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Initiation of a doubly diffusive convection in a stable halocline

by Melvin E. Stern

ABSTRACT

A vertically stable density stratification with temperature gradient $\tilde{T}_z < 0$, salinity gradient $\tilde{S}_z < 0$, and with density compensating horizontal $T/S$ gradients is unstable to lateral intrusions because the molecular heat diffusivity is much larger than the salt diffusivity. Previous marginal instability theory is extended to the supercritical regime. The fastest growing vertical wavelength ($\delta z$) is obtained in a model of a $T/S$ front with finite lateral width ($L_\delta$) and lateral salinity variation ($\Delta S$). The value of $\delta z = \Delta S/|\tilde{S}_z|$ varies only slightly with the parameters, and this result is applied to the smallest “step” size observed in the main thermocline of the Weddell Sea. Nonlinear considerations show that the laterally divergent heat/salt flux produced by the instability forces a mean vertical circulation which enhances the compensating lateral $T/S$ gradients, thereby accelerating the intrusive instability, and eventually producing local density ratios sufficient to initiate strong vertical convection. This suggests that weak isopycnal $T/S$ gradients are necessary to initiate the steps which subsequently merge into larger layers.

1. Introduction

Double diffusive convection leading to well mixed density layers is often observed in polar haloclines where temperature and salinity increase with depth. Under the ice in the Canada basin, for example, Robertson et al. (1995) found layers 1 m–10 m thick separated by interfaces sufficiently thin so that the directly computed molecular heat flux agreed with the bulk empirical laws (e.g., Kelley et al., 2003) based on the $T/S$ difference between layers; in these observations double diffusion was the dominant vertical heat transfer mechanism. Here and elsewhere the $T/S$ notation represent temperature/salinity in nondimensional buoyancy units (i.e., the expansion/contraction coefficients are absorbed in $T/S$). Larger steps have been observed in ice-free (austral summer) portions of the Weddell Sea (Fig. 1, Foster and Carmack, 1976), and Muench et al. (1990) found a dominant size of $\sim 10$ m, and fewer steps $\sim 1$ m thick; the former were consistent with the aforementioned empirical laws, but no explanation was available for the smallest steps. Furthermore, these laws stem from laboratory experiments (Turner, 1973) wherein a given heat flux is imposed at a (lower) boundary of a stable salt stratification, and consequently $T/S$ must

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adjust to transmit this flux. But in Figure 1 there is no “imposed” flux (in summertime), and since the overall density ratio is substantially less than the critical value (Veronis, 1965) required to initiate (any) double diffusive convection, it is therefore unclear as to how the observed ocean layers can be initiated and maintained. Veronis (1965) suggested that a finite amplitude disturbance such as occurs in the well known Rayleigh convection problem might lower the critical density ratio ($R$) below the value given by linear stability theory, but this critical $R$ is still well removed from the overall value in Figure 1. Veronis’s suggestion will, however, be revived here by relating it to the effect produced by isopycnal $T/S$ gradients like those in Figure 2. This was obtained from a drifting ice station and it shows density compensated intrusions and $T/S$ fronts with $O(1 \text{ km})$ lateral width, correlated (Robertson et al., 1995) with $\sim 1 \text{ m}$ thick layers and $\sim 1 \text{ cm}$ thick interfaces.

A similar effect occurs in laboratory experiments (Thorpe et al., 1969) where a stable salt gradient is heated at a vertical wall. This tilts the initially horizontal isohalines producing density-compensated lateral gradients, from which regularly spaced layers appear if a critical Rayleigh number is exceeded. Figure 3a illustrates the well known

![Figure 1. STD profiles [Foster and Carmack, 1976] in austral summer at the center of the Weddell Sea. Note the warm surface layer above the main thermocline. The minimum step size observed by Muench et al. (1990) was 1 m.](image-url)
instability mechanism for the case where there are no lateral boundaries. The Thorpe et al. (1969) calculation is, however, based on a bounded model (Fig. 3b), and uses a linear normal mode theory to compute the condition for instability of the uniform lateral $T/S$ gradients. Chen (1974) extended this by considering the linear initial value problem for a model in which a vertical wall has its temperature suddenly raised by $\Delta T$; the resulting cell size is expressed as a multiple of the “Chen scale” $\sim |\Delta T/\tilde{S}_z| = |\Delta S/\tilde{S}_z|$, where $\Delta S$ is the induced lateral salinity variation. Note that in our model the overall vertical salinity gradient $\tilde{S}_z$ will remain unchanged even when layers form.

We must emphasize here the difference between our starting model and the ocean since the former employs the molecular conductivity, diffusivity and viscosity ($K_T$, $K_S$, $\nu$, respectively), and the reader may be puzzled to see how these can enter into the ocean problem. The point is that it is impermissible to use eddy coefficients at this stage of the problem since there is no mechanism provided for any small scale turbulence. On the other hand, in the lateral instability for a salt finger favorable vertical stratification (e.g.,

Figure 2. A map of potential temperature (from Robertson et al., 1995) $\theta$ showing the horizontal extent of intrusions [using the ice-relative velocity $u_{rel}$ at 200 m to determine distance]. Double diffusive steps were found in the permanent halocline at $z \leq 300$ m. See text for details.
Merryfield, 2000) one must use the relevant eddy diffusivities (e.g., Stern and Simeonov, 2002) in the parametric theory of the larger scale lateral intrusions (Toole and Georgi, 1981).

The plan of the paper is as follows. The Thorpe et al. (1969) calculation for Figure 3b is reviewed in Section 2 and extended to supercritical $\Delta T = \Delta S$ in order to relate the fastest growing vertical wavelength to the cross-frontal $\Delta T$ (e.g., Fig. 3). The asymptotic result obtained for $K_s \to 0, \Delta S \to 0$ in Section 2 leads to a relatively simple nonlinear theory which indicates (Section 4) how well mixed “steps” tend to form.

The development of this nonlinear theory requires a prior consideration (Section 3) as to how the vertically averaged lateral $T/S$ gradients are modified by the lateral eddy flux convergences. This effect is absent in all previous lateral intrusion studies (e.g., Merryfield,
2000) using an unbounded model with lateral uniformity and symmetry. The mean field effect will, however, be important in more realistic frontal models (e.g., Fig. 3c), aspects of which are briefly discussed in the Appendix.

As will be seen, the nonlinear calculation (Section 4) applies only up to the time when the folding isohalines produce local density ratios equal to the Veronis critical value for initiation of strong turbulent convection, a topic which is beyond our scope.

Also beyond our scope is a discussion of the role of baroclinic currents (Kelley, 2003), although the Section 2b contains a short discussion of the quantitative influence of the Coriolis parameter on the cell size.

2. Linear theory for the bounded model (Fig. 3b)

a. Perturbation equations

The mean field gradients $T_z, S_z, S_x = T_x$ are constant in the undisturbed state of Figure 3b, and although there is no mean geostrophic current the effect of the Coriolis parameter $f$ on the periodic perturbations will be considered. Since the lateral distance $L_\infty$ is assumed to be much larger than the vertical wavelength, the temporal ($t$) evolution of the quasi-hydrostatic perturbation satisfies the longwave equations; this also implies that lateral diffusion is negligible compared to vertical diffusion. As will be seen, the resulting eigenfunction equations have constant coefficients, and the $x$-boundary conditions of vanishing normal velocity will be satisfied by superposition of $\exp(\lambda t + imz + ikx)$ modes with suitable $k$, where $\lambda$ is growth rate and $2\pi/m$ is the vertical wave length. If ($\hat{u}, \hat{v}, \hat{w}, \hat{\rho}, \hat{T}, \hat{S}$) denote the respective mode amplitudes of $x$-velocity, $y$-velocity, $z$-velocity, pressure, temperature, and salinity, then the respective perturbation momentum and continuity equations are

$$\hat{u}(\lambda + vm^2) - f\hat{v} = -ik\hat{\rho}, \quad (2.1a)$$
$$\hat{v}(\lambda + vm^2) = -f\hat{u}, \quad (2.1b)$$
$$\hat{w}\hat{\rho} = -g(\hat{T} - \hat{S}), \quad (2.1c)$$
$$k\hat{u} + mv\hat{w} = 0. \quad (2.1d)$$

From the balance of vertical diffusion, vertical advection, and lateral advection (with $T_x = S_x$) the respective heat/salt conservation equations yield

$$\hat{T} = -\frac{\hat{w}T_z + \hat{u}S_z}{\lambda + Kz\eta m^2}, \quad \hat{S} = -\frac{\hat{w}S_z + \hat{u}T_z}{\lambda + Kz\eta m^2}. \quad (2.2a,b)$$

By eliminating $\hat{v}, \hat{\rho}$ from (2.1a,b,c) and then using (2.2a,b) there results

$$\hat{u}[\lambda + vm^2 + f^2/(\lambda + vm^2)] = -g\left(\frac{k}{m}\right)\left[\frac{\hat{w}S_z + \hat{u}S_z}{\lambda + Kz\eta m^2} - \frac{\hat{w}T_z + \hat{u}T_z}{\lambda + Kz\eta m^2}\right].$$

The eigenvalue equation for $\lambda$ obtained using Eq. (2.1d) and simplification is:
\[
(\lambda + vm^2)^2(\lambda + K_s m^2)(\lambda + K_T m^2) + f^2(\lambda + K_s m^2)(\lambda + K_T m^2)
= -g \left( \frac{k}{m} \right) (\lambda + vm^2) \left[ \frac{k}{m} T_z (\lambda + K_s m^2) - \frac{k}{m} S_z (\lambda + K_T m^2) + S_z (K_T - K_s) \right].
\] (2.3)

Marginal stability is considered in this section, and when \( \lambda = 0 \) in (2.3) we get
\[
v^2 m^4 K_s K_T + f^2 K_s K_T + \left[ \frac{k}{m} \left( \frac{T_z K_s - S_z K_T}{S_z} \right) + S_z (K_T - K_s) \right] = 0
\] (2.4)

If \((k_1, k_2)\) denote the two \(k\)-roots of this, then the aforementioned boundary conditions are satisfied by
\[
\hat{u} = e^{ik_1 x} - e^{ik_2 x}, \quad k_2 - k_1 = 2\pi/L_*,
\] (2.5a,b)

(or by any integral multiple of \(2\pi/L_*\)). Special cases of (2.4) are discussed below.

b. \(f = 0, \lambda = 0\) (Thorpe et al., 1969)

First to be considered is the nonrotating neutral instability problem. Let
\[
a = k/m, \quad \varepsilon = \frac{S_z}{\overline{S}_z}
\] (2.6a,b)
\[
H = \frac{-g\overline{S}_z}{K_s v} > 0, \quad R = \frac{T_z}{S_z} < 1, \quad \tau = \frac{K_s}{K_T}.\] (2.6c,d)

Eq. (2.4) then yields the quadratic
\[
a^2(R\tau - 1) + a\varepsilon(1 - \tau) - \frac{m^4}{H} = 0
\] (2.7)

for the two nondimensional \(k\)-roots, viz. \((a_1, a_2)\). The discriminant of (2.7) and \(a_2 - a_1 = 2\pi/mL_*\) yields:
\[
\frac{4\pi^2}{L_*^2} = \frac{m^2\varepsilon^2(1 - \tau)^2 - 4(m^2)^3(1 - R\tau)/H}{(1 - R\tau)^2}.
\]

This is minimized at \(m = m_c\), where
\[
m_c^2 = \frac{|\varepsilon|}{\sqrt{12}} \left( \frac{1 - \tau}{1 - R\tau} \right)^{1/2} = |\varepsilon|(1 - \tau) \left[ \frac{-g\overline{S}_z}{12(1 - R\tau)K_s v} \right]^{1/2},
\] (2.8a)

and therefore a neutral mode exists if
\[
\frac{4\pi^2}{L_*^2} \leq \frac{2}{3\sqrt{12}} \left| \frac{S_z}{\overline{S}_z} \right|^3 (1 - \tau)^3 \left( \frac{-g\overline{S}_z}{K_s v} \right)^{1/2} (1 - R\tau)^{-5/2}.
\] (2.8b)
Note that the presence of compensated lateral gradients can cause instabilities even if $T_z > 0$, as is the case in the upper part of Figure 1.

c. $f \neq 0, \lambda = 0, \tau \rightarrow 0$ with $R = O(1)$

Even though we are concerned with motions on a scale $L_* \sim 1$ km, a brief indication of the influence of rotation on the diffusive instability is in order. Eq. (2.4) shows that it is only necessary to add the term $f^2/v^2 m^4$ to (2.7). In addition, our focus on the ocean problem suggests setting $\tau \ll 1$ and $R = O(1)$, thereby yielding the asymptotic wavenumber equation

$$a^2 - \varepsilon a + (m^4 + f^2/v^2)H^{-1} = 0.$$  

The two roots ($a_1, a_2$) of this satisfy

$$a_2 - a_1 = [\varepsilon^2 - 4H^{-1}(m^4 + f^2/v^2)]^{1/2},$$

and the boundary condition then gives

$$\frac{4\pi^2}{L_* m^2} + \frac{4(f^2 + v^2 m^4)}{(-gS_z)vK_S} = \varepsilon^2$$  

(2.9)

This is minimized at

$$m_c^2 = \left[ \frac{\pi^2}{2L_*^2} \frac{(-gS_z)}{vK_S} \right]^{1/3}$$  

(2.10)

and therefore instability occurs if

$$\left( 1 + \frac{1}{2} \right) \left[ \frac{4\pi^2}{L_*^2} \right] \left[ \frac{\pi^2}{2L_*^2} \frac{(-gS_z)}{vK_S} \right]^{-1/3} \leq \frac{S_x^2}{S_z} - \frac{4K_S}{v} \frac{f^2}{(-gS_z)}.$$  

(2.11)

If $S_x$ is replaced by

$$\Delta S = L_* S_x,$$  

(2.12)

Eq. (2.11) becomes

$$\frac{\Delta S}{S_z} \geq L_* \left[ \frac{6\pi^2}{L_*^2} \left[ \frac{\pi^2}{2L_*^2} \frac{(-gS_z)}{vK_S} \right]^{-1/3} \frac{4K_S}{v} \frac{f^2}{(-gS_z)} \right]^{1/2}.$$  

These results may also be written as

$$\frac{6\pi^2}{m_c^2} \left( \frac{\Delta S}{S_z} \right)^{-2} = 1 - \frac{4S_z^2}{S_x^2} \frac{f^2}{(-gS_z)} \frac{K_S}{v},$$  

(2.13)
and this relates the marginally stable wavelength $2\pi/m_c$ to $f$ and the lateral salinity variation.

For typical ocean parameters and for the molecular diffusivities given below:

$$f = 10^{-4} \text{ s}^{-1}, \quad -g\bar{S}_z = 3 \times 10^{-5} \text{ s}^{-2}, \quad \bar{S}_z/\bar{S}_x = 2 \times 10^{-3}$$

$$K_s = 10^{-5} \text{ cm}^2/\text{s}, \quad \nu = 10^{-2} \text{ cm}^2/\text{s}.$$  

(2.14a)

the value of the rotational term in (2.13) is

$$\frac{4\bar{S}_z^2 f^2}{\bar{S}_x^2 (-g\bar{S}_z)} \frac{K_s}{\nu} = \frac{1}{3},$$

(2.14b)

which is not negligible compared to unity.

d. Supercritical lateral gradients ($f = 0$)

For $f = 0$ and finite $\lambda$ Eq. (2.3) is nondimensionalized using

$$\lambda = \hat{\xi} K_s m^2, \quad k/m = [\bar{S}_z/(\bar{S}_z)] a,$$

(2.15a,b)

$$C = 1 - \bar{T}/\bar{S}_z > 0, \quad \tau = K_s/K_T, \quad \sigma = \nu/K_T,$$

(2.15c)

$$\Delta S \equiv L_\gamma \bar{S}_z, \quad \mu \equiv m \left( -\frac{\Delta S}{\bar{S}_z} \right), \quad \eta \equiv \frac{-g\bar{S}_z}{K_T \nu} \left( \frac{\bar{S}_z}{\bar{S}_x} \right)^2 \Delta S^4 > 0.$$  

(2.15d)

Then (2.3) becomes

$$a^2 \left[ 1 + \hat{\xi} C - \tau (1 - C) \right] + a (1 - \tau) + (1 + \hat{\xi}/\sigma)(1 + \hat{\xi} + \tau) \eta^{-1} \mu^4 = 0.$$  

(2.16)

The two $a$-roots of this must satisfy the boundary condition

$$\mu(a_2 - a_1) = 2\pi,$$

(2.17)

[or an integral multiple of this]. When the discriminant of (2.16) is used for $(a_2 - a_1)$ we get the following quartic equation for (the growth rate) $\hat{\xi}$ as a function of the (vertical) wavenumber $\mu$:

$$\mu^2 \left( (1 - \tau)^2 - 4 [1 + \hat{\xi} C - \tau (1 - C)] (1 + \hat{\xi}/\sigma) (1 + \hat{\xi} + \tau) \frac{\mu^4}{\eta} \right)$$

$$= 4\pi^2 [1 + \hat{\xi} C - \tau (1 - C)]^2.$$  

(2.18)

For highly supercritical $\eta$ a quartic equation (2.18) for $\hat{\xi}$ must be solved, but it is much simpler to assume a real $\hat{\xi}$ and solve the cubic equation for $\mu^2 > 0$. A search is then made for the maximum $\hat{\xi}$ which gives a positive root $\mu^2$. The result for $\tau = 0.01$ (Fig. 4) shows that as $\eta$ increases, the vertical length $(2\pi/m_*)$ of maximum growth rate ($\lambda$) is nearly a constant multiple of the “Chen scale.” For $R = \bar{T}/\bar{S}_z = 0.8$ the value of $m = m_*$ at large $\eta$ is
whereas the maximum growth rate increases by two orders of magnitude in the range

\[ 25 < \eta^{1/2} \equiv \left( \frac{-g \overline{S}_z}{K_T v} \right)^{1/2} \left( \frac{\overline{S}_z}{\overline{S}_z} \right) \left( \frac{\Delta S}{\overline{S}_z} \right)^{2}. \]  

The reason for this large increase is that when \( \eta \) is near the minimum critical value the growth rate is proportional to \( K_S \), whereas (3.14b) is independent of \( K_S \) for \( \eta^{1/2} > 25 \). As indicated in the conclusion Eq. (2.19a) can be used to relate observed layer thickness to frontal strength \( \Delta S \).

Especially important for the subsequent (Section 4) \textit{nonlinear} calculation is the limit of Eq. (2.18) when

\begin{equation}
m_{\infty} = 0.7 \left( \frac{-\overline{S}_z}{\Delta S} \right),
\end{equation}

Figure 4. The nondimensional maximum growth rate (right axis) and the corresponding nondimensional vertical wavelength (left axis) as a function of \( \eta \). Three different density ratios are considered and the results for the growth rate (wavelength) are denoted by \( \bullet \), \( \square \), \( \triangle \) for \( 1 - R = 0.1 \), 0.2, and 2.0, respectively. In the latter case the temperature increases upwards.
\[ \tau \to 0, \text{ and } \eta_2 = \left( \frac{\Delta S}{S_z} \right)^4 \left( -g \frac{\bar{S}_z}{K_S \nu} \frac{\bar{S}_z}{S_z} \right)^2 = \tau^{-1} \eta = (O(1)) \] (2.20a)

and

\[ \hat{\zeta} = \tau (h \eta_2 - 1), \] (2.20b)

where \( h \) is an \( O(1) \) constant. Then the limit of (2.18) gives

\[ \mu^2 (1 - 4h \mu^4) = 4\pi^2, \] (2.21)

and the dimensional growth rate is

\[ \lambda = \hat{\zeta} k_l m^2 = K_S \mu^2 \left( \frac{\Delta S}{S_z} \right)^2 (\eta_2 h - 1), \]

or

\[ \frac{\lambda}{K_s (\Delta S/S_z)^2} = \frac{\eta_2}{4} (\mu^{-2} - 4\pi^2 \mu^{-4}) - \mu^{+2}. \] (2.22)

We note in passing that this is positive only if \( 4/\eta_2 < \mu^{-4} - 4\pi^2 \mu^{-6} \), and since the latter has a maximum at \( \mu_C^2 = 6\pi^2 \), it follows that the condition for \( \lambda > 0 \) is

\[ \eta_2 = \left( -\frac{g \bar{S}_z}{K_S \nu} \frac{\bar{S}_z}{S_z} \right)^2 \left( \frac{\Delta S}{S_z} \right)^4 \geq (12)(36)\pi^4. \] (2.23)

This is equivalent to (2.8)–(2.9) when \( K_S \to 0 \). For \( \tau \to 0, \lambda \to 0 \) the quadratic equation (2.16) also gives \( a_2 + a_1 = -1 \) and \( a_2 - a_1 = \sqrt{2/3} \), or \( a_2 = -(1 - \sqrt{2/3})/2 \), and \( a_1 = (-1 - \sqrt{2/3})/2 \). Therefore for either \( k_{1,2} = k_1 \) or \( k_{1,2} = k_2 \) the amplitude ratio of temperature/salinity of the two components of the normal mode is

\[ \frac{K_S \left( k_{1,2} \bar{T}_z - m \bar{S}_z \right)}{K_T \left( k_{1,2} \bar{S}_z - m \bar{S}_z \right)} = \tau \left( \frac{a_{1,2} \bar{T}_z / \bar{S}_z + 1}{a_{1,2} + 1} \right). \] (2.24a)

where \( a_{1,2} = k_{1,2}/m(-\bar{S}_z/\bar{S}_z) \). We therefore recover the important qualitative result

\[ \frac{\text{temperature perturbation}}{\text{salinity perturbation}} = O(\tau) \to 0, \] (2.24b)

implying that the salinity “torque” balances the viscous term in the vorticity equation, while the thermal torque is negligible (\( \tau \to 0 \)). We can also show that as a multiple of the streamfunction amplitude the salinity normal mode is proportional to

\[ \frac{vm^4}{g} \left[ e^{ik_1x} - e^{ik_2x} \right] e^{imz} e^{\lambda t} \]

so that the vertical integral of the horizontal salt flux is proportional to
\[
\frac{2 \nu m^4}{g} \text{Re} \left( \text{Im} [e^{ik_1x} - e^{ik_2x}] \left[ \frac{e^{ik_1x}}{ik_1} - \frac{e^{ik_2x}}{ik_2} \right] \right) \quad (2.25) \\
= \frac{2 \nu m^4}{g} \left( \frac{m}{k_1} + \frac{m}{k_2} \right) \left[ 1 - \cos (k_1 - k_2)x \right] |\hat{\psi}|^2 e^{2\lambda t}.
\]

This is a negative definite for \((\overline{S}_x > 0)\) because

\[
\left( \frac{m}{k_1} + \frac{m}{k_2} \right) = \left( \frac{\overline{S}_x}{S} \right) \left( \frac{1}{a_1} + \frac{1}{a_2} \right)
\]

and because both \(a_1\) and \(a_2\) are negative. As expected the eddy salt flux (2.25) is from the high end \((x = L_*)\) to the low end \((x = 0)\), with a maximum at \(x = L_*/2\), a result which will be used in Section 4.

3. A theorem on the mean vertical circulation

a. Nonrotating case

The amplifying mode produces a divergent eddy flux [Eq. (2.25)] which will modify the vertically averaged [hereafter denoted by \(\langle \cdot \rangle\)] \(T, S\) fields, and we want to see how this affects the amplitude of the disturbance. Accordingly, we let

\[
S = \overline{S}_z + x\overline{S}_x + S_d, \quad S_d = S'(x, z, t) + \langle S \rangle,
\]

\[
T = \overline{T}_z + x\overline{T}_x + T_d, \quad T_d = T'(x, z, t) + \langle T \rangle,
\]

\[
w = w'(x, z, t) + \langle w \rangle,
\]

\[
\langle S' \rangle = \langle T' \rangle = \overline{S'} = \overline{T'} = 0 = \langle w' \rangle = \overline{w},
\]

where \(S', T', w'\) are periodic disturbances in a specified fundamental vertical wavelength, but the overall vertically averaged \(\overline{T}_z, \overline{S}_z\) do not change. Since the vertical average of \(u\overline{S}_x = 0\), the equations for vertically averaged heat, salt and (primitive) vorticity are then given respectively by

\[
\frac{\partial}{\partial t} \langle T \rangle + \langle w \rangle \frac{\partial \overline{T}}{\partial z} + \frac{\partial}{\partial x} \langle u' T' \rangle = 0 \quad (3.1a)
\]

\[
\frac{\partial}{\partial t} \langle S \rangle + \langle w \rangle \frac{\partial \overline{S}}{\partial z} + \frac{\partial}{\partial x} \langle u' S' \rangle = 0, \quad (3.1b)
\]

\[
g \frac{\partial}{\partial x} (\langle T \rangle - \langle S \rangle) = \left( \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right) \frac{\partial \langle w \rangle}{\partial x} - \frac{\partial^2}{\partial x^2} \langle u' w' \rangle, \quad (3.2)
\]

the last term being the Reynolds stress. Elimination of \(\langle T \rangle, \langle S \rangle\) yields

\[
\frac{\partial}{\partial x} \langle w \rangle g (\overline{T}_z - \overline{S}_z) + g \frac{\partial^2}{\partial x^2} \langle u'(T' - S') \rangle = -\frac{\partial}{\partial t} \frac{\partial}{\partial x} \left[ \left( \frac{\partial}{\partial t} - v \frac{\partial^2}{\partial x^2} \right) \langle w \rangle - \frac{\partial}{\partial x} \langle u' w' \rangle \right]. \quad (3.3)
\]
For hydrostatic motions \[\frac{\partial w}{\partial t} \approx g(T' - S')\] with time scales large compared to the buoyancy period the first (second) term on the right-hand side of Eq. (3.3) is small compared to the first (second) term on the left, and thus the right-hand side may be set equal to zero. Since \(\langle \vec{w} \rangle = 0\) and \(\partial /\partial x \langle u' (T' - S') \rangle = 0\), Eq. (3.3) reduces to the important result

\[
\langle w \rangle = \frac{- \frac{\partial}{\partial x} \langle u' (T' - S') \rangle}{T_z - S_z}.
\]  

(3.4)

It then follows from (3.1a,b) that \(\partial (T - S)/\partial t = 0\), i.e., “overall” density compensation in the longwave theory is insured at all time; and (3.1b) becomes

\[
\frac{\partial \langle S \rangle}{\partial t} = - \frac{\partial}{\partial x} \langle u' T' - u' S' \rangle + \frac{\partial}{\partial x} \langle u' S' \rangle.
\]

(3.5)

In many convection problems a “down-gradient” eddy flux, such as the last term in (3.5), is a stabilizing (amplitude limiting) effect, but in the present problem it may be offset by the \(\langle w \rangle\) term. This rotates the isotherms (isohalines) and modifies \(\langle S_x \rangle = \langle T_x \rangle\), thereby affecting the perturbation.

**b. Density compensation with Coriolis parameter \(f\)**

This result is easily generalizeable to the rotating case when a mean baroclinic shear is absent; it is only necessary to add a velocity component \(V(x, z, t)\) in the \(y\) direction to the other nonlinear components \(U, W\). The \(y\)-independent hydrostatic solutions are

\[
\begin{align*}
\frac{dU}{dt} - fV &= -\partial p/\partial x + \nu \partial^2 U/\partial z^2 \\
\frac{dV}{dt} + fU &= \nu \partial^2 V/\partial z^2 \\
0 &= -\partial p/\partial z - \rho g \\
\partial U/\partial x + \partial W/\partial z &= 0
\end{align*}
\]

where \(p, \rho\) are the dimensional pressure and density. Eliminating \(p\) gives

\[
\frac{\partial}{\partial z} \left( \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} U^2 + \frac{\partial}{\partial z} (WU) \right) - f \frac{\partial V}{\partial z} - \nu \frac{\partial^3 U}{\partial z^3} = g \frac{\partial \rho}{\partial x}.
\]

As done previously we look for \(z\)-periodic \((U, V, W, \ldots)\), in which case the vertical average of all terms on the left-hand side must vanish, thereby yielding

\[
0 = \langle \partial \rho / \partial x \rangle.
\]

Thus “overall” density compensation in this theory is maintained at all times if it exists initially. Since (3.1a,b) still hold, the mean circulation formula (3.4) still applies. Is there a
more general consideration indicating that the convection of $T/S$ finite structure tends to produce stronger lateral $T/S$ gradients than density gradients?

4. Nonlinear equations for $\tau \to 0$ and $\Delta S \to 0$

a. Asymptotic equations

To obtain a relatively simple nonlinear solution we introduce a nondimensionalization which is guided by the foregoing asymptotic results for $\tau \to 0$. For physical and motivational reasons explained below, let

$$x = L_\ast x_0, \quad z = [\Delta S/(-\overline{S_c})]z_0, \quad t = t_0[\nu^{-1}[\Delta S/(-\overline{S_c})]^3 g \Delta S / L_\ast]^{-1}L_\ast$$

(4.1a,b,c)

$$S_d = (\Delta S)S_0 = \Delta S[S'_0(x_0, z_0, t_0) + \langle S_0 \rangle], \quad T_d = \tau \Delta S T'_1 + \Delta S \langle S_0 \rangle$$

(4.2a,b,c)

$$u = \frac{(g \Delta S)}{L_\ast} v^{-1}[\Delta S/(-\overline{S_c})]^3 u_0, \quad w = \frac{(g \Delta S)}{L_\ast^2} v^{-1}(\Delta S/\overline{S_c})^4 [w'_0 + \langle w_0 \rangle]$$

(4.3a,b)

$$\tau = K_S / K_T \to 0, \quad K_S / \nu \to 0, \quad \eta_2 = \frac{(-g \overline{S_c})}{K_S \nu} \left(\frac{\overline{S_c}}{\overline{S_c}} \right)^2 \left(\frac{\Delta S}{\overline{S_c}} \right)^4 = O(\eta_2)_{\text{crit}}$$

(4.4a,b,c)

and for fixed $L_\ast$, let us decrease $\overline{S_c}$ and $K_S$ with $\eta_2 = \text{constant}$.

In (4.1a,b) the chosen $x$-$z$ scaling factors are obvious. The scaling in Eq. (4.3a) will make the viscous term balance the salt buoyancy term in the vorticity equation. In Eqs. (4.2) $S_d, T_d$ are the total departures from the undisturbed fields with gradients ($\overline{S_c}, \overline{S_c}), (T_c, \overline{T_c}),$ and $\langle S_0 \rangle$ is the modified vertically averaged field. The thermal buoyancy ($\tau T'_1$) in (4.2b) is negligible to leading order, although the thermal mean field $\langle T \rangle$ is always equal to the salinity mean field. The scaling in Eq. (4.1c) will make the time dependent term in the salinity equation of the same order as the advection term. When that equation is divided by the product of the undisturbed $\overline{S_c}$ and $(g \Delta S / L_\ast) v^{-1}[\Delta S/(-\overline{S_c})]^3$ we get the asymptotic salt equation

$$\frac{\partial S_0}{\partial t_0} + u_0 - w_0 + u_0 \frac{\partial S_0}{\partial x_0} + w_0 \frac{\partial S_0}{\partial z_0} = \frac{1}{\eta_2} \frac{\partial^2 S_0}{\partial z_0^2}$$

(4.5)

$$w_0 = w'_0 + \langle w_0 \rangle, \quad S_0 = S'_0 + \langle S_0 \rangle.$$  

(4.6)

[The separation of the linear ($w_0$) and nonlinear ($w_0 \partial S_0 / \partial z$) advective terms in (4.5) is merely a matter of convenience; likewise, the negative sign on $w_0$ is merely due to the convention of scaling salinity with $-\overline{S_c}$.] The vertically averaged horizontal velocity $u_0$ is equal to zero and the vorticity and continuity equations are now respectively given by

$$- \frac{\partial^2 u_0}{\partial z_0^2} = \frac{\partial S'_0}{\partial x_0}$$

(4.7)
\[ \frac{\partial u_0}{\partial x_0} + \frac{\partial w_0}{\partial z_0} = 0. \]  

(4.8)

Note that \( T' \) does not appear in this system, but if desired the leading temperature perturbation \( (T'_1) \) can be obtained by setting \( (1/\eta)\partial^2 T'_1/\partial z^2 \) equal to the sum of \( u_0 - w'_0(\overline{T}_z S_z) \) plus the fluctuating nonlinear temperature advection term. To complete the system we need (3.1b) and (3.4), the nondimensional asymptotic form of which are

\[
\frac{\partial \langle S_0 \rangle}{\partial t_0} + (-1)\langle w_0 \rangle = -\frac{\partial}{\partial x_0} \langle u_0 S_0 \rangle, \tag{4.9a}
\]

\[
\langle w_0 \rangle = \frac{\partial}{\partial x_0} \langle u_0 S_0 \rangle \left/ \left( 1 - \frac{\overline{T}_z}{\overline{S}_z} \right) \right., \tag{4.9b}
\]

or

\[
\frac{\partial \langle S_0 \rangle}{\partial t_0} = \frac{R}{1 - R} \frac{\partial}{\partial x_0} \langle u_0 S_0 \rangle. \tag{4.9c}
\]

These clearly show that the down-gradient eddy flux [r.h.s. of (4.9a)] is completely offset by the induced mean vertical circulation which tends to rotate the mean isohalines and isotherms, thereby converting vertical gradients into horizontal ones. For diagnostic purposes a “power integral” can also be obtained by multiplying (4.5) with \( S_0 = S'_0 + \langle S \rangle \) and averaging over all \( (x, z) \):

\[
\frac{1}{2} \frac{\partial}{\partial t} \left\{ \langle (S'_0)^2 \rangle + \langle S_0' \rangle^2 \right\} + \langle u_0 S'_0 \rangle - \langle w'_0 S'_0 \rangle - \langle w \rangle \langle S'_0 \rangle = -\langle (\partial S'_0/\partial z)^2 \rangle \eta^{-1}. \tag{4.10}
\]

Since \( \partial S'_d/\partial z = \Delta S[\Delta S/(\overline{S}_z)]^{-1}(\partial S_0/\partial z_0) = -\overline{S}_z \partial S_0/\partial z_0 \) the dimensional value of the density ratio at any time is

\[
\frac{\overline{T}_z}{\overline{S}_z} \frac{\partial S_d}{\partial z} = \frac{\overline{T}_z}{\overline{S}_z} \frac{1}{1 - \frac{\partial S_0}{\partial z_0}} = \frac{R}{1 - \frac{\partial S_0}{\partial z_0}}.
\]

Therefore the nonlinear calculation for \( R > 0 \) will only be continued up to the time when \( \partial S_0/\partial z_0 \approx 1 - R \), at which the vertical density gradient becomes unstable and strong vertical convection (Veronis, 1965) is expected to produce well mixed layers. An additional restriction is due to the fact that the nondimensional coefficient of the time dependent term in the momentum equation is \( O(\eta^2) \); and since \( \eta^2 = \tau \eta \) this limits the maximum supercritical value of \( \eta_2 \) (cf. Fig. 4) that can be used in the following.

b. Numerics

The foregoing equations were solved using the pseudo-spectral Galerkin method with Fourier/Chebyshev expansions in the vertical/horizontal. The \( N_1 \times N_2 \) grid had regular
z-spacing and Gauss-Lobatto collocation points in the horizontal [0, 1] interval. Two of the resulting ordinary differential equations are replaced by Neumann boundary conditions for 

\[ u_0(x = 0, 1) = 0 = dS^0_0/dx. \]

Temporal integration was performed with a fourth order Runge-Kutta scheme and a time step \( \Delta t_0 \). For initialization the previously obtained eigenfunctions with a small amplitude were used; Figure 5a shows the velocity vectors for our first run with \( \tau = 0.01, R = \frac{\mathbf{T}^0}{S^0} = 0.6, \eta_2 = 2\eta_{cr} = (24)(36)\pi^4 \). This is twice the marginal stability value, and the value of \( m = 8.37 \) is approximately the fastest growing wavenumber. The subsequent exponential growth (not shown) using \( \Delta t_0 = 0.05 \) agreed with the linear analytic result, and at \( t_0 = 22 \times 10^2 \) (Fig. 5b) the maximum velocity vector has increased by a factor of 25. Note that the counter-rotating cells in Figure 5a give way to co-rotating cells (Fig. 5b). This occurs because \( \partial \rho/\partial x(u'S^0_0) \) is negative-maximum half way between the center (\( x = 1/2 \)) and the end wall (\( x = 1 \)), so that \( \langle w \rangle > 0 \) therein. The isopycnals (total values) overturned (Fig. 6a) shortly before this time, so that the condition for the onset of strong vertical convection is reached. The amplitude normalized plot of \( \partial w/\partial x \) at the center of the gyre (Fig. 6b) indicates that the lateral width of the disturbance shrinks by a factor of \( (0.09/0.02)^2 \) during the run, suggesting that the intrusive instability is entraining the horizontal gradient into the central region, rather than diffusing it outwards. To further elucidate the nonlinear effect \( \eta_2 = 12 \times 36 \times \pi^4 \) was reduced to its critical value (marginal stability) with vertical wave number \( m = \sqrt{6\pi^2} \). The initial amplitude was chosen three times larger than in the previous runs, and \( \Delta t_0 = 0.02 \) was used. As expected the amplitude of the mode remained constant initially, but the time independent eddy salt flux in (4.9a,b) gives rise to a mean field \( \langle S^0 \rangle \) which increases linearly with time, thereby producing a lateral gradient with a supercritical \( \eta \); this then causes a temporal amplification of the perturbation, as is clear from the plot (Fig. 7a) of all the terms in the power integral. The most convincing evidence of the \( T/S \) frontogenetical tendency appears in Figure 7b where the plot of \( \langle S^0 \rangle \) shows the temporally increasing gradient at the center [a “diffusive” effect, on the other hand, would decrease the gradient here and increase it at the end points.] Unfortunately, we are unable to evaluate the extent of the subsequent frontogenesis because of the onset of large Reynolds number turbulence.

The weak velocities at the walls [\( x = 0, 1 \)] suggest that these walls are mere artifacts of the numerical calculation, so that the model in Figure 3c will probably give the same global behavior, viz. intrusions amplifying a high gradient region until some transition to a statistically stable regime is reached. We have only achieved such a regime in our last run with \( \eta_2 \) = half the critical value; Figure 8 shows that this state is stable to finite amplitude effects, so that vertical convection will not form even though \( \mathbf{T}^0 \) is negative.

5. Conclusions

We have shown that when the isopycnal variation of salinity exceeds a certain value, as given by the Thorpe et al. (1969) critical number
Figure 5. (a) The velocity eigenvectors at time $t_0 = 0$ in our first calculation with the nonlinear asymptotic model for $\eta_2 = 2 \eta_c$, and density ratio $R = 0.6$. This state corresponds to the fastest growing normal mode of the bounded model with a vertical wavelength of 0.75. (b) Total velocity vectors at time $t_0 = 22 \times 10^2$. Note the relatively small vectors at the vertical walls. The numerical domain had 33 and 16 grid points in the horizontal and the vertical direction, respectively. The shown results have been interpolated (for graphical purposes) on a grid with a regular horizontal spacing. Note the co-rotating cells at this time.
Figure 6. (a) The overturning isopycnals at time $t_0 = 21 \times 10^2$ in the supercritical run of Figure 5. The total nondimensional density is $\rho(x_0, z_0) = S'_0(x_0, z_0) + (R - 1) z_0$. (b) The horizontal length scale of the disturbance at the center of the gyre as a function of time in the supercritical run. This length scale is defined as the inverse of the horizontal gradient of the vertical velocity normalized by the r.m.s. amplitude of the vertical velocity. This shows that the widths of the cells shrink towards the center of the front.
Figure 7. (a) The terms in the salinity variance power integral (4.11) and their sum (dashed line) as a function of time in our second run for marginal stability parameters \( \eta_2 = \eta_c \) and \( R = 0.6 \). The terms were multiplied by \( 10^6 \). The model was initialized with the marginally stable normal mode whose vertical wavelength is equal to \( \sqrt{2/3} \) (and whose initial amplitude was 3 times larger than the one in the first run). The isopycnals overturn at the end of this run because of the positive feedback from the mean field modification. (b) The vertically averaged salinity profiles \( \langle S_0 \rangle \) as a function of \( x_0 \) for \( t_0 \times 10^{-2} = 0, 1, 2, 3, 4 \) (denoted by +, ■, □, ●, and ○) illustrate the initially linear growth of the mean field in the above critical run. The same grid sizes as in Figure 5 were used.
lateral intrusions amplify [Fig. 3], and the preferred asymptotic ($\eta \rightarrow \infty$) vertical wavelength is

$$\delta_z \equiv 0.7 \Delta S / \bar{S}_z.$$ 

In the longwave theory the amplifying disturbance produces laterally converging $T/S$ fluxes which induce a mean vertical velocity $\langle w \rangle$ that preserves density compensations [i.e., $\langle T_x \rangle = \langle S_x \rangle$] at all $(x, t)$. The $\langle w \rangle$ relation (3.4) was used to compute the individual
mean field modifications (e.g., \( \langle S \rangle \)) in a nonlinear calculation initialized with the fastest growing mode. For the relatively simple asymptotic model with \( K_S \to 0, \Delta S \to 0 \) the horizontal velocity \( u \propto u_0 \exp \lambda t \) continues to increase exponentially in the nonlinear stage, and the eddy salt flux produces frontogenesis. The surfaces of (total) salinity and temperature fold after some time \( \Delta t_* \), thereby increasing the local value of the density ratio \( (\bar{T}/\bar{S})_z \) to the Veronis critical value for the onset of strong small scale vertical convection. Prior to this the lateral scale of the circulation cells shrink [Eq. 5b], and these cells change from counter-rotating (at \( t = 0 \)) to co-rotating (Fig. 5b). This effect [due to \( \langle w \rangle \)] probably accounts for the co-rotating cells observed in the Ruddick et al. (1999, Fig. 4) sugar/salt experiment, even though the latter are driven by salt finger convection.

Although statistical equilibrium has not been reached in our numerical calculation, some order of magnitude estimates can be made which are relevant to the small scale fronts in the polar oceans. The finite displacement \( \Delta x_* \), of a parcel in time \( \Delta t_* \) computed from \( dx/dt = u \sim \hat{u}_0 \exp \lambda t \) is

\[
\Delta x_* \sim (\hat{u}_0/\lambda) \exp(\lambda \Delta t_*) \sim u/\lambda
\]

Therefore for highly supercritical conditions (\( \eta^{1/2} \gg 1 \)) and large amplitude we have \( \Delta x_* = A u/\lambda \), where \( A = O(1) \) is a constant independent of \( \Delta S \). Furthermore, we assume that folding occurs in a time \( \Delta t \), in which a parcel travels a distance \( \Delta x \sim L_* \) with velocity \( u \); and therefore

\[
\Delta t_* \sim 1/\lambda.
\]

This means that an initially unstable front will develop strong small scale vertical convection in an interval of the same order of magnitude as the e-folding time of the fastest growing lateral mode.

From Muench et al. (1990) and Robertson et al. (1995) the following typical values are obtained for the Weddell Sea:

\[
R = 0.7, \quad \partial \bar{S}/\partial z = -0.3 \%/100 \text{ m} = 3 \times 10^{-5} \%/\text{cm}^{-1}
\]

\[
\Delta S = 0.005 \%/\text{cm}, \quad L_* \sim 4 \times 10^5 \text{ cm}.
\]

From these we calculate

\[
\left| \frac{\bar{S}_x}{\bar{S}_z} \right| = \frac{0.005(4 \times 10^5)^{-1}}{3 \times 10^{-5}} \approx 4 \times 10^{-3}
\]

\[
\eta^{1/2} \sim (0.4 \times 10^{-3})(\frac{5}{3} \times 10^2)\left(\frac{3 \times 10^{-2}}{10^{-5}}\right)^{1/2} \approx 500
\]

\[
\Delta t_* \sim 5 \text{ days}
\]

vertical wavelength [Eq. (2.19a)] \( \approx 1 \text{ meter} \).
Thus the smallest observed step size in the Weddell halocline might be due to a density front ~ 4 km wide across which the salinity difference was as small as 0.003‰; in O (1 week) the 1 m intrusions would produce strong vertical convection, [as occurs in Turner’s (1973) experiments]; and this might be followed by the growth of the thicker mixed layers (~10 m). In the absence of a significant surface heat flux forcing, the small scale lateral intrusive instability might be necessary to maintain the very thin interfaces separating the observed mixed layers. The physical importance of these interfaces is due to the molecular heat flux which resupplies the turbulent flux in the thick layer.

Some additional work (see the Appendix) has been done which indicates that the artificial lateral boundary conditions used in the previous calculation does not greatly influence the results, but it would be desirable to perform additional runs relaxing these conditions (as suggested in Figs. 3c,d). In order to include a nonpassive $T$-effect one would want to extend the parametric range of the numerics beyond the $\tau \to 0$ asymptote used herein. Finally, we mention that the role of a baroclinic shear flow is of primary oceanographic concern (Kelley et al., 2003).

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APPENDIX

The effect of relaxing the lateral boundary conditions

Are convective modes ($\lambda =$ real) possible with nonuniform lateral gradients (Fig. 3c)?

To answer this question consider an arbitrary lateral salinity gradient $\bar{S}_x$ between boundaries at $x_1$, $x_2$ which may be at infinity. The linearized version of (4.5), (4.7), (4.8) for $\bar{S}_x \equiv 1 + \frac{\partial S'}{\partial x_0}$ then leads to the following eigenfunction problem for $S'_0$ with amplitude $\hat{S}$, and for the streamfunction with amplitude $\hat{\psi}$:

$$\left( \lambda + \frac{m^2}{\eta_2} \right) \hat{S} - \hat{\psi}_x - im\bar{S}_x \hat{\psi} = 0,$$

$$m^4\hat{\psi} = \hat{S}_x,$$

or

$$m^4(\lambda + m^2/\eta_2)\hat{\psi}_x - \hat{\psi}_{xx} - im \frac{d}{dx} (\hat{\psi} \bar{S}_x) = 0. \quad (A1)$$

with $\hat{\psi} = 0$ at $x_1$, $x_2$. The previous equation is now multiplied by the complex conjugate $\hat{\psi}^*$, averaged, and the imaginary part taken. If $\lambda =$ real then

$$\int dx \hat{\psi}^* \hat{S}_{xx} + \text{Re} \int dx \hat{\psi}^* \hat{\psi}_x \bar{S}_x = 0. \quad (A2)$$
The last term is equivalent to the integral of

\[ \frac{1}{2} (\hat{\psi}_x \hat{\psi}_x + \hat{\psi}_x \hat{\psi}_x)S_x = \frac{d}{dx} (\hat{\psi}_x \hat{\psi}_x)S_x \]

and this reduces to \(-1/2\) times the first integral in (A2). It follows that

\[ \int dx |\hat{\psi}_x|^2 S_{xx} = 0 \]

is a necessary condition for \(\text{Im } \lambda = 0\). This means that in a more realistic frontal problem (e.g., Fig. 3c) a pure convective mode is possible if \(S_x\) and \(|\hat{\psi}|^2\) are both symmetric about the midpoint of the \(x\)-interval.

In order to show that such modes actually occur we solved the initial value problem corresponding to Eq. (A1) for \(S_x = \pi/2 \sin \pi x\) in \(0 \leq x \leq 1\), using an initialization consisting of the fastest growing normal mode for the previous \(S_x = 1\) problem. The discretization of the Fourier-Chebychev representation was also the same as previous. In this initial value calculation most of the adjustment to the new normal mode occurs in the first 400 time units, and at \(t = 2500\) the amplifying \(\psi\) mode is quite similar to that obtained for uniform \(S_{x'}\).
Next we made a full nonlinear calculation in which the boundaries were extended to 
$-1/2 \leq x \leq 3/2$, using $\bar{S}_x = \pi/2 \sin \pi x$ in $0 \leq x \leq 1$ and $\bar{S}_x = 0$ outside. We again initialized using the $\bar{S}_x = 1$ normal mode in $0 \leq x \leq 1$, and outside this we set $S(x, z) = S(0, z)$ or $S(1, z)$. Even though the initial amplitude was the same as previously the isopycnals overturned (Fig. 9) significantly earlier (at $t = 7 \times 10^2$ rather than $t = 22 \times 10^2$). The instability is focused at the center $x = 1/2$ suggesting frontogenesis with little influence in the far field. It must be strongly emphasized however that this situation is likely to change at a later stage of the nonlinear evolution when the overturning isopycnals produce density lenses (boluses) which have a natural gravitational tendency to collapse and spread laterally (i.e., away from the front); this might produce diffusion to the far field, as occurs in the Ruddick et al. (1999) experiment.

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