

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[4]Since (a) $\phi(0) < 0$; (b) $\phi(\infty) > 0$ {express $\phi(\tau)$ as $e^{b\tau}(1 + \frac{\theta}{1-\theta} e^{gt-\frac{b}{\theta}\tau}) - \frac{1}{1-\theta} e^{gt}$} and (c) $\phi^1(\tau)$ vanishes only once for $\tau > 0$.

[5]Hint: show that $e^{(b-g^*)T} > 1$, so that $b > g$, and also $e^{g^*T} > 1$, so that $g^* > 0$, for all $\theta > 0$, $\theta \neq 1$.

is given by (from 6.1 and 6.6)

$$s^*(t) = 1 - \frac{1}{\left(\frac{b}{B} - \frac{\theta}{\theta-1}\right) e^{b(1-\frac{1}{\theta})t} + \frac{\theta}{\theta-1}}$$

during the "growth phase". For the non-classical solution, this gives us (see 7.3)

$$s^*(t) = 1 - \frac{1}{\left(e^{\frac{b}{\theta}T - gT} - \frac{\theta}{\theta-1} \right) e^{b(1-\frac{1}{\theta})t} + \frac{\theta}{\theta-1}}$$

For the classical solution, it can be shown {by putting $K^*(T) = e^{gT}$ in 6.6}

$$B = \frac{b(\theta-1)}{\theta} \cdot \frac{e^{bT} - e^{gT}}{e^{bT} - e^{\frac{b}{\theta}T}},$$

whence

$$S^*(t) = 1 - \frac{1}{\frac{\theta}{\theta-1}\left(\frac{e^{bt} - e^{\frac{b}{\theta}T}}{e^{bT} - e^{gT}} - 1\right) e^{b(1-\frac{1}{\theta})t} + \frac{\theta}{\theta-1}}$$

APPENDIX 2: ESTIMATION OF 'v' BY TINBERGEN

Tinbergen estimates 'v' by using Frisch's work on what Frisch calls the "flexibility of marginal utility," $\check{w}$. The concept of 'v' differs from the concept of 'w' in that the former refers to the elasticity of marginal utility with respect to surplus (real) consumption while the latter is concerned with real income. In the rough measurement of 'v' that Tinbergen attempts the distinction between real income and real consumption has apparently been ignored. Assuming that 'v' is constant for all magnitudes of surplus consumption 'z' ($c-\bar{c}$ in Tinbergen's notation) so that $\check{w} = v.c/c-\bar{c}$ changes only as the ratio $c/c-\bar{c}$ changes, and assuming subsistence consumption to be a constant, an estimate of 'v' is suggested from any two estimates of '$\check{w}$', say w_1 and w_2, as follows: $\check{w}_1 = v.c_1/c_1-\bar{c}$, and $\check{w}_2 = v.c_2/c_2-\bar{c}$, whence $v = (c_1-c_2)/\left(\frac{c_1}{\check{w}_1} - \frac{c_2}{\check{w}_2}\right)$.

With $\check{w}_1 = 1$ (American workers) and $\check{w}_2 = 3.5$ (French workers) and $c_1 = 2c_2$, as assumed by Tinbergen, the above yields Tinbergen's estimate of 'v' as .6.

Since any pair of c's (and the corresponding $\check{w}$'s) would yield one such estimate of 'v', this procedure of estimating 'v' would be valid only if the assumption of constancy of 'v' were empirically borne out, yielding the same estimate of 'v' from different pairs of c's except for random variations. This, however, is not the case: estimation of 'v' for different pairs of c's with corresponding $\check{w}$'s taken from Frisch's 1931 study shows a systematic variation of 'v' with respect to different such pairs. This is shown in Tables A and B.

Table A			Table B		
'c'	'w'	'v' from consecutive pairs of 'c'	'c'	'w'	'v' from consecutive pairs of 'c'
75%	6.40		2.40	.617	
80%	4.52	.83	2.62	.559	.28
85%	3.55	.80	2.90	.510	.28
90%	2.96	.77	3.17	.467	.24
95%	2.55	.72	3.48	.428	.22
100%	2.25	.69	3.80	.396	.21
105%	2.03	.68	4.16	.362	.18
110%	1.85	.64	4.55	.333	.17
115%	1.71	.64	5.00	.312	.19
120%	1.59	.60	5.40	.294	.17
125%	1.50	.63	5.91	.278	.17
130%	1.41	.56	6.50	.261	.16
135%	1.34	.58			
140%	1.28	.57			
145%	1.22	.52			
150%	1.18	.60			

Table A shows the different values of 'v' that are obtained by pairing consecutive c's (and corresponding w's) given in the columns of Frisch[1] as estimates from the Paris material, 1920-22; Table B gives the same for the U.S. material, 1918-19. In both these tables 'v' is seen to rise with significant consistency as lower and lower consumption levels are taken, giving a range from .52 to .83 for Paris workers and from .16 to .28 for U.S. workers. If we assume that the lowest U.S. consumption level represented in Table B was higher than the highest Paris consumption level represented in Table A, and also assume following Tinbergen that Paris and U.S. workers had

[1]Frisch, R., New Methods of Measuring Marginal Utility, Verlag Von J.C.B. Mohr (Paul Siebeck) 1932, pp. 32 and 64 (tables 2 and 5).

the same utility function, then the consistency in the variation of 'v' holds even if the two tables were combined. The monotonic rise in 'v' with fall in the corresponding levels of consumption gives us a reason to expect that 'v' would have exceeded unity if still lower consumption levels were investigated; this suggests that for consumption levels pertaining in underdeveloped countries with which most optimal growth studies are really concerned, 'v' would in fact be greater than unity if it were to be empirically estimated, following the above procedure, from data more directly relevant to situations in these countries.

The more important point that comes out from the tables presented above is that an estimate of 'v' the way Tinbergen does it would be as arbitrary as anything, depending as it would so heavily on the specific pair of consumption levels that are used for the estimation. This not only means that empirical justification for regarding 'v' as less than unity, always, is not established; it also means that the assumption of constancy of 'v' itself is not consistent with observed market behaviour of individuals.

This does not necessarily compel us to discard the constancy postulate for 'v' in the social welfare function under study. The social welfare function does not necessarily have to be that directly consistent, if at all, with revealed market behaviour of individuals or groups of individuals. If, however, the constancy potulate is retained, the possibility of estimation of the parameter 'v' from observed market behaviour would not be in the nature of things, and social choice of the value of this parameter has to seek its rationalization elsewhere.